Chapter 1

Introduction

Many phenomena, such as the basic laws of motion or the shape of water in an overfull cup, can be modeled either as a system of equations or as an optimization problem. Complementarity problems generalize square systems of equations by constraining the variables and functions to lie within specified bounds and imposing a complementarity condition between matched variable, function pairs. The complementarity relationship is also a key component of the first-order optimality conditions for linear and nonlinear optimization problems. In this case, the complementarity relationship enforces the condition that the multiplier, or the shadow price, on a constraint is allowed to be nonzero only when the constraint is active.

Square systems of equations and the first-order optimality conditions for optimization problems are just two types of complementarity problem. Modeling and solving the complementarity problems encountered in economics and engineering are the foci of this book. Some of the applications developed are Nash games, Wardropian (traffic) and Walrasian (economic) equilibria, and the pricing of American options.

Modeling environments play a central role when developing large, complex applications because of the conveniences they offer. These environments readily manage large volumes of data, interface with many computational packages capable of solving the models constructed, and automatically generate any necessary derivative information. Modeling languages, such as AMPL and GAMS, are tailored to expressing algebraic systems of equations and optimization problems in an efficient manner. Over time, these languages have evolved and adapted as new problem classes have been explored. Complementarity problems are one such addition where modeling languages have added facilities to denote the complementarity relationship between a variable and function. All the example complementarity problems in this book are written in both the AMPL and GAMS modeling languages.

Formally, given a function $F : \Re^n \to \Re^n$, the nonlinear complementarity problem is to find an $x \in \Re^n$ such that for every component i:

$$x_i \geq 0,\ F_i(x) \geq 0, \tag{1.1}$$

and either

$$x_i = 0 \text{ or } F_i(x) = 0. \tag{1.2}$$

Condition (1.2) is equivalent to the product of x_i and $F_i(x)$ being equal to zero for each i, that is

$$x_i F_i(x) = 0.$$

Since the nonnegativity constraints (1.1) imply that $x_i F_i(x) \geq 0$, all of the these pairwise products can be summed together in the more succinct form

$$x^T F(x) = 0. \tag{1.3}$$

This relationship should be thought of as the variable x complementing the function F. Throughout this book, a shorthand notation for the complementarity conditions (1.1) and (1.3) is used that derives from the fact that x and $F(x)$ are considered orthogonal if (1.3) holds. Thus, a complementarity problem involving x and F will be written in standard form as:

$$0 \leq x \quad \perp \quad F(x) \geq 0.$$

Here, the inequalities are assumed to hold componentwise and the $\perp$ sign signifies that (1.3) holds. The mixed complementarity problem generalizes the nonlinear complementarity problem to the case where the variables have both lower and upper bounds. The adoption of standard form in this book is made to facilitate the translation of an algebraic problem description into a working model in any modeling language supporting the complementarity paradigm. All nonlinear and mixed complementarity problems can be written in standard form.

Square systems of equations are related to nonlinear complementarity problems. Consider, for example, the following square system of equations involving a single variable:

$$\min(x, x^2 - 4) = 0,$$

where the min function selects the minimum of the two arguments. Classical methods, such as Newton's method, cannot be directly applied to this system due to the nondifferentiability of the min function. However, this system of equations can be reformulated as a nonlinear complementarity problem that can be solved by applying an extension of Newton's method. In particular, the nonlinear complementarity problem

$$0 \leq x \quad \perp \quad x^2 - 4 \geq 0$$

is equivalent to the original nonsmooth system of equations. This complementarity problem is also square, involving one variable x and the single smooth function $F(x) = x^2 - 4$. Even though F has two zeros, the complementarity problem has the unique solution $x = 2$. The min function, as demonstrated by this example, is one function that can be used to transform a complementarity problem into a square, although typically nonsmooth, system of equations.

The remainder of this chapter motivates the complementarity relationship by using an example transportation problem. The transportation problem is a linear optimization problem where demand for a single good must be satisfied by suppliers at a minimal transportation cost. A linear complementarity problem for the first-order optimality conditions for this problem is developed in Section 1.1. The complementarity problem is refined in Sections 1.3–1.5 by adding nonlinear functions to model congestion, non-fixed supply and demand, and taxes and subsidies to demonstrate the benefits of utilizing complementarity relationships. The problems resulting from these refinements are no longer the first-order optimality conditions for a linear optimization problem. In general, no optimization problem exists whose first-order optimality conditions correspond to the given complementarity problem. The notion of a mixed complementarity problem is then introduced in Section 1.7 in order to incorporate capacities constraining the amount of the good that can be shipped along each road in the network and to model price controls. Finally, Section 1.8 discusses the use of equations in mixed complementarity problems. To demonstrate the translation from an algebraic problem in standard form using the $\perp$ notation to a working model, both AMPL and GAMS code for the transportation problem are provided throughout this chapter.

Chapter 2 then discusses some of the theory associated with complementarity problems including results on the existence, uniqueness, and stability of solutions. These results provide insight into what can be expected when modeling an application. The remainder of Part I concentrates on economic and engineering applications of complementarity. Salient features of a variety of applications are drawn out to show commonalities and differences that are of importance when formulating and solving real-world problems. This part concludes in Chapter 5 with general modeling guidelines and further information on aspects of the AMPL and GAMS modeling languages related to complementarity. Part II then details how solutions to complementarity problems are calculated when using the PATH algorithm.

1.1 Transportation Model

The transportation problem is a linear optimization problem used to determine a schedule for shipping a good from warehouses to retailers by utilizing a network of roads at minimal transportation cost. The given data for the problem consists of the amount of the good available at the warehouses, the amount demanded at the retail centers, the network of roads, and the per-unit shipping cost along each road. Typically, the network of roads is provided by a directed graph where each arc in the graph represents a road.

Mathematically, the underlying transportation network is given as a set of nodes $\mathcal{N}$ and arcs $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. The nodes are partitioned into two disjoint sets, the suppliers (warehouses) $\mathcal{S} \subseteq \mathcal{N}$ and the demand centers (retailers) $\mathcal{D} \subseteq \mathcal{N}$. The available supply of the good is denoted by s_i and the required demand is denoted by d_j for each $i \in \mathcal{S}$ and $j \in \mathcal{D}$, with $s_i \geq 0$ and $d_j \geq 0$. The per-unit shipment cost for using arc $(i, j) \in \mathcal{A}$ is denoted by $c_{i,j}$.

Example 1.1.

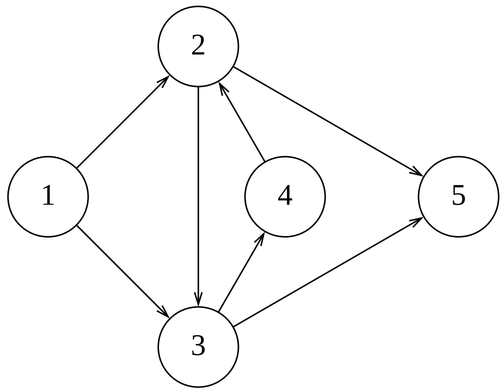

Figure 1.1. *Directed graph for Example 1.1.*

arc	cost
(1,2)	0.1
(1,3)	0.2
(2,3)	0.3
(2,5)	0.4
(3,4)	0.5
(3,5)	0.6
(4,2)	0.7

Table 1.1. *Cost data for Example 1.1.*

For a concrete illustration, the given network consists of the nodes and arcs

$$\mathcal{N} = \{1, 2, 3, 4, 5\}$$
$$\mathcal{A} = \{(1, 2), (1, 3), (2, 3), (2, 5), (3, 4), (3, 5), (4, 2)\}\,.$$

A graphical representation of the network is found in Figure 1.1. For this example, node 1 is a warehouse with $s_1 = 5$ and node 5 is a retailer with $d_5 = 5$. The remaining nodes are transshipment nodes and can be thought of as either warehouses with zero supply or retail centers with zero demand. These transshipment nodes can be thought of as staging points that the good can flow through. Sample per-unit cost data for using each road is found in Table 1.1.

