Shape from Shading

- People "see" variations in brightness over a region as a "shaded" surface in 3D

Shading – variation in brightness due to surface orientation change

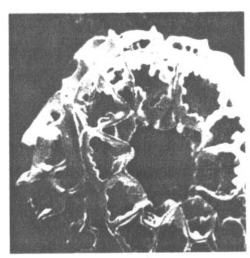
Shadows – variation in brightness due to illumination discontinuity

Goal: Given a single intensity image, recover surface orientation of the surface patch associated with each pixel

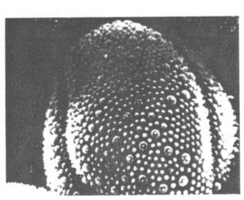
→ Assume: Brightness changes due to orientation changes, not illumination or surface reflectance changes (violated w/ facial makeup!)

→ Application: Astronomy – reconstruct planet surface from image(s)

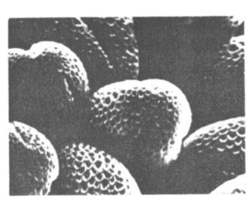
(a) CASTANOPSIS (X 3500)



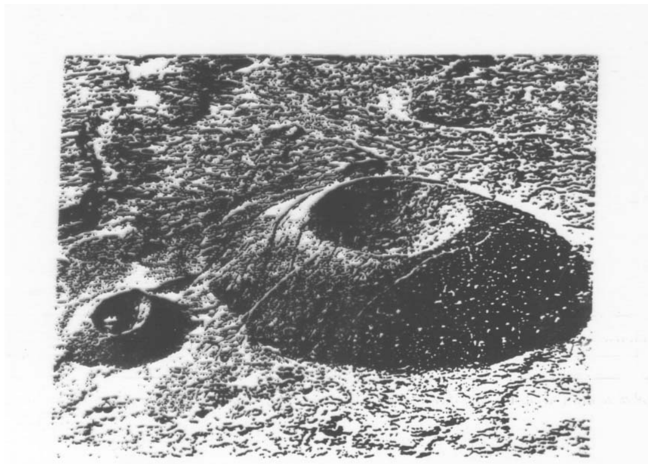
(b) DRIMYS (X 3200)

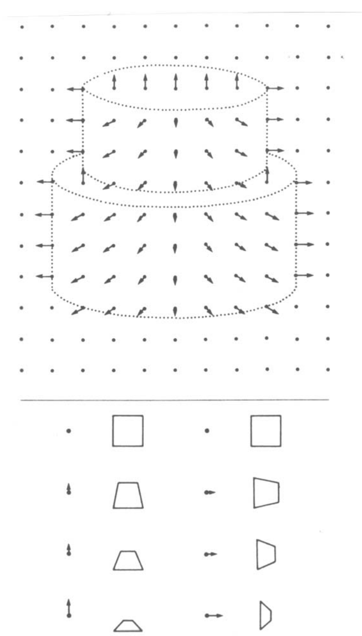


(c) FLAX (X 1000)

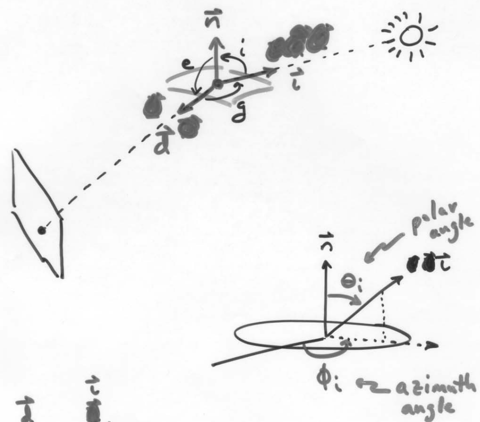


(d) WALLFLOWER (X 1800)





Representing Surface Orientation



$f(\vec{d}, \vec{n}, \vec{i})$

$f(i, e, g) \qquad (g = \phi_e - \phi_i)$

$f(\theta_i, \phi_i, \theta_e, \phi_e)$

All are "object-centered" representations

BRDF is the ratio of the radiance in the outgoing direction to the incident irradiance

We assume:

1) Radiance leaving a point is due only to radiance arriving at the <u>same</u> point
2) All light leaving surface at a given wavelength is due to light arriving at the <u>same</u> wavelength
3) Surfaces do <u>not</u> generate light internally
4) Constant illumination strength at point source

$$L = E_0 \, f_{BRDF}(\theta_i, \phi_i, \theta_e, \phi_e)$$

$$\Rightarrow \text{BRDF:} \quad f_{BRDF}(\theta_i, \phi_i, \theta_e, \phi_e) = \frac{L(\theta_e, \phi_e)}{E(\theta_i, \phi_i)}$$

