9.2 Adding New Variables or Constraints

Suppose we have solved a problem with an optimal basis B and we desire to add an extra variable with constraint matrix column $a \in \mathbf{R}^m$ and objective coefficient $\pi \in \mathbf{R}$, that is we now have

$$\begin{array}{ll} \min & p'x + \pi x_{l+1} \\ \text{subject to} & \mathcal{A}x + a x_{l+1} = b \\ & x, x_{l+1} \geq 0 \end{array}$$

To check whether adding this column affects the basis, we just calculate the corresponding reduced cost entry. If the entry is negative, that is,

$$\pi - p'_{\mathrm{B}} \mathcal{A}_{\cdot\mathrm{B}}^{-1} a < 0$$

then the basis is changed, otherwise it remains optimal. For details of how to proceed when the basis changes, see Section 9.3.

Example 9-2-1. If we were to add the column

$$a = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad \pi = 2$$

to the problem given in Example 9-1-1, then

$$2 - \begin{bmatrix} 1 & 1.5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 0.5 > 0$$

so the basis remains optimal.

Now consider the case where a new inequality constraint $a'x \geq \beta$ is added to the original problem, that is

$$\begin{array}{ll} \min & p'x \\ \text{subject to} & \mathcal{A}x = b \\ & a'x \geq \beta \\ & x \geq 0 \end{array}$$

Define a slack variable x_{l+1} corresponding to this constraint and then construct the tableau corresponding to the basis $\tilde{\mathrm{B}} := \mathrm{B} \cup \{l+1\}$, B representing the optimal basis from the original problem. This implies that

$$\tilde{A}_{\cdot\tilde{\mathrm{B}}} = \begin{bmatrix} \mathcal{A}_{\cdot\mathrm{B}} & 0 \\ a'_{\mathrm{B}} & -1 \end{bmatrix}$$

Hence

$$\tilde{A}_{\cdot\tilde{B}}^{-1} = \begin{bmatrix} \mathcal{A}_{\cdot B}^{-1} & 0 \\ a'_B \mathcal{A}_{\cdot B}^{-1} & -1 \end{bmatrix}$$

resulting in the corresponding tableau

$$\begin{array}{lc|c|c|} & & x_N & 1 \\ \cline{3-4} x_B & = & -\mathcal{A}_{\cdot B}^{-1}\mathcal{A}_{\cdot N} & \mathcal{A}_{\cdot B}^{-1}b \\ x_{n+m+1} & = & -a'_B\mathcal{A}_{\cdot B}^{-1}\mathcal{A}_{\cdot N} + a'_N & a'_B\mathcal{A}_{\cdot B}^{-1}b - \beta \\ \cline{3-4} z & = & p'_N - p'_B\mathcal{A}_{\cdot B}^{-1}\mathcal{A}_{\cdot N} & p'_B\mathcal{A}_{\cdot B}^{-1}b \\ \cline{3-4} \end{array} \tag{9.4}$$

Note that the bottom row is unchanged because

$$\begin{aligned} p'_N - p'_{\tilde{B}}\tilde{\mathcal{A}}_{\cdot\tilde{B}}^{-1} \begin{bmatrix} \mathcal{A}_{\cdot N} \\ a'_N \end{bmatrix} &= p'_N - \begin{bmatrix} p'_B & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{\cdot B}^{-1} & 0 \\ a'_B\mathcal{A}_{\cdot B}^{-1} & -1 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{\cdot N} \\ a'_N \end{bmatrix} \\ &= p'_N - p'_B\mathcal{A}_{\cdot B}^{-1}\mathcal{A}_{\cdot N} \end{aligned}$$

Since the bottom row of the new tableau is nonnegative, the corresponding point is dual feasible. Either the value of x_{l+1} is nonnegative and the tableau is optimal, or it is negative. In the latter case, dual simplex pivots can be performed until an optimal solution is found or it is determined that the problem is primal infeasible. This is an instance when the current basis has to be changed.

Example 9-2-2. Suppose we add the constraint

$$x_1 + x_2 \geq 5$$

to the problem given as Example 9-1-1. Then all we need to calculate is

$$a'_B\mathcal{A}_{\cdot B}^{-1}b - \beta = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 5 = 1$$

so the problem is still feasible and hence optimal. (This is fairly obvious; $x_1 = 2$, $x_2 = 4$ is the basic feasible solution and hence the extra constraint is already satisfied at this point.) If instead the constraint added was

$$x_1 + x_2 \geq 7$$

then

$$a'_B\mathcal{A}_{\cdot B}^{-1}b - \beta = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 7 = -1$$

so the problem is no longer feasible. We perform a dual simplex pivot on this row by calculating the corresponding row

$$-a'_{\rm B}\mathcal{A}_{\cdot\rm B}^{-1}\mathcal{A}_{\cdot\rm N} + a'_{\rm N} = -\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 1 & 0 & -1\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\end{bmatrix}$$

then do the dual simplex ratio test (recall that the bottom row $c' = (1.5, 0.5, 0.5)$) that determines variable x_4 as the entering variable. Performing this pivot gives a new basis of $\tilde{B} = \{1, 2, 4\}$ with $\tilde{N} = \{3, 5, 6\}$. It is then easy to show that this leads to primal feasibility and hence optimality.

If instead of adding an inequality constraint, we wish to add the equation

$$a'x = \beta$$

to the problem, the same analysis that led to (9.4) can be used, with the caveat that the variable x_{l+1} is now an artificial variable and hence must be removed from the basis. To effect this, we can simply perform a dual simplex pivot to remove x_{l+1} from the basis, followed by further dual simplex pivots to regain dual optimality.

Example 9-2-3. If we add the constraint

$$x_1 + x_2 = 5$$

to the standard example, then as we calculated above

$$a'_{\rm B}\mathcal{A}_{\cdot\rm B}^{-1}b - \beta = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2\\ 4\end{bmatrix} - 5 = 1$$

Since for feasibility of the equation, this value must be zero, we perform a dual simplex pivot on the row to remove x_6 from the basis. As above, the corresponding row of the tableau is calculated as:

$$-a'_{\rm B}\mathcal{A}_{\cdot\rm B}^{-1}\mathcal{A}_{\cdot\rm N} + a'_{\rm N} = -\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 1 & 0 & -1\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\end{bmatrix}$$

and hence either the first or second nonbasic (x_3 or x_4) can be chosen to enter the basis. It can be seen that allowing x_3 to enter the basis, leads to an optimal basis $\mathrm{B} = \{1, 2, 3\}$.

Exercise 9-2-4. Solve the following problem using the revised simplex method. (You may want to try a starting basis of $\{2, 4\}$).

$$\min p'x \text{ subject to } \quad \mathcal{A}x = b, x \geq 0$$

where

$$p = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Use the techniques of this section to perform the following.

1. Change b to

$$\tilde{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Is the same basis still optimal? It the same solution optimal? Justify.

2. Go back to the original data and add the extra variable x_{l+1} with cost coefficient $\pi = -1$ and column $a = [3\ 1]'$ so that the new constraints are

$$\mathcal{A}x + ax_{n+m+1} = b$$

   What is the optimal solution and optimal value now?

3. Add the constraint $x_1 + x_2 \geq 0.5$, to the original data and determine the optimal solution and the optimal value. What is the corresponding dual solution and its value?

We now turn to the general case where the perturbation does affect the basis, the subject of parametric programming.