The decision variables used to compute an optimal shipment schedule are the quantities of the good shipped along each arc. These quantities are denoted by $x_{i,j}$ for each $(i, j) \in \mathcal{A}$ and are constrained to be nonnegative since negative shipments would correspond to a retailer sending the good back to a warehouse. The vector x represents all of these quantities. In any feasible schedule, the amount of the good shipped out of each warehouse must not exceed the sum of the available supply and the amount shipped to the warehouse from other nodes. Algebraically,

$$\sum_{j:(i,j)\in\mathcal{A}} x_{i,j} \quad \leq \quad \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i \quad \forall\, i \in \mathcal{S}$$

The demand at the retail centers must also be satisfied. Therefore, the amount shipped to each retailer must be greater than or equal to the sum of the demand and the amount of the good reshipped to other nodes:

$$\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} \quad \geq \quad \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i \quad \forall\, i \in \mathcal{D}.$$

A linear optimization problem to find an optimal shipment schedule can then be written as

$$\begin{array}{lllll} \min_{x\geq 0} & \sum_{(i,j)\in\mathcal{A}} c_{i,j}x_{i,j} & & & \\ \text{subject to} & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} & \leq & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i & \forall\, i \in \mathcal{S} \\ & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i & \leq & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} & \forall\, i \in \mathcal{D}. \end{array} \tag{1.4}$$

A condition guaranteeing this problem has a feasible schedule is for a route to exist between every warehouse and retailer, and for the available supply to exceed the demand, $\sum_{i\in\mathcal{S}} s_i \geq \sum_{j\in\mathcal{D}} d_j$. To ensure that the objective function value cannot approach negative infinity, $c_{i,j}$ should be nonnegative for each $(i,j) \in \mathcal{A}$. That is, every road should have a usage cost.

Example 1.2.

The optimization problem for the sample network in Example 1.1 is written algebraically as

$$\begin{array}{rlll} \min_{x\geq 0} & 0.1x_{1,2} + 0.2x_{1,3} + 0.3x_{2,3} + 0.4x_{2,5} + 0.5x_{3,4} + 0.6x_{3,5} + 0.7x_{4,2} & & \\ \text{subject to} & x_{1,2} + x_{1,3} & \leq & 5 \\ & x_{2,3} + x_{2,5} & \leq & x_{1,2} + x_{4,2} \\ & x_{3,4} + x_{3,5} & \leq & x_{1,3} + x_{2,3} \\ & x_{4,2} & \leq & x_{3,4} \\ & 5 & \leq & x_{2,5} + x_{3,5}. \end{array}$$

The unique solution to this optimization problem is $x_{1,2} = 5$, $x_{2,5} = 5$, and all other quantities zero. In general, an infinite number of shipment schedules with the same optimal objective function value can exist for a given problem.
Implementations of this linear program in AMPL and GAMS are given in Figure 1.2 and Figure 1.3, respectively.

The derivation of a complementarity problem from the transportation model begins by associating with each constraint a shadow price, alternatively termed a dual variable or multiplier. These multipliers are the marginal prices on changes to the corresponding constraint. The marginal prices on the supply and demand constraints are represented by p. For $i \in \mathcal{S}$, p_i is the market value of the good at warehouse i, while for $j \in \mathcal{D}$, p_j is the retail value of the good at center j.

Consider first the supply nodes $i \in \mathcal{S}$. If there is excess supply at warehouse i, $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i > \sum_{j:(i,j)\in\mathcal{A}} x_{i,j}$, then in a competitive marketplace, no rational

```
set NODES;                          # Nodes in the network
set ARCS within NODES cross NODES;  # Arcs in the network

param s{NODES} >= 0;                # Amount of good available
param d{NODES} >= 0;                # Amount of good demanded
param c{ARCS};                      # Transportation cost

var x{ARCS} >= 0;                   # Shipment quantity

minimize cost: sum {(i,j) in ARCS} c[i,j]*x[i,j];

subject to
conserve {i in NODES}:              # Conservation of material
    sum{(j,i) in ARCS} x[j,i] + s[i] >= sum{(i,j) in ARCS} x[i,j] + d[i];
```

Figure 1.2. *AMPL model for Example* **??**.

```
$include network-mcp.dat

alias(NODES,i,j);

         variables cost;
positive variables x(NODES,NODES);

equations objective,
          conserve(NODES);

objective..
  cost =e= sum(ARCS(i,j), c(i,j)*x(i,j));

conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =g=
  sum(j$ARCS(i,j), x(i,j)) + d(i);

model network / objective, conserve /;

solve network using lp minimizing cost;
```

Figure 1.3. *GAMS model for Example* **??**.

person is willing to pay for more of the good to sit idle at the warehouse. Therefore, $p_i = 0$ in this case. Alternatively, if warehouse i clears, $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i = \sum_{j:(i,j)\in\mathcal{A}} x_{i,j}$, then a rational person might be willing to pay for additional supply of the good. Hence, $p_i \geq 0$ in this case. These two conditions are written succinctly as:

$$0 \le p_i \quad \perp \quad \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i \ge \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} \quad \forall\, i \in \mathcal{S},$$

where the $\perp$ notation is understood to mean that at least one of the adjacent inequalities must be satisfied as an equality. That is, either $0 = p_i$, the first case, or $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i = \sum_{j:(i,j)\in\mathcal{A}} x_{i,j}$, the second case.

Similarly, at each node $i \in \mathcal{D}$, the demand must be satisfied in any feasible solution and the retail price must be nonnegative, $p_i \ge 0$. Furthermore, if the retailer is over-supplied, $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} > \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i$, then in a competitive marketplace the retail price p_i will be driven down to zero. Summing these relationships gives the following complementarity condition:

$$0 \le p_i \quad \perp \quad \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} \ge \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i \quad \forall\, i \in \mathcal{D}.$$

The final component to the derivation of the complementarity problem for the transportation model is the profitability of using arc (i,j) to transport the good. The profitability of using $(i,j) \in \mathcal{A}$ is $p_j - (p_i + c_{i,j})$, the retail price for the good at the destination minus the delivery price. The delivery price is the sum of the amount paid for the good at the origin node and the transportation cost for using arc (i,j). If $p_j > (p_i + c_{i,j})$, then any rational person will buy the good at i and ship it to j to make a profit, thus forcing down the market price p_j. This situation cannot occur at a solution, so $p_j \le p_i + c_{i,j}$. There are now two cases to consider. If $p_j < p_i + c_{i,j}$, a loss is incurred by using arc (i,j). Therefore, any rational person will not utilize the arc and $x_{i,j} = 0$. When $p_j = p_i + c_{i,j}$, neither a profit nor a loss is realized and $x_{i,j} \ge 0$. Combining these two situations gives the complementarity condition:

$$0 \le x_{i,j} \quad \perp \quad p_i + c_{i,j} \ge p_j \quad \forall\, (i,j) \in \mathcal{A}.$$

The collection of these three relationships form the linear complementarity problem

$$\begin{array}{lllll} 0 \le p_i & \perp & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i & \ge & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i & \forall\, i \in \mathcal{N} \\ 0 \le x_{i,j} & \perp & p_i + c_{i,j} & \ge & p_j & \forall\, (i,j) \in \mathcal{A}, \end{array} \tag{1.5}$$

where the separate supply and demand constraints have been combined together into a single type of constraint by assuming that $s_i = 0$ for $i \notin \mathcal{S}$ and $d_i = 0$ for $i \notin \mathcal{D}$. The conditions (1.5) are the complementary slackness conditions for the linear optimization problem (1.4). For linear optimization problems the complementary slackness conditions are both necessary and sufficient for x to be an optimal solution to (1.4). This particular complementarity problem can have zero, one, or an infinite number of solutions.

Example 1.3.