Some Surface Reflectance functions

1. Lambertian Surfaces
   - Perfect matte surfaces; ideal diffuse surfaces
   - Examples: cotton cloth, matte paint
   - Radiance leaving the surface does not depend on the illumination direction

     ⇒ BRDF is independent of outgoing direction and incoming direction

     ⇒ Surface appears equally bright in all directions

$$f_{BRDF} = \frac{\rho}{\pi}$$

ρ ← constant, called albedo

π ← area of a hemisphere of directions, θ_e, ϕ_e

$$\Rightarrow L = \frac{\rho}{\pi} E_0$$

Combining relation between light source radiance and surface irradiance

$$E_0 = I \cos\theta_i = I \vec{l}^T \vec{n}$$

$$\Rightarrow L = \rho \cos\theta_i$$

↖ "effective albedo" $(= I\rho)$

More specifically,

$$\begin{cases} E = kL \\ L = \frac{\rho}{\pi} E_0 \\ E_0 = I \cos\theta_i \end{cases}$$

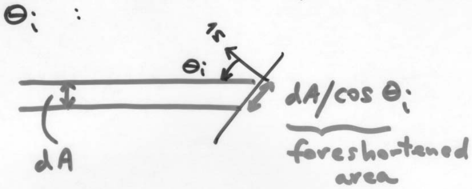
← assumes narrow field of view or optical system has been rectified

$$\Rightarrow E = \frac{\rho}{\pi} k I \cos\theta_i$$

$$= \rho \cos\theta_i$$

Factors Affecting Lambertian Reflection

1. A beam of light hitting a plane covers an area inversely proportional to Θ_i :



2. Amount of light reflected from area dA toward viewer is proportional to $\cos\Theta_e$

   <u>but</u> amount of surface area seen is inversely proportional to $\cos\Theta_e$

   $\Rightarrow$ these 2 factors cancel and amount of light seen is independent of viewing direction, Θ_e

2. Specular Surfaces

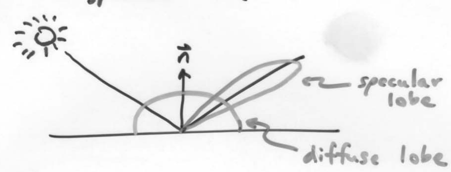
- Ideal mirror surfaces :

  Radiation arriving along a particular direction (Θ_i, ϕ_i) can leave only along the specular direction $(\Theta_i, \phi_i + \pi)$

$$f_{BRDF} = \begin{cases} 1, & \text{if } i = e \text{ and } g = i + e \\ & (\text{i.e., } \Theta_e = \Theta_i \text{ and } \phi_e = \phi_i + \pi \\ 0, & \text{otherwise} \end{cases}$$

3. Lambertian + Specular Surfaces

- Reflectance is a combination of a diffuse (Lambertian) component and a specular component, in the form of a specular lobe :



- Example : plastics

$$f_{BRDF} = \rho_d + \rho_s \underbrace{\cos^n |\theta_e - \theta_s|}_{\text{size of specular lobe}}$$

ρ_s ← specular albedo

where θ_s = specular direction ($= \theta_i$)

Called <u>Phong model</u>

$$\Rightarrow E = I\left[\rho_d \cos\theta_i + \rho_s \cos^n|\theta_e - \theta_i|\right]$$

4. Moon Dust

$$E = I \frac{\cos\theta_i}{\cos\theta_e}$$

At full moon $\theta_i \approx \theta_e$

$\Rightarrow E = k$

$\Rightarrow$ moon appears flat!

Problem: BRDF defined in terms of a <u>local</u> coordinate system on surface (i.e., wrt $\vec{n}$) $\Rightarrow$ can't relate visible brightness in image to viewer.

Solution: Change to viewer-centered surface representation:

Monge patch

$(x, y, Z(x,y))$ — i.e. image point (x,y) at depth $Z(x,y)$

+ assume:

* Point light source at ∞
* orthographic projection
  $\Rightarrow$ viewing direction is constant
  phase angle, g, is constant

Gradient Space

Surface defined by $z = Z(x,y)$

$\Rightarrow$ surface normal

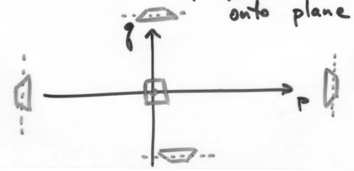
$$\vec{n} = \left(\frac{\partial Z(x,y)}{\partial x}, \frac{\partial Z(x,y)}{\partial y}, +1\right)^T$$

