CS 547 Lecture 16: The M/G/1 Queue

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It's finally time to derive the average residence time in the M/G/1 queue. Recall our master tagged customer equation:

$$\overline{R} = U\overline{z} + (\overline{Q} - U)\overline{s} + \overline{s}$$

If we can find the value of the average residual service time, $\overline{z}$, we can solve the equation for $\overline{R}$.

In the previous lecture, we established that the value of $\overline{z}$ depends on the distribution of the interrupted service time, which we denoted $f_Y(x)$. In general, customers are more likely to arrive during a long service period, which skews $f_Y(x)$ in favor of longer service times. The actual distribution of the interrupted service period is given by the mass-weighted service time distribution.

$$\begin{aligned} f_Y(x) &= \text{Prob. a newly arriving job finds a job of length } x \text{ in service} \\ &= \text{Fraction of total server load due to jobs of length } x \\ &= f_S^{mass-weight}(x) \\ &= \frac{x f_S(x)}{\overline{s}} \end{aligned}$$

Suppose a new customer arrives to find a job of length x already in service. On average, customers that arrive to find a job in service will tend to arrive halfway through the job's service period. Therefore, the expected residual service time of any customer that arrives during a job of length x is $\frac{x}{2}$.

$$E[\text{residual service time} \mid \text{finding a job of length } x \text{ in service}] = \frac{x}{2}$$

We can now use total probability to remove the conditioning and calculate the value of $\overline{z}$.

$$\begin{aligned} \overline{z} &= \int_0^\infty \frac{x}{2} P(\text{finding a job of length } x \text{ in service})\, dx \\ &= \int_0^\infty \frac{x}{2} \frac{x f_S(x)}{\overline{s}}\, dx \\ &= \frac{1}{2\overline{s}} \int_0^\infty x^2 f_S(x)\, dx \end{aligned}$$

The integral is simply the second moment of the service time distribution, $E[s^2] = \overline{s^2}$.

$$\overline{z} = \frac{\overline{s^2}}{2\overline{s}}$$

Surprisingly, it turns out that the average residual service time depends only on the first two moment of the service time distribution. Therefore, we can solve for the average residence time in the M/G/1 queue given only the mean and the variance of the service times – we don't need to know the actual distribution!

We can rewrite the result to use the coefficient of variation instead of the second moment.

$$\overline{z} = \frac{\overline{s}}{2}(1 + c_s^2)$$

Substituting this result into the master residence time equation,

$$\overline{R} = U\frac{\overline{s}}{2}(1 + c_s^2) + (\overline{Q} - U)\overline{s} + \overline{s}$$

Solving for $\overline{R}$:

$$\overline{R} = \overline{s} + \frac{U\overline{s}(1 + c_s^2)}{2(1 - U)}$$

This is the formula for the average residence time in the M/G/1 queue. To use it, we need to know three things:

- the average service time, $\overline{s}$
- the coefficient of variation in service times, c_s^2 (which can be calculated from $\overline{s}$ and the variance)
- the utilization, U, which depends on $\overline{s}$ and the Poisson arrival rate λ