The complementarity problem formulation for the optimization problem in Example 1.1 is

$$
\begin{array}{llll}
0 \le p_1 & \perp & 5 & \ge & x_{1,2} + x_{1,3} \\
0 \le p_2 & \perp & x_{1,2} + x_{4,2} & \ge & x_{2,3} + x_{2,5} \\
0 \le p_3 & \perp & x_{1,3} + x_{2,3} & \ge & x_{3,4} + x_{3,5} \\
0 \le p_4 & \perp & x_{3,4} & \ge & x_{4,2} \\
0 \le p_5 & \perp & x_{2,5} + x_{3,5} & \ge & 5 \\
0 \le x_{1,2} & \perp & p_1 + 0.1 & \ge & p_2 \\
0 \le x_{1,3} & \perp & p_1 + 0.2 & \ge & p_3 \\
0 \le x_{2,3} & \perp & p_2 + 0.3 & \ge & p_3 \\
0 \le x_{2,5} & \perp & p_2 + 0.4 & \ge & p_5 \\
0 \le x_{3,4} & \perp & p_3 + 0.5 & \ge & p_4 \\
0 \le x_{3,5} & \perp & p_3 + 0.6 & \ge & p_5 \\
0 \le x_{4,2} & \perp & p_4 + 0.7 & \ge & p_2.
\end{array}
$$

A solution to this complementarity problem is $x_{1,2} = 5$, $x_{2,5} = 5$, $p_2 = 0.1$, $p_5 = 0.5$, and all other variables are set to zero. The prices are not unique for this problem because $x_{1,2} = 5$, ${}_{2,5} = 5$, $p_1 = 0.1$, $p_2 = 0.2$, $p_5 = 0.6$, and all other variables set to zero is an alternate solution.

By looking more closely at conditions (1.5), further insight into complementarity problems can be gained. A solution of (1.5) identifies the arcs used to transport the good. The arcs to use do not need to be specified a priori, the solution itself indicates them. This property represents the key contribution of a complementarity problem over a system of equations. If the arcs to send flow down are known, a simple system of linear equations can be solved to find the quantities and marginal prices. However, the key to the modeling power of complementarity is that the choice of which inequalities in (1.5) to satisfy as equations is part of the problem description.

The conditions (1.5) are also the necessary and sufficient optimality conditions for a related problem in the marginal prices p

$$
\begin{array}{lll}
\max_{p \ge 0} & \sum_{i \in \mathcal{D}} d_i p_i - \sum_{i \in \mathcal{S}} s_i p_i & \\
\text{subject to} & p_i + c_{i,j} \ge p_j & \forall\, (i,j) \in \mathcal{A}
\end{array}
\tag{1.6}
$$

termed the dual linear optimization problem. Hence the nomenclature dual variables for the marginal prices. Intuitively, this problem attempts to maximize the profit, that is the revenue received at the demand centers minus the amount paid by the suppliers, subject to the rational pricing constraints.

Example 1.4.

For the transportation problem in Example 1.1, the dual linear optimization problem is:

$$\begin{array}{rl} \min_{p\geq 0} & 5p_5 - 5p_1 \\ \text{subject to} & p_1 + 0.1 \geq p_2 \\ & p_1 + 0.2 \geq p_3 \\ & p_2 + 0.3 \geq p_3 \\ & p_2 + 0.4 \geq p_5 \\ & p_3 + 0.5 \geq p_4 \\ & p_3 + 0.6 \geq p_5 \\ & p_4 + 0.7 \geq p_2. \end{array}$$

An optimal solution for the dual problem is the prices $p_1 = 0$, $p_2 = 0.1$, $p_5 = 0.5$, and all other prices set to zero. The solution to this problem not unique since $p_1 = 0.1$, $p_2 = 0.2$, $p_5 = 0.6$, and all other prices set to zero is also optimal.

Three representations for the same transportation problem have now been developed. The original minimization problem uses the quantities shipped along the arcs as the decision variables, while the dual linear optimization problem uses the marginal prices. The linear complementarity problem has both the quantities and marginal prices as variables. Even though the complementarity problem has more variables and constraints than the corresponding optimization problems, the real power of this modeling format is the new problem instances it enables a modeler to create. Examples of how to extend the simple linear complementarity problem (1.5) to investigate other facets of the problem are discussed in Sections 1.3–1.8.

1.2 Model Translation

Translating an algebraic description of a complementarity problem into a working AMPL or GAMS model requires that the equalities and inequalities be written using the syntax for the chosen modeling language and that the variables and functions in complementarity relationships be conveyed by using a matching mechanism. The syntax for defining the constraints and the matching mechanisms differ in AMPL and GAMS, so the transportation models are developed in both languages.

1.2.1 AMPL Implementation

AMPL uses the `complements` keyword to associate variables and functions in complementarity relationships at the constraints are defined. An implementation of the model for Example 1.1 using this notation is given below:

```
set NODES;                          # Nodes in the network
set ARCS within NODES cross NODES;  # Arcs in the network

param s{NODES} >= 0;                # Amount of good available
param d{NODES} >= 0;                # Amount of good demanded
param c{ARCS};                      # Transportation cost

var p{NODES};                       # Marginal prices
var x{ARCS};                        # Shipment quantity
```

```
subject to
conserve {i in NODES}:             # Conservation of material
    0 <= p[i] complements sum{(j,i) in ARCS} x[j,i] + s[i] >=
                          sum{(i,j) in ARCS} x[i,j] + d[i];

subject to
rational {(i,j) in ARCS}:          # Rational pricing
    0 <= x[i,j] complements p[i] + c[i,j] >= p[j];
```

The first component of the model is the definition of the sets $\mathcal{N}$ and $\mathcal{A}$, where $\mathcal{A}$ is declared as a subset of $\mathcal{N} \times \mathcal{N}$. Once these sets are declared, the problem data, called parameters, are declared over their appropriate set domains, followed by the declaration of the variables p and x. All variables in the model are declared without restrictions because their bounds are part of the definition of the complementarity relationships.

The definition of the complementarity constraints using the `complements` keyword then follows. Each complementarity condition must specify *exactly* two inequalities. In this case, these inequalities correspond to one bound on the variable and one inequality for the complementary function to define a nonlinear complementarity relationship. Any ordering of the inequalities can be used during the matching. For example, an alternative declaration would be to reorder the `conserve` constraint as

```
conserve {i in NODES}:              # Conservation of material
    sum{(j,i) in ARCS} x[j,i] + s[i] >=
    sum{(i,j) in ARCS} x[i,j] + d[i] complements 0 <= p[i];
```

Internally, the constraints are rewritten to produce a nonlinear complementarity problem in standard form.

To define a particular instance of a model, data definitions for the declared sets and parameters need to be given. These definitions are typically contained in a separate file. An example data file for Example 1.1 is given below:

```
param : NODES :   s   d =
          1       5   0
          2       0   0
          3       0   0
          4       0   0
          5       0   5;

param : ARCS : c    u =
        1  2   0.1  2
        1  3   0.2  4
        2  3   0.3  5
        2  5   0.4  5
        3  4   0.5  5
        3  5   0.6  2
        4  2   0.7  5;
```

In this data file, the elements of the sets $\mathcal{N}$ and $\mathcal{A}$ are specified when the parameters using these sets as their domain are defined.

The model can be solved using the following command:

```
ampl network-mcp.cmd
```

The `network-mcp.cmd` file contains the following sequence of AMPL statements that set the solver, load the model and data files, and solve the problem instance.

```
option solver pathampl;
model network-mcp.mod;
data network-mcp.dat;
solve;
```

Once the problem is solved, the values for the variables at the solution computed can be viewed by using the appropriate `display` statements.