Define $\begin{cases} p = \frac{\partial Z}{\partial x} \\ q = \frac{\partial Z}{\partial y} \end{cases}$

$\Rightarrow$ $\vec{n} = (p, q, 1)$

(p, q) called gradient of $Z(x,y)$

Gradient Space = 2D space of all points (p,q)
= projection of hemisphere onto plane ($\mathbb{R}^2$)



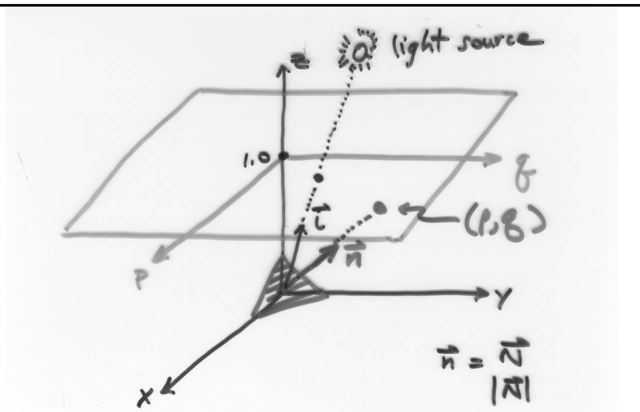
Gradient Space Properties

- A point in gradient space corresponds to a plane with surface normal $\vec{n}$
- Origin of gradient space represents all surfaces with normals parallel to the optical axis
- Edge-on planes are represented at ∞ (i.e., equator of hemisphere maps to points at ∞ in (p,q)-space
- Slant, tilt representation of surface orientation defined by polar coordinate form of (p_0, q_0):

$$\begin{cases} \sigma = \sqrt{p_0^2 + q_0^2} \\ \tau = \tan^{-1}(q_0/p_0) \end{cases}$$

σ ⇝ how much surface inclined wrt viewer

τ ⇝ direction of fastest rate of change



- (p,q) is independent of position of plane
- All parallel planes in scene have the <u>same</u> (p,q)

New Goal: Given $I(x,y)$ recover (p,q)

Given image point (x,y), surface point is $(x, y, Z(x,y))$

Surface point has unit normal:

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{(p, q, 1)^T}{\sqrt{1+p^2+q^2}}$$

Light source unit normal:

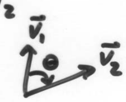
$$\hat{\imath} = \frac{\vec{\imath}}{|\vec{\imath}|} = \frac{(p_s, q_s, 1)^T}{\sqrt{1+p_s^2+q_s^2}}$$

Viewer unit normal:

$$\hat{\ell} = \frac{\vec{\ell}}{|\vec{\ell}|} = (0, 0, 1)^T$$

Recall for 2 vectors $\vec{v}_1$, $\vec{v}_2$

$$\cos\theta = \hat{v}_1 \cdot \hat{v}_2$$



So,

$$\cos\theta_i = \hat{n}\cdot\hat{\imath} = \frac{1 + p p_s + q q_s}{\sqrt{1+p^2+q^2}\,\sqrt{1+p_s^2+q_s^2}}$$

$$\cos\theta_e = \frac{1}{\sqrt{1+p^2+q^2}}$$

$$\cos g = \frac{1}{\sqrt{1+p_s^2+q_s^2}}$$

* Use these relations with specific surface reflectance functions to "compile" specific reflectance maps.

With assumptions and (p,q) parameterization

$$\boxed{E(x,y) = R(p,q)}$$

Image Irradiance Equation (image irrad.)

where R: orientation $\to$ intensity

called Reflectance Map

using normalized R,

$$R: \mathbb{R} \times \mathbb{R} \to [0,1]$$

Reflectance Map relates image irradiance, $E(x,y)$, to surface orientation, (p,q), for a given light source direction and given surface reflectance function

1. Lambertian Reflectance

$$E = I_0 \rho \cos\theta_i$$

$$= I_0 \rho \left(\frac{1 + p p_s + q q_s}{\sqrt{1+p^2+q^2}\,\sqrt{1+p_s^2+q_s^2}} \right)$$

$$\Rightarrow R(p,q) \propto \frac{1 + p p_s + q q_s}{\sqrt{1+p^2+q^2}\,\sqrt{1+p_s^2+q_s^2}}$$

