
Volume 42B, issue 9, November 2008
ISSN 0191-2615
ELSEVIER
TRANSPORTATION
RESEARCH
AN INTERNATIONAL JOURNAL
Part B: Methodological
Editor-in-Chief: Fred L. Mannering

Available online at www.sciencedirect.com

ScienceDirect

Transportation Research Part B 42 (2008) 823–842

TRANSPORTATION RESEARCH PART B

www.elsevier.com/locate/trb

A link-node complementarity model and solution algorithm for dynamic user equilibria with exact flow propagations

Xuegang (Jeff) Ban [a,*], Henry X. Liu [b,1], Michael C. Ferris [c,2], Bin Ran [d,3]

[a] California Center for Innovative Transportation (CCIT), Institute of Transportation Studies (ITS), University of California, Berkeley, 2105 Bancroft Way, Suite 300 Berkeley, CA 94720, United States

[b] Department of Civil Engineering, University of Minnesota, Twin Cities, 122 Civil Engineering Building, 500 Pillsbury Drive S.E., Minneapolis, MN 55455, United States

[c] Computer Sciences Department, University of Wisconsin at Madison, 1210 West Dayton Street, Madison, WI 53706, United States

[d] Department of Civil and Environmental Engineering, University of Wisconsin at Madison, 1212 Engineering Hall, 1415 Engineering Drive Madison, WI 53706, United States

Abstract

In this paper, we propose a link-node complementarity model for the basic deterministic dynamic user equilibrium (DUE) problem with single-user-class and fixed demands. The model complements link-path formulations that have been widely studied for dynamic user equilibria. Under various dynamic network constraints, especially the exact flow propagation constraints, we show that the continuous-time dynamic user equilibrium problem can be formulated as an infinite dimensional mixed complementarity model. The continuous-time model can be further discretized as a finite dimensional non-linear complementarity problem (NCP). The proposed discrete-time model captures the exact flow propagation constraints that were usually approximated in previous studies. By associating link inflow at the beginning of a time interval to travel times at the end of the interval, the resulting discrete-time model is predictive rather than reactive. The solution existence and compactness condition for the proposed model is established under mild assumptions. The model is solved by an iterative algorithm with a relaxed NCP solved at each iteration. Numerical examples are provided to illustrate the proposed model and solution approach. We particularly show why predictive DUE is preferable to reactive DUE from an algorithmic perspective.

Keywords: Dynamic traffic assignment; Dynamic user equilibrium; Dynamic network loading; Nonlinear complementarity problem

* Corresponding author. Tel.: +1 510 642 5112; fax: +1 510 642 0910.

E-mail addresses: xban@berkeley.edu (X. (Jeff) Ban), henryliu@umn.edu (H.X. Liu), ferris@cs.wisc.edu (M.C. Ferris), bran@engr.wisc.edu (B. Ran).

[1] Tel.: +1 612 625 6347; fax: +1 612 626 7750.

[2] Tel.: +1 608 262 4281; fax: +1 608 262 9777.

[3] Tel.: +1 608 262 0052; fax: +1 608 262 5199.

doi:10.1016/j.trb.2008.01.006

1. Introduction

Dynamic traffic assignment (DTA) models, which can predict future dynamic traffic states in a short-term fashion, have been extensively studied for decades. DTA studies have been particularly accelerated in the last 15 years since the advent of the intelligent transportation systems (ITS). In this paper, we are interested in the so-called dynamic user equilibrium (DUE) that is the fundamental yet most challenging problem of DTA. Although two distinct approaches have dominated the methodologies applied to DTA research: the simulation based (microscopic/mesoscopic) approach and the analytical (macroscopic) approach, in this paper, we focus ourselves on the analytical DTA models, especially those with the variational inequality (VI) formulation.

VI has been applied for long to model various traffic interactions for static traffic assignment problems (Smith, 1979; Dafermos, 1980), and has been shown to be more capable of modeling and computing dynamic network equilibria than the constrained optimization approach (Ran and Boyce, 1996; Chen, 1999; Peeta and Ziliaskopoulos, 2001). Friesz et al. (1993) were among the first to model DUE with VI, and Ran and Boyce (1994, 1996) extensively studied the issues of applying VI to formulate and solve DTA problems. The VI approach has also been used for DTA studies by Lo et al. (1996), Ran et al. (1996), Heydecker and Verlander (1999), to name just a few. Recently, Friesz et al. (2001) and Friesz and Mookherjee (2006) developed a differential variational inequality (DVI) technique to model DUE in continuous-time. Although formulated continuously in the temporal domain, most DUE models (e.g., those by Friesz et al. (1993) and Ran and Boyce (1996)) were solved by time discretization. This is because, to date, solving continuous-time DUE models directly for practical transportation networks is still not feasible. However, many discrete models were only treated as part of the solution procedure without rigorous investigations on their mathematical properties. Chen and Hsuen (1998) were among the first to investigate explicitly on discrete-time VI models for DTA. Bliemer (2001) and Bliemer and Bovy (2003) further improved the model by Chen and Hsuen (1998) and proposed a link-path based quasi-variational inequality (QVI) formulation. Lo and Szeto (2002) integrated the cell transmission model (CTM) into DUE which was formulated as a route based VI problem. Due to the nature of CTM (Daganzo, 1994, 1995a), the model by Lo and Szeto (2002) is discretized in both temporal and spatial domains.