1.2.2 GAMS Implementation

Complementarity problems are written in GAMS by defining the complementary function using the standard equation definitions. The variables are then matched to the appropriate functions using the '.' symbol during the `model` statement to indicate a complementarity relationship. The left-hand term is the function, while the right-hand term is the complementary variable. All the equation and variable declarations and definitions are made using standard GAMS syntax. The transportation complementarity problem from Example 1.1 is given below:

```
$include network-mcp.dat

alias(NODES,i,j);

positive variables p(NODES),
                   x(NODES,NODES);

equations conserve(NODES),
          rational(NODES,NODES);

conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =g=
  sum(j$ARCS(i,j), x(i,j)) + d(i);

rational(ARCS(i,j))..
  p(i) + c(i,j) =g= p(j);

model network / conserve.p, rational.x /;

solve network using mcp;
```

GAMS models are procedural rather than declarative, and therefore do not typically separate the model description from the data. For convenience however, the sets and parameters of the GAMS model for this transporation example are defined in a separate data file called `network-mcp.dat` and incorporated into the model by using the `include` facility. The use of the `include` facility allows the same data to be used in multiple models. The contents of this file follow:

```
set NODES / 1*5 /;
set ARCS(NODES,NODES) / 1.2, 1.3, 2.3, 2.5, 3.4, 3.5, 4.2 /;

table ndata(NODES,*)
        s       d
1       5
5               5 ;

table adata(NODES,NODES,*)
        c       u
1.2     0.1     2
1.3     0.2     4
2.3     0.3     5
2.5     0.4     5
3.4     0.5     5
3.5     0.6     2
4.2     0.7     5;

parameter s(NODES);         s(NODES) = ndata(NODES,'s');
parameter d(NODES);         d(NODES) = ndata(NODES,'d');
parameter c(NODES,NODES);   c(ARCS)  = adata(ARCS,'c');
```

The remainder of this model first declares aliases `i` and `j` to the `NODES` set using the `alias` command. The alias command essentially provides indices that can be used to access the individual elements in the `ARCS` set and over which the summations can be made. Variables p and x are then defined. The `positive` prefix ensures each of these variables is nonnegative.

Following these declarations, the functions F_i are defined by using the standard GAMS syntax for defining equations. Thus the equation

```
conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =g=
  sum(j$ARCS(i,j), x(i,j)) + d(i);
```

actually defines (the first $|\mathcal{N}|$) components of the function F as

$$F_i(x) = \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i - \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} - d_i$$

These functions F_i are then made complementary to the p variables in the ensuing `model` statement. For nonlinear complementarity problems, `positive variables` are complementary to `=g=` inequalities. The GAMS compiler issues an error during the model generation if a `positive variable` is matched to a `=l=` inequality.

Alternatively, the function can be defined in an equation using `=e=`. Therefore, the `conserve` constraint could also be written as

```
conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =e=
  sum(j$ARCS(i,j), x(i,j)) + d(i);
```

The resulting problem is equivalent to the problem above since the the meaning of the complementarity relationship is determined based on the bounds for the matched variable. Internally, the equations are rewritten so that all terms appear on the left-hand side of the `=e=` exactly as shown above. The resulting term becomes the function complementary to the matched variable and the `=e=` is ignored. Therefore, the equation must be oriented correctly to preserve the meaning of the relationship when the function is created. Hence, while the following is syntactically correct for the `conserve` constraint, the actual meaning of the resulting complementarity problem is incorrect:

```
conserve(i)..
  sum(ARCS(i,j), x(i,j)) + d(i) =e=
  sum(ARCS(j,i), x(j,i)) + s(i);
```

The can be verified algebraically by writing the constraint in the nonlinear complementarity form

$$0 \leq p_i \quad \perp \quad \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i \geq \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i \quad \forall\, i \in \mathcal{N},$$

which is inconsistent with the description given in Example 1.1.

The `solve` statement indicates the model to be solved and the model type. The model type is `mcp` for complementarity problems. The default solver for `mcp` models is applied unless an alternate solver is specified.

1.3 Congestion

Congestion increases the transportation cost for heavily used roads. Congestion could be modeled, for example, by making the cost of traveling along a road a function of the amount shipped along the roads:

$$0 \leq x_{i,j} \quad \perp \quad p_i + c_{i,j}(x) \geq p_j \quad \forall\, (i,j) \in \mathcal{A}$$

for some cost function, $c_{i,j}(\cdot)$. A typical cost function for modeling congestion is $c_{i,j}(x) := A_{i,j} + B_{i,j}(x_{i,j})^4$, where $A_{i,j}$ and $B_{i,j}$ are positive constants. In this case, $A_{i,j}$ is the cost of using the arc when there is no congestion and $B_{i,j}(x_{i,j})^4$ models the effect of congestion. As expected, this function increases with an increased amount of flow along the road. Typically, the constants are chosen to fit data collected on the "real" network.

Example 1.5.

Assuming that congestion can be uniformly modeled as $c_{i,j}(x) = c_{i,j} + (x_{i,j})^4$, the complementarity problem from Example 1.1 can be written to handle the congestion

variable	solution
$x_{1,2}$	2.5120
$x_{1,3}$	2.4880
$x_{2,3}$	0.0000
$x_{2,5}$	2.5120
$x_{3,4}$	0.0000
$x_{3,5}$	2.4880
$x_{4,2}$	0.0000
p_1	0.0002
p_2	39.2377
p_3	39.1877
p_4	38.5377
p_5	78.7753

Table 1.2. *Solution to Example 1.3.*

effects as

$$\begin{array}{llll}
0 \le p_1 & \perp & 5 & \ge & x_{1,2} + x_{1,3} \\
0 \le p_2 & \perp & x_{1,2} + x_{4,2} & \ge & x_{2,3} + x_{2,5} \\
0 \le p_3 & \perp & x_{1,3} + x_{2,3} & \ge & x_{3,4} + x_{3,5} \\
0 \le p_4 & \perp & x_{3,4} & \ge & x_{4,2} \\
0 \le p_5 & \perp & x_{2,5} + x_{3,5} & \ge & 5 \\
0 \le x_{1,2} & \perp & p_1 + 0.1 + (x_{1,2})^4 & \ge & p_2 \\
0 \le x_{1,3} & \perp & p_1 + 0.2 + (x_{1,3})^4 & \ge & p_3 \\
0 \le x_{2,3} & \perp & p_2 + 0.3 + (x_{2,3})^4 & \ge & p_3 \\
0 \le x_{2,5} & \perp & p_2 + 0.4 + (x_{2,5})^4 & \ge & p_5 \\
0 \le x_{3,4} & \perp & p_3 + 0.5 + (x_{3,4})^4 & \ge & p_4 \\
0 \le x_{3,5} & \perp & p_3 + 0.6 + (x_{3,5})^4 & \ge & p_5 \\
0 \le x_{4,2} & \perp & p_4 + 0.7 + (x_{4,2})^4 & \ge & p_2.
\end{array}$$

A solution to this complementarity problem is given in Table 1.2.
The AMPL and GAMS models for this problem are very similar to the models for Example 1.1. The only change is to the rational pricing function where the nonlinear cost function is defined. The implementations are given in Figure 1.4 and Figure 1.5.

A sufficient condition for the complementarity problem with congestion effects to have a solution is for the shipment constraints to be feasible and for the cost function to be strongly monotone on this feasible region. When these conditions are satisfied, the quantities shipped are guaranteed to be unique. However, no such guarantee can be made that the marginal prices are unique at a solution.

Sometimes the resulting complementarity problem with congestion effects is equivalent to a nonlinear optimization problem in the quantities, where the derivative of the objective function is the (marginal) cost function in the complementarity problem and the (linear) network constraints are as before. For equivalence to hold,

```
set NODES;                          # Nodes in the network
set ARCS within NODES cross NODES;  # Arcs in the network

param s{NODES} >= 0;                # Amount of good available
param d{NODES} >= 0;                # Amount of good demanded
param c{ARCS};                      # Transportation cost

var p{NODES};                       # Marginal prices
var x{ARCS};                        # Shipment quantity

subject to
conserve {i in NODES}:              # Conservation of material
    0 <= p[i] complements sum{(j,i) in ARCS} x[j,i] + s[i] >=
                          sum{(i,j) in ARCS} x[i,j] + d[i];

subject to
rational {(i,j) in ARCS}:          # Rational pricing
    0 <= x[i,j] complements p[i] + c[i,j] + x[i,j]^4 >= p[j];
```

Figure 1.4. *AMPL model for Example 1.3.*

the objective function must be convex on the feasible region. This situation is the case for the example congestion function. However, other models of congestion can be used for which there is no equivalent optimization problem because either the cost function is not integrable or the integral is non-convex. In the non-convex case, each solution to the complementarity problem is a critical point for the optimization problem, but not necessarily a local minimizer.