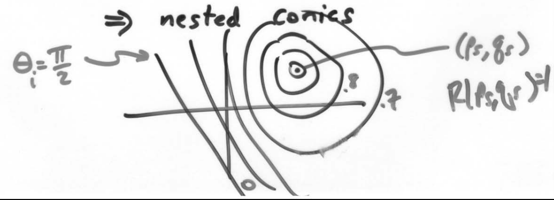
Maximized at $(p,q) = (p_s, q_s)$

Lines of constant R (isobrightness contours)

$\Rightarrow R(p,q) = c$

$\Rightarrow (1 + p p_s + q q_s)^2 = c^2 (1+p^2+q^2)(1+p_s^2+q_s^2)$

$\Rightarrow$ nested conics



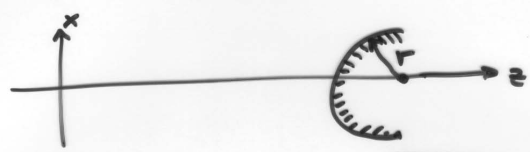
2. Lambertian surface with (point) light source near viewer

$\Rightarrow \theta_i \approx \theta_e \,, \quad \phi_i \approx \phi_e$

$\Rightarrow (p_s, q_s) \approx (0,0)$

$$\Rightarrow R(p,q) = \frac{1}{\sqrt{1+p^2+q^2}}$$

1D Example of Lambertian Sphere



surface: $z = z_0 + \sqrt{r^2 - x^2} \,, \; x \leq r$

$$\Rightarrow p = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{r^2-x^2}}$$

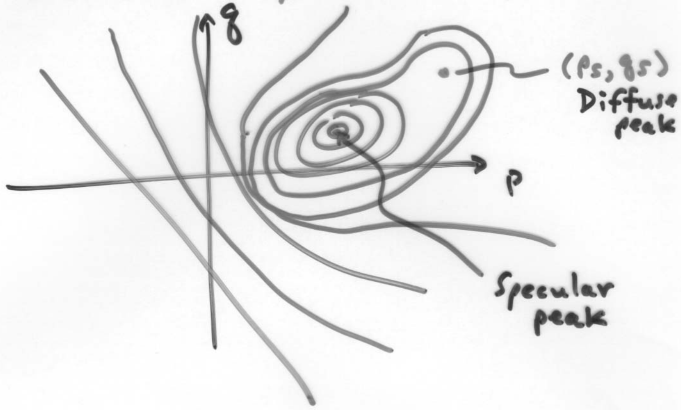
$$\Rightarrow R(p) = \frac{1}{\sqrt{1+p^2}}$$

$$= \frac{\sqrt{r^2-x^2}}{r}$$

$\Rightarrow E(0) = 1$

$E(r) = 0$

3. Lambertian + Specular Surface



4. Moon surface

$$R(p,q) = \frac{\cos \theta_i}{\cos \theta_e}$$

$$= \frac{\dfrac{1 + p p_s + q q_s}{\sqrt{1+p^2+q^2}\,\sqrt{1+p_s^2+q_s^2}}}{\dfrac{1}{\sqrt{1+p^2+q^2}}}$$

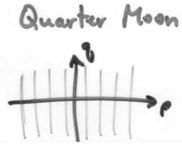
$$= \frac{1 + p p_s + q q_s}{\sqrt{1 + p_s^2 + q_s^2}}$$

Full Moon $\Rightarrow (p_s, q_s) \approx 0$

$\Rightarrow R(p,q) = 1$

Quarter Moon $\Rightarrow (p_s, q_s) = (1, 0)$

$\Rightarrow R(p,q) = \dfrac{1+p}{\sqrt{2}}$



Computing $R(p,q)$

* Experimentally from test objects; vary viewpoint and light source direction; interpolate from many samples

* Theoretically from known BRDF and known light source direc.

Problem: Given source direction (p_s, q_s) and surface reflectance function (BRDF), so that we know $R(p,q)$,
can we find unique surface orientation (p,q) from a single image intensity, $E(x,y)$?

No. Image Irradiance Equation gives 1 equation in 2 unknowns.

Photometric Stereo

Idea: Use multiple point light sources to resolve surface orientation ambiguity, and multiple images.