1.4 Supply and Demand

Supply and demand in (1.5) are independent of the marginal prices p. Since the prices p are variables in the complementarity formulation (1.5), the constant supply and demand can easily be replaced with functions $s(p)$ and $d(p)$ to more accurately model them. Typically, the amount supplied is high when the price for the good is high and low when the price of the good is low. An inverse relationship usually holds for the demand, where the demand is low when the price is high and high when the price is low.

Clearly, any function of p could be added to model the supply and demand. For example, a linear demand function could be expressed using

$$\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} \geq \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i(1-p_i) \quad \forall\, i \in \mathcal{D}.$$

This demand function is strange if p_i exceeds 1 because a negative amount of demand is indicated. Other more reasonable examples for $d(p)$ can also be derived. For example, if the demand for the commodity is isoelastic, then the demand function

```
$include network-mcp.dat

alias(NODES,i,j);

positive variables p(NODES),
                   x(NODES,NODES);

equations conserve(NODES),
          rational(NODES,NODES);

conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =g=
  sum(j$ARCS(i,j), x(i,j)) + d(i);

rational(ARCS(i,j))..
  p(i) + c(i,j) + power(x(i,j),4) =g= p(j);

model network / conserve.p, rational.x /;

solve network using mcp;
```

Figure 1.5. *GAMS model for Example 1.3.*

for the commodity can be written as

$$d_i(p) := \frac{\bar{d}_i}{(p_i)^{\alpha_i}}$$

for fixed $\bar{d}_i$ and $\alpha_i \geq 0$. In this case, the resulting complementarity problem becomes nonlinear in the variables p and the function is undefined when $p_i = 0$.

One function that can be used to model variable supply is

$$s_i(p) := \bar{s}_i(p_i)^{\beta_i}$$

for fixed $\bar{s}_i$ and $\beta_i \geq 1$. As expected, when the marginal price of supply increases, the amount supplied also increases.

To guarantee that a solution exists to the complementarity problem with non-fixed supply and demand, the rational price constraints must be feasible, and the supply function minus the demand function must be a strongly monotone function of the marginal prices. In this case, the marginal prices at the solution are unique. However, the quantities are not guaranteed to be unique.

Example 1.6.

Assuming that the supply is xxxx at node 1 and demand is isoelastic at node 5, the complementarity problem from Example 1.1 can be written to model non-fixed

variable	solution 1	solution 2
$x_{1,2}$	3.0481	3.0481
$x_{1,3}$	0.0000	0.0000
$x_{2,3}$	0.0000	0.0000
$x_{2,5}$	3.0481	3.0481
$x_{3,4}$	0.0000	0.0000
$x_{3,5}$	0.0000	0.0000
$x_{4,2}$	0.0000	0.0000
p_1	0.7808	0.7808
p_2	0.8808	0.8808
p_3	0.9733	0.9733
p_4	1.0000	1.2000
p_5	1.2808	1.2808

Table 1.3. *Solutions to Example 1.4.*

supply and demand as

$$\begin{array}{llll}
0 \le p_1 & \perp & 5(p_1)^2 & \ge & x_{1,2} + x_{1,3} \\
0 \le p_2 & \perp & x_{1,2} + x_{4,2} & \ge & x_{2,3} + x_{2,5} \\
0 \le p_3 & \perp & x_{1,3} + x_{2,3} & \ge & x_{3,4} + x_{3,5} \\
0 \le p_4 & \perp & x_{3,4} & \ge & x_{4,2} \\
0 \le p_5 & \perp & x_{2,5} + x_{3,5} & \ge & \frac{5}{(p_5)^2} \\
0 \le x_{1,2} & \perp & p_1 + 0.1 & \ge & p_2 \\
0 \le x_{1,3} & \perp & p_1 + 0.2 & \ge & p_3 \\
0 \le x_{2,3} & \perp & p_2 + 0.3 & \ge & p_3 \\
0 \le x_{2,5} & \perp & p_2 + 0.4 & \ge & p_5 \\
0 \le x_{3,4} & \perp & p_3 + 0.5 & \ge & p_4 \\
0 \le x_{3,5} & \perp & p_3 + 0.6 & \ge & p_5 \\
0 \le x_{4,2} & \perp & p_4 + 0.7 & \ge & p_2.
\end{array}$$

Two solutions to this complementarity problem are given in Table 1.3. The presence of the transshipment nodes prevents the supply minus demand function from being strongly monotone. Hence, the model can have multiple solutions with different prices.
The AMPL and GAMS models for this problem are very similar to the models for Example 1.1. The only change is to the conservation function where the nonlinear supply and demand functions are defined. The implementations are given in Figure 1.6 and Figure 1.7. A restriction is used in the definition of the `conserve` constraint so that a nonlinear demand is included only in nodes with a nonzero fixed demand constant. Moreover, the values for the prices have been initialized to one so that all the functions are defined at the starting point provided to the algorithm.

These modifications to the supply and demand can be thought of as introducing a nonlinear objective function in the dual linear optimization problem. This

```
set NODES;                          # Nodes in the network
set ARCS within NODES cross NODES;  # Arcs in the network

param s{NODES} >= 0;                # Fixed supply constant
param d{NODES} >= 0;                # Fixed demand constant
param c{ARCS};                      # Transportation cost

var p{NODES} := 1;                  # Marginal prices
var x{ARCS};                        # Shipment quantity

subject to
conserve {i in NODES}:              # Conservation of material
    0 <= p[i] complements sum{(j,i) in ARCS} x[j,i] + s[i]*p[i]^2 >=
                          sum{(i,j) in ARCS} x[i,j] +
                          if (d[i] > 0) then d[i]/p[i]^2 else 0;

subject to
rational {(i,j) in ARCS}:          # Rational pricing
    0 <= x[i,j] complements p[i] + c[i,j] >= p[j];
```

Figure 1.6. *AMPL model for Example 1.3.*

```
$include network-mcp.dat

alias(NODES,i,j);

positive variables p(NODES),
                   x(NODES,NODES);

equations conserve(NODES),
          rational(NODES,NODES);

conserve(i)..
  sum(ARCS(j,i), x(j,i)) + s(i)*power(p(i),2) =g=
  sum(ARCS(i,j), x(i,j)) + (d(i)/power(p(i),2))$d(i);

rational(ARCS(i,j))..
  p(i) + c(i,j) =g= p(j);

model network / conserve.p, rational.x /;

p.l(i) = 1;

solve network using mcp;
```

Figure 1.7. *GAMS model for Example 1.3.*

interpretation is possible when the supply and demand functions are integrable. For the equivalence to hold, the integral of the demand function must be concave on the feasible region and the integral of the supply function must be convex on the feasible region. Not all supply and demand functions considered are integrable. If the resulting objective function is not convex, then each solution to the complementarity problem corresponds to a critical point of the optimization problem, but not necessarily a local minimizer.

1.5 Taxes and Tariffs

Another feature that can be added independently to (1.5) are taxes and tariffs. A tax increases the real cost of the good. In the case where an externally specified tax is applied at the origin node (an export tax), the first general inequality in (1.5) is replaced by

$$p_i(1+t_i) + c_{i,j} \geq p_j \quad \forall\,(i,j) \in \mathcal{A}.$$

Essentially, the price a consumer sees is increased by the tax amount $t_i > 0$. A tax at the destination, essentially an import tax or tariff, results in the inequality becoming

$$p_i + c_{i,j} \geq p_j(1+t_i) \quad \forall\,(i,j) \in \mathcal{A},$$

where $t_i > 0$ is the tax rate.

All of these modifications change the dual constraints in such a way that the resulting linear complementarity problem is no longer the optimality conditions for a linear or nonlinear optimization problems. The reason is that the technology matrix in the dual problem is modified for the tax effects. However, the technology matrix in the primal problem remains unchanged.

Example 1.7.

Assuming that a 3% excise tax is applied for all good leaving node 2 and a 5% import tariff is applied on all goods entering node 5, the complementarity problem from Example 1.1 can be written to model these effects as

$$\begin{array}{llcll}
0 \leq p_1 & \perp & 5 & \geq & x_{1,2} + x_{1,3} \\
0 \leq p_2 & \perp & x_{1,2} + x_{4,2} & \geq & x_{2,3} + x_{2,5} \\
0 \leq p_3 & \perp & x_{1,3} + x_{2,3} & \geq & x_{3,4} + x_{3,5} \\
0 \leq p_4 & \perp & x_{3,4} & \geq & x_{4,2} \\
0 \leq p_5 & \perp & x_{2,5} + x_{3,5} & \geq & 5 \\
0 \leq x_{1,2} & \perp & 1.00p_1 + 0.1 & \geq & 1.00p_2 \\
0 \leq x_{1,3} & \perp & 1.00p_1 + 0.2 & \geq & 1.00p_3 \\
0 \leq x_{2,3} & \perp & 1.03p_2 + 0.3 & \geq & 1.00p_3 \\
0 \leq x_{2,5} & \perp & 1.03p_2 + 0.4 & \geq & 1.00p_5 \\
0 \leq x_{3,4} & \perp & 1.00p_3 + 0.5 & \geq & 1.05p_4 \\
0 \leq x_{3,5} & \perp & 1.00p_3 + 0.6 & \geq & 1.05p_5 \\
0 \leq x_{4,2} & \perp & 1.00p_4 + 0.7 & \geq & 1.00p_2.
\end{array}$$

A solution to this complementarity problem is given in Table 1.4. **Model changed – need updated solution.**

variable	solution
$x_{1,2}$	5.0000
$x_{1,3}$	0.0000
$x_{2,3}$	0.0000
$x_{2,5}$	5.0000
$x_{3,4}$	0.0000
$x_{3,5}$	0.0000
$x_{4,2}$	0.0000
p_1	0.0793
p_2	0.2108
p_3	0.2779
p_4	0.0000
p_5	0.8393

Table 1.4. *Solution to Example 1.5.*

```
set NODES;                          # Nodes in the network
set ARCS within NODES cross NODES;  # Arcs in the network

param s{NODES};                     # Amount of good available
param d{NODES};                     # Amount of good demanded
param c{ARCS};                      # Transportation cost

param ex{NODES};                    # Excise tax rates
param im{NODES};                    # Import tax rates

var p{NODES};                       # Marginal prices
var x{ARCS};                        # Shipment quantity

subject to
conserve {i in NODES}:              # Conservation of material
    0 <= p[i] complements sum{(j,i) in ARCS} x[j,i] + s[i] >=
                          sum{(i,j) in ARCS} x[i,j] + d[i];

subject to
rational {(i,j) in ARCS}:          # Rational pricing
    0 <= x[i,j] complements (1 + ex[i])*p[i] + c[i,j] >= (1 + im[j])*p[j];
```

Figure 1.8. *AMPL model for Example 1.5.*

The AMPL and GAMS models for this problem are very similar to the models for Example 1.1. The only change is to the conservation function where the taxes are included. The implementations are given in Figure 1.8 and Figure 1.9.

Furthermore, the tax rates can be made decision variables in the model. To

```
$include network-mcp.dat

parameter ex(NODES) / 2 0.03 /;
parameter im(NODES) / 3 0.05 /;

alias(NODES,i,j);

positive variables p(NODES),
                   x(NODES,NODES);

equations conserve(NODES),
          rational(NODES,NODES);

conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =g=
  sum(j$ARCS(i,j), x(i,j)) + d(i);

rational(ARCS(i,j))..
  (1+ex(i))*p(i) + c(i,j) =g= (1+im(j))*p(j);

model network / conserve.p, rational.x /;

solve network using mcp;
```

Figure 1.9. *GAMS model for Example 1.5.*

maintain the squareness of the system, an additional complementarity condition needs to be added. For example, the constraint

$$0 \leq t_i \quad \perp \quad t_i p_i \textstyle\sum_{j:(i,j)\in\mathcal{A}} x_{i,j} \geq T_i \quad \forall\, i \in \mathcal{N}$$

could be added to require that the revenue from the import tax be at least T_i for node i. This latter constraint is nonlinear in the prices and quantities. Moreover, if $T_i > 0$, the inequality can only be satisfied when $t_i > 0$. Hence, the tax rate t_i will be chosen to make the tax revenue from this supplier exactly T_i.

An important observation to make is that with the inclusion of any of these tax modifications, the resulting complementarity problem is no longer the optimality conditions for a linear or nonlinear optimization problem. In many cases, there is no optimization problem whose first-order optimality conditions correspond to the complementarity conditions. The new situations that can be modeled with by using the complementarity formulation are a key benefit of this problem class.

1.6 Mixed Complementarity Problems

The notion of a complementarity relationship can be generalized to situations in which the variable is constrained by both lower and upper bounds. Consider for a moment the first-order optimality conditions for the bound constrained optimization problem

$$\min_{\ell \le x \le u} f(x),$$

where $\ell \le u$ and $f : \Re \to \Re$. If the lower bound is active at a solution, then f must be nondecreasing, so $\nabla f(x) \ge 0$. However, if the upper bound is active at a solution, then f must be nonincreasing, so $\nabla f(x) \le 0$. If neither bound is active at a solution then x must be a critical point, so $\nabla f(x) = 0$. These conditions motivate the definition of a mixed complementarity problem.

In particular, the mixed complementarity problem written in standard form is to find an $x \in \Re^n$ such that

$$\ell \le x \le u \quad \perp \quad F(x)$$

for given function $F : \Re^n \to \Re^n$ and lower and upper bounds, $\ell \in \bar{\Re}^n$ and $u \in \bar{\Re}^n$ with $\ell \le u$. The $\perp$ sign denotes a mixed complementarity relationship between the left variable and right function.

The mixed complementarity relationship states that for given lower and upper bounds, $\ell \in \bar{\Re}^n$ and $u \in \bar{\Re}^n$ with $\ell \le u$, and function $F : \Re^n \to \Re^n$ at least one of the following conditions hold for each $i \in \{1, \ldots, n\}$:

1. $\ell_i = x_i$ and $F_i(x) \ge 0$.
2. $\ell_i < x_i < u_i$ and $F_i(x) = 0$.
3. $x_i = u_i$ and $F_i \le 0$.

That is, the inequality satisfied by the complementary function depends on which of the variable bounds, if any, are active. If the lower bound is active, then the function must be nonnegative, while if the upper bound is active, then the function must be nonpositive. If neither of the variable bounds is active, then the function must be satisfied as an equation. The first-order conditions for the example bound-constrained optimization problem are recovered as

$$\ell \le x \le u \quad \perp \quad \nabla f(x),$$

where f is the objective function.

The inequality relationships required by the function F cannot be determined a priori. It is only once the solution x is known that the required relationships on F are determined. This is a fundamental property of complementarity problems and underpins the power of the format as a modeling tool.

Since the lower and upper bounds are in the set of extended reals,

$$\bar{\Re} := \Re \cup \{\infty\} \cup \{-\infty\},$$

they can take on infinite values. Some special cases of the mixed complementarity problem are of interest. Firstly, the nonlinear complementarity problem is recovered as a special case when the lower bounds are the zero vector and the upper bounds are all set to infinity. A second example is when all the lower bounds are $-\infty$ and all the upper bounds are ∞. In this case, the variables have no bounding constraints and are called free variables. Since no bound constraints can be active at the solution, every F_i must be equal to zero, and thus this mixed complementarity problem with infinite bounds corresponds to the set of nonlinear equations $F(x) = 0$. When the variable is fixed, that is $\ell_i = u_i$, then the corresponding function is unrestricted. The variable x_i can be fixed at this value and the function F_i eliminated from the problem.

In the following sections further extensions of the transportation model are developed that show the utility of the more general mixed complementarity format.