Fixed camera, fixed scene $\Rightarrow$ correspondence problem is trivial!

$$\begin{cases} E_1(x,y) = R_1(p,q) \\ E_2(x,y) = R_2(p,q) \\ \vdots \end{cases}$$

Example: Lambertian Surfaces

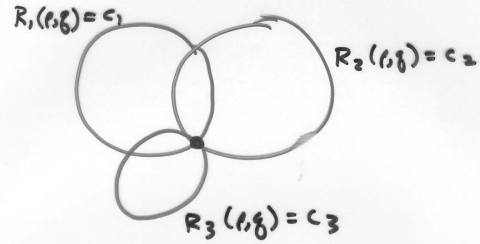
3 light sources: $\bar{s}_1, \bar{s}_2, \bar{s}_3$

$$\Rightarrow \begin{cases} E_1(x,y) = \rho\, \bar{s}_1 \cdot \bar{n} \\ E_2(x,y) = \rho\, \bar{s}_2 \cdot \bar{n} \\ E_3(x,y) = \rho\, \bar{s}_3 \cdot \bar{n} \end{cases}$$

or

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \rho \underbrace{\begin{bmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{bmatrix}}_{3\times 3 \text{ matrix } S} \hat{n}^T$$

$$\Rightarrow \rho \hat{n}^T = \underbrace{S^{-1}}_{\text{known}} \underbrace{\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}}_{\text{measured}} = |\vec{n}|^T$$



Implement as a <u>3D Table Lookup</u>:

$T(c_1, c_2, c_3) = (p,q)$

+ Can be generalized to non-Lambertian surfaces

* In general, unknown "scale" factor on intensity $\Rightarrow$ assume pixel (0,0) at height $z(0,0) = 0$, then compute height relative to that point using recovered (p,q)'s. (<u>relative</u> <u>depth</u> <u>map</u>)

Comments on Photometric Stereo

* Reflectance and Illumination must be known a priori
* Local method
* No surface inter-reflections
* Has been applied to problems such as astronomy and industrial inspection

Solving the Image Irradiance Eq.

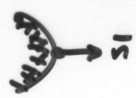
$E(x,y) = R(p,q)$

Nonlinear, 1st-order PDE

Add constraints such as:

Boundary conditions

At depth discontinuities, can solve uniquely for (p,q)



$\bar{n}$ $\qquad \bar{n} = \bar{e} \times \bar{d}$

where $\bar{e}$: edge direct.

Surface smoothness

Nearby image points have similar surface orientations

Global Optimization Approach

* Brightness error:

$$B = \iint_{\text{Image}} \left(E(x,y) - R(p,q)\right)^2 dx\,dy$$

* Non-smoothness error:

$$S = \iint \left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2 + \left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 dx\,dy$$

where $\frac{\partial p}{\partial x} \approx \Delta_x p(i,j) = p(i+1,j) - p(i,j)$

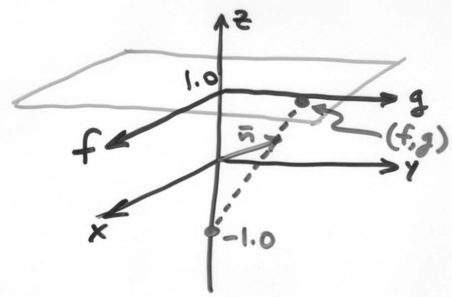
etc.

Since p, q can be infinite (when $\theta = 90°$) parameterize as

$$f = \frac{2p}{1+\sqrt{1+p^2+q^2}}$$

$$g = \frac{2q}{1+\sqrt{1+p^2+q^2}}$$

$\Rightarrow$ Reflectance Map: $R(f,g)$



Minimize

$$\iint \left(E(x,y) - R(f,g)\right)^2 + \lambda\left(f_x^2 + f_y^2 + g_x^2 + g_y^2\right) dx\,dy$$

* Solve, for example, by differentiating wrt f, g, set equal to 0, solve iteratively using Gauss-Seidel algorithm

* Use known f, g values at occluding boundary points + singular points to constrain solution

$$\begin{cases} f_{k,l}^{(n+1)} = \bar{f}_{k,l}^{(n)} + \lambda\left(E_{k,l} - R\left(f_{k,l}^{(n)}, g_{k,l}^{(n)}\right)\right) \frac{\partial R}{\partial f} \\ g_{k,l}^{(n+1)} = \bar{g}_{k,l}^{(n)} + \lambda\left(E_{k,l} - R\left(f_{k,l}^{(n)}, g_{k,l}^{(n)}\right)\right) \frac{\partial R}{\partial g} \end{cases}$$

* Decrease λ over time to remove smoothness constraint
* No convergence guaranteed

Local Method

* Assume each point lies on a sphere locally
  $\Rightarrow z(x,y) = \sqrt{R^2 - x^2 - y^2}$
  Estimate local curvature $\frac{1}{R}$ using $E_x, E_y, E_{xx}, E_{yy}, E_{xy}$
* Add other types of local surface primitive types
  plane, saddle, cylinder, ...
* Assume locally linear reflectance map
  $R(p,q) = k_1 + k_2 p + k_3 q$