1.7 Capacity Constraints and Price Controls

Many times, finite capacities are placed upon the amount of the good that can be shipped along arcs to model physical limitations. These capacities can be used to model quotas, for example. Lower bounds other than zero can also be imposed to handle contractual obligations. These capacities are added to the primal transportation problem (1.4) by constraining the quantities:

$$\begin{array}{lllll} \min & \sum_{(i,j)\in\mathcal{A}} c_{i,j}x_{i,j} & & & \\ \text{subject to} & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i & \leq & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i & \forall\, i \in \mathcal{N} \\ & \ell \leq x \leq u, & & & \end{array}$$

where $\ell \in \Re_+^{|\mathcal{A}|}$ is a nonnegative vector containing the lower bounds on the quantity shipped along each arc, and $u \in \bar{\Re}_+^{|\mathcal{A}|}$ is a nonnegative vector containing the upper bounds, possibly infinite, on the quantity shipped along each arc. The resulting optimization problem only makes sense if $\ell \leq u$. Finite capacities create a barrier which means that profit can be realized when utilizing the arc.

The derivation of the first order conditions for this linear optimization problem proceeds as before by associating marginal prices with the constraints. The multipliers are matched to the primal constraints with the complementarity conditions

$$0 \leq p_i \quad \perp \quad \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i \quad \geq \quad \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i \quad \forall\, i \in \mathcal{N}$$

The difference occurs in the bounded quantities that are matched to the rational price constraint, $p_i + c_{i,j} - p_j$. As before, the minimal amount of flow is sent along the arc if a loss would result. That is, if $p_j < p_i + c_{i,j}$, then $x_{i,j} = \ell_{i,j}$. In the zero profit regime, $p_i + c_{i,j} = p_j$, any feasible flow, $\ell_{i,j} \leq x_{i,j} \leq u_{i,j}$, can be sent along the arc. The final case occurs when a profit is realized by utilizing the arc. In this case, the maximal amount of flow is sent along the arc. That is, if $p_j > p_i + c_{i,j}$, then $x_{i,j} = u_{i,j}$. These three cases are succinctly written as

$$\ell_{i,j} \leq x_{i,j} \leq u_{i,j} \quad \perp \quad p_i + c_{i,j} - p_j \quad \forall\, (i,j) \in \mathcal{A},$$

where the $\perp$ symbol denotes the mixed complementarity relationship.

Example 1.8.

arc	$\ell_{i,j}$	$u_{i,j}$
(1,2)	0	2
(1,3)	0	4
(2,3)	0	5
(2,5)	0	5
(3,4)	0	5
(3,5)	0	2
(4,2)	0	5

Table 1.5. *Bound data for Example 1.7*

Given the bound data in Table 1.5, the mixed complementarity problem for the network in Example 1.1 is written as:

$$\begin{array}{lcl}
0 \le p_1 \le \infty & \perp & 5 - x_{1,2} - x_{1,3} \\
0 \le p_2 \le \infty & \perp & x_{1,2} + x_{4,2} - x_{2,3} - x_{2,5} \\
0 \le p_3 \le \infty & \perp & x_{1,3} + x_{2,3} - x_{3,4} - x_{3,5} \\
0 \le p_4 \le \infty & \perp & x_{3,4} - x_{4,2} \\
0 \le p_5 \le \infty & \perp & x_{2,5} + x_{3,5} - 5 \\
0 \le x_{1,2} \le 2 & \perp & p_1 + 0.1 - p_2 \\
0 \le x_{1,3} \le 4 & \perp & p_1 + 0.2 - p_3 \\
0 \le x_{2,3} \le 5 & \perp & p_2 + 0.3 - p_3 \\
0 \le x_{2,5} \le 5 & \perp & p_2 + 0.4 - p_5 \\
0 \le x_{3,4} \le 5 & \perp & p_3 + 0.5 - p_4 \\
0 \le x_{3,5} \le 2 & \perp & p_3 + 0.6 - p_5 \\
0 \le x_{4,2} \le 5 & \perp & p_4 + 0.7 - p_2.
\end{array}$$

A solution to this mixed complementarity problem is $x_{1,2} = 2$, $x_{1,3} = 3$, $x_{2,5} = 3$, $x_{3,4} = 1$, $x_{3,5} = 2$, $x_{4,2} = 1$, $p_1 = 0$, $p_2 = 1.4$, $p_3 = 0.2$, $p_4 = 0.7$, and $p_5 = 1.8$, and all other variables are set to zero. The prices are not unique in this case, since for example the same positive number can be added to each price variable as an alternative solution.

The AMPL and GAMS models for this problem are given in Figure 1.10 and Figure 1.11. This AMPL model places both inequalities defining the complementarity relationship on the variable with the notation:

```
0 <= x[i,j] <= u[i,j]  complements p[i] + c[i,j] - p[j];
```

The upper bounds on the variables are added to the GAMS model with the following statement:

```
x.up(ARCS) = adata(ARCS,'u');
```

```
set NODES;                          # Nodes in the network
set ARCS within NODES cross NODES;  # Arcs in the network

param s{NODES} >= 0;                # Amount of good available
param d{NODES} >= 0;                # Amount of good demanded
param c{ARCS};                      # Transportation cost
param u{ARCS};                      # Arc capacities

var p{NODES};                       # Marginal prices
var x{ARCS};                        # Shipment quantity

subject to
conserve {i in NODES}:              # Conservation of material
    0 <= p[i] complements sum{(j,i) in ARCS} x[j,i] + s[i] >=
                          sum{(i,j) in ARCS} x[i,j] + d[i];

subject to
rational {(i,j) in ARCS}:          # Rational pricing
    0 <= x[i,j] <= u[i,j] complements p[i] + c[i,j] - p[j];
```

Figure 1.10. *AMPL model for Example 1.7.*

```
$include network-mcp.dat

alias(NODES,i,j);

positive variables p(NODES),
                   x(NODES,NODES);

equations conserve(NODES),
          rational(NODES,NODES);

conserve(i)..
  sum(j$ARCS(j,i), x(j,i)) + s(i) =g=
  sum(j$ARCS(i,j), x(i,j)) + d(i);

rational(ARCS(i,j))..
  p(i) + c(i,j) =g= p(j);

model network / conserve.p, rational.x /;

x.up(ARCS) = adata(ARCS,'u');
solve network using mcp;
```

Figure 1.11. *GAMS model for Example 1.7.*

Price supports and price ceilings can also be incorporated into the dual linear optimization problem (1.6):

$$\begin{array}{rl} \max & \sum_{i\in\mathcal{D}} d_i p_i - \sum_{i\in\mathcal{S}} s_i p_i \\ \text{subject to} & p_i + c_{i,j} \geq p_j \ \forall\, (i,j) \in \mathcal{A} \\ & \ell \leq p \leq u, \end{array}$$

where $\ell \in \Re_+^{|\mathcal{N}|}$ and $u \in \bar{\Re}_+^{|\mathcal{N}|}$ with $\ell \leq u$ are the nonnegative lower and upper bounds on the prices, respectively. A finite upper bound on the demand price implies that the corresponding retailer might be under-supplied. The mixed complementarity relationship generated by this modification states that if a retailer $i \in \mathcal{D}$ is over-supplied, then the price a rational person is willing to pay is minimal. That is, if $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} > \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i$, then $p_i = \ell_i$. If the retailer is supplied the exact amount needed, $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} = \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i$, then any feasible price can be paid, $\ell_i \leq p_i \leq u_i$. The new case occurs when the demand exceeds supply, $\sum_{j:(j,i)\in\mathcal{A}} x_{j,i} < \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i$, in which case the price is at the ceiling, $p_i = u_i$. Summarizing these three conditions leads to the mixed complementarity relationship:

$$\ell_i \leq p_i \leq u_i \quad \perp \quad \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i - \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} - d_i \quad \forall\, i \in \mathcal{D}.$$

While upper bounds on the prices for the suppliers can be added, the resulting problem makes little sense because a warehouse could then ship more of the good than is available.

Another possibility is to add an upper bound on the tax rate for the endogenous tax formulation in Section 1.5. This modification leads to the mixed complementarity relationship:

$$0 \leq t_i \leq u_i \quad \perp \quad t_i p_i \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} - T_i \quad \forall\, i \in \mathcal{N}.$$

If T_i is thought of as a desired revenue level, then the complementarity relationship determines the tax rate t_i needed to obtain this outcome. When $T_i > 0$ and $u_i > 0$, the complementarity condition implies that the tax rate is positive at a solution. Furthermore, either the desired tax revenue is achieved exactly or the tax rate is at its upper bound.

1.8 Revenue and Transshipment

The mixed complementarity relationship also enables equations to be placed into the model. In fact, a square system of nonlinear equations is a special case of the mixed complementarity problem in which the lower bounds on the variables are all set to negative infinity and the upper bounds are set to positive infinity. A nonlinear system of equations is described by a function $F : \Re^n \to \Re^n$. The solution is a vector $x \in \Re^n$ such that $F(x) = 0$. In mixed complementarity notation, the problem is written as

$$x \quad \perp \quad F(x) = 0$$

These equations can be used in the transportation problem to make the problem easier to understand. For example, a function for the shipment cost was need when constructing a model with congestion effects. Introducing a variable $c_{i,j}$ for each arc $(i,j) \in \mathcal{A}$, the nonlinear shipment cost can be obtained by adding the equation

$$c_{i,j} \quad \perp \quad c_{i,j} = A_{i,j} + B_{i,j}(x_{i,j})^4.$$

The rational cost constraint can then be written as

$$0 \leq x_{i,j} \ \perp \ p_i + c_{i,j} \geq p_j.$$

Another possibility is to make equality hold on the transshipment nodes in the primal transportation problem. In this case, the linear optimization problem (1.4) can be written as:

$$\begin{array}{llcll}
\min_{x \geq 0} & \sum_{(i,j)\in\mathcal{A}} c_{i,j} x_{i,j} & & & \\
\text{subject to} & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} & \leq & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i & \forall\, i \in \mathcal{S} \\
& \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i & \leq & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} & \forall\, i \in \mathcal{D} \\
& \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} & = & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} & \forall\, i \in \mathcal{T},
\end{array}$$

where $\mathcal{T}$ is the set of nodes with $s_i = 0$ and $d_i = 0$. The marginal prices on the transshipment constraints then become unrestricted in the mixed complementarity formulation

$$\begin{array}{lclcll}
0 \leq p_i & \perp & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} + s_i & \geq & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} & \forall\, i \in \mathcal{S} \\
0 \leq p_i & \perp & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} & \geq & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} + d_i & \forall\, i \in \mathcal{D} \\
p_i & \perp & \sum_{j:(j,i)\in\mathcal{A}} x_{j,i} & = & \sum_{j:(i,j)\in\mathcal{A}} x_{i,j} & \forall\, i \in \mathcal{T} \\
0 \leq x_{i,j} & \perp & p_i + c_{i,j} & \geq & p_j & \forall\, (i,j) \in \mathcal{A}.
\end{array}$$

For these transportation problems, the introduction of transshipment nodes does not effect the solution to the problem when the per-unit cost of using each arc is positive. In general, changing an inequality to an equation can cause significant changes in the solution.

For completeness, the dual linear optimization problem when the transshipment nodes are represented using equations is:

$$\begin{array}{lll}
\max_p & \sum_{i\in\mathcal{D}} d_i p_i - \sum_{i\in\mathcal{S}} s_i p_i & \\
\text{subject to} & p_i + c_{i,j} \geq p_j & \forall\, (i,j) \in \mathcal{A} \\
& p_i \geq 0 & \forall\, i \in \mathcal{S} \cup \mathcal{D}.
\end{array}$$

Note that the dual variables corresponding to the transshipment nodes are free.

Example 1.9.

This example needs to be changed – subsidies are not longer in the text. All of the effects discussed can be incorporated into the transportation problem in

Example 1.1. One example of such a problem is:

$$
\begin{array}{lcl}
0 \le p_1 \le \infty & \perp & 5 + (p_1)^2 - x_{1,2} - x_{1,3} \\
p_2 \text{ free} & \perp & x_{1,2} + x_{4,2} = x_{2,3} + x_{2,5} \\
p_3 \text{ free} & \perp & x_{1,3} + x_{2,3} = x_{3,4} + x_{3,5} \\
p_4 \text{ free} & \perp & x_{3,4} = x_{4,2} \\
0 \le p_5 \le 2 & \perp & x_{2,5} + x_{3,5} - \frac{5}{(p_5)^2} \\
0 \le x_{1,2} \le 2 & \perp & p_1 + 0.1 + (x_{1,2})^4 - p_2 \\
0 \le x_{1,3} \le 4 & \perp & p_1 + 0.2 + (x_{1,3})^4 - p_3 \\
0 \le x_{2,3} \le 5 & \perp & (1 + t_2)p_2 + 0.3 + (x_{2,3})^4 - p_3 \\
0 \le x_{2,5} \le 5 & \perp & (1 + t_2)p_2 + 0.4 + (x_{2,5})^4 - p_5 \\
0 \le x_{3,4} \le 5 & \perp & (1 - t_3)p_3 + 0.5 + (x_{3,4})^4 - p_4 \\
0 \le x_{3,5} \le 2 & \perp & (1 - t_3)p_3 + 0.6 + (x_{3,5})^4 - p_5 \\
0 \le x_{4,2} \le 5 & \perp & p_4 + 0.7 + (x_{4,2})^4 - p_2 \\
t_2 \text{ free} & \perp & t_2 = 0.02 + 0.0016 r_2 \\
t_3 \text{ free} & \perp & t_3 = 0.05 \\
r_2 \text{ free} & \perp & r_2 = p_2(x_{2,3} + x_{2,5}).
\end{array}
$$

A solution to this problem is $x_{1,2} = 0.7840$, $x_{1,3} = 0.7168$, $x_{2,5} = 0.7840$, $x_{3,5} = 0.7168$, $p_1 = 0.5479$, $p_2 = 1.0256$, $p_3 = 1.0119$, $p_4 = 1$, $p_5 = 1.8253$, and all other variables begin equal to zero.

1.9 Optimal Taxation

1.9.1 Optimal Tolling and Taxation

The combination of optimal tolling and optimal taxation in the unified spatial price equilibrium model leads to an equilibrium problem with equilibrium constraints if there are either multiple tolling authorities or the tolling authority and government are separate entities, since individually seek to maximize their profit subject to the follower constraints.

The case where the government taxes both the revenue generated by the sale of goods and the revenue generated by the tolling authorities leads to a heirarchical equilibrium problem. The government is at the top level and maximized revenue subject to the tolling authorities maximizing their revenue subject to the follower selection or routes. There are now three levels: the government, the tolling authorities, and the individual companies.

These types of models would be a nice addition to highlight current limitations in theory and practice. For example, the equilibrium problem with equilibrium constraints may have no pure solution. To my knowledge, the heirarchical equilibrium problem has not been considered in the literature.

1.10 Accessing Models

1.10.1 AMPL

1.10.2 GAMS

1.10.3 Matlab

1.11 Notes and References

Complementarity problems are also known as equilibrium problems and box-constrained variational inequalities. The term complementarity comes from the orthogonality condition used when describing the nonlinear complementarity problem. An in depth study of linear complementarity is found in [49], while the extensive theory of nonlinear complementarity problems and variational inequalities is detailed in [?].

A survey of many complementarity applications is given in [87]. More example problems can be found in [59, 86, 87].

The first order optimality conditions were derived in [164], but later found to appear in the master's thesis [145]. These conditions are fundamental to many of the primal-dual methods [305] and form the basis of many interior point algorithms for linear, quadratic, and convex optimization

Modeling languages [20, 95] began as matrix generators for linear optimization problems. The have advanced much beyond these origins. AMPL[96] and GAMS[25] are the most widely used modeling systems incorporating complementarity. These facilities for describing complementarity relations can be found in [65, 76]. Other tools, such as AIMMS [19], MATLAB [196], and NEOS [51, 81], can also be used to communicate a complementarity problem. Further information on the GAMS syntax can be found in [261].

An early version of the derivation of the first order optimality conditions was developed in [84]. Alternative supply and demand functions for the transportation problem are described in [261] and details about complementarity problems for more general transportation models can be found in [63, 80]. Nonintegrable congestion functions are used in [99].

Add reference on modeling water in an overfull cup.