As a special case of VI, the non-linear complementarity problem (NCP) has been extensively studied by the mathematical programming community (see Facchinei and Pang (2003) and references therein). Efficient solution approaches have been developed during the last decade for solving large scale NCPs (Cao and Ferris, 1996; Billups et al., 1997; Ferris et al., 1999). Generally, solving an NCP is much easier than a regular VI and NCP formulations have been used by some researchers in modeling and solving DUE (Ran and Boyce, 1996; Akamatsu, 2001; Wie et al., 2002). In particular, Wie et al. (2002) formulated the discrete-time DUE with departure time choice as an NCP. It was also pointed out in Wie et al. (2002) that continuous-time and discrete-time DUE models are significantly different. Generally, the former are infinite dimensional mathematical programming problems, while the latter are finite dimensional mathematical programming problems. The NCP based DUE model by Wie et al. (2002), nevertheless, applied approximate flow propagations and projected the link exit flows to two neighboring time grids. Therefore, the exact flow propagation of DUE (Astarita, 1996) was not fully respected. Further, the linear programming based solution approach in Wie et al. (2002) is similar to the Frank-Wolfe (FW) algorithm. Due to the well-known convergence problem of FW, such a method may not be effective for solving DUE, especially for producing accurate solutions.

In this paper, we formulate the discrete-time DUE problem with exact flow propagations as a link-node based NCP. This new link-node NCP formulation complements the link-path DUE models that have been widely studied in the literature. The special structure of the proposed model may also facilitate the development of more efficient algorithms for solving DUE. We specifically consider the basic discrete-time DUE problem in this paper, which is deterministic and single-user-class, and has fixed travel demands.

In what follows we will start with the continuous-time DUE and formulate it as an infinite dimensional mixed complementarity problem (MiCP) with side constraints. Based on this MiCP model, we adopt the discretization scheme by Astarita (1996) and prove that an NCP formulation exists for the discrete-time DUE with exact flow propagations. Also, by associating link inflow at the beginning of a given time interval to travel times at the end of the interval in DUE route choice, the proposed NCP model represents predictive DUE (Heydecker and Verlander, 1999). We further prove the solution set of the proposed NCP model is non-empty and compact. To solve the model, we develop an iterative algorithm with a relaxed NCP solved at each iteration. Numerical studies are provided in this paper to illustrate the proposed model and demonstrate the efficiency of the solution algorithm. We show that, at least for the proposed NCP model, the predictive DUE is algorithmically more preferable than reactive DUE.

2. Continuous-time DUE model

In this section, we introduce the link-node based continuous-time formulation for DUE. In the literature, Friesz et al. (1993), Ran and Boyce (1996), and Bliemer and Bovy (2003) have extensively studied the link-path based continuous-time DUE models.

Assume a given transportation network can be represented as a connected and directed graph, denoted as $G(N, A)$, where N is the set of nodes and A is the set of links (arcs). Since we are dealing with dynamic (or time-varying) traffic flows, we denote T' as the total study period and $t \in [0, T']$. Also denote R and S as origin and destination node sets, respectively. Throughout this paper, we will use index $a \in A$ to denote a link, index $i \in N$ or $j \in N$ to denote a node, and index $s \in S$ to denote a destination. Moreover, we only consider the basic DUE problem, i.e., the deterministic and single-user-class DUE with fixed demands. Further, $d_{is}(t)$ denotes the (fixed) travel demand rate from node i to destination s at time instant t. We conventionally set $d_{ss}(t) = 0$ $\forall s \in S, t \in [0, T']$.

2.1. DUE condition

The link-node model is derived from the so-called (link-node based) DUE condition that describes the optimality condition of the DUE problem. The DUE condition is a dynamic extension to the Wardrop's first principle (1952) for the static case and can be stated as follows:

> *If, from each decision node to every destination node at each instant of time, the actual travel times for all the routes that are being used are equal and minimal, then the dynamic traffic flow over the network is in a travel time based dynamic user equilibrium (DUE) state.*

In this condition, a "decision node" with respect to a given destination can be any node in the network which either generates OD trips (i.e., an origin node) or is traversed by flows heading to the destination (i.e., an intermediate node). Hence, a destination node is not a decision node of itself. Mathematically, the DUE condition can be expressed as follows:

$$0 \leqslant u_{as}(t) \perp \{\tau_a(t) + \pi_{h_a s}[t + \tau_a(t)] - \pi_{l_a s}(t)\} \geqslant 0 \quad \forall a, s, t. \tag{1}$$

Here $u_{as}(t)$ denotes the inflow rate to link a with respect to destination s at time instant t. $\pi_{l_a s}(t)$ and $\pi_{h_a s}(t)$ denote, respectively, the minimum travel time from the tail node (l_a) and head node (h_a) of link a to destination s at time t. We conventionally set $\pi_{ss}(t) = 0$ $\forall s \in S$, $t \in [0, T']$. $\tau_a(t)$ is the travel time for link a at time instant t. In addition, "$\perp$" in (1) means "perpendicular" so that $x \perp y \Longleftrightarrow x^T y = 0$. Note that the DUE condition can also be conveniently expressed using the dynamic programming (DP) notation. For details, one can refer to Han and Heydecker (2006).

2.2. Dynamic network constraints

The dynamic network constraints describe the defining set of the DUE problem, which must be satisfied by any feasible solution. Five types of constraints have been identified in the literature (Ran and Boyce, 1996; Bliemer and Bovy, 2003), namely mass balance constraints, flow conservation constraints, flow propagation constraints, first-in-first-out (FIFO) constraints, and other definitional constraints. In the following, we simply list these constraints without further discussions (except the flow propagation constraints in Section 2.2.4). For details, one can refer to Ran and Boyce (1996) and Bliemer and Bovy (2003).

2.2.1. Mass balance constraints

Mass balance constraints define the relationship among the link flow (denoted as $x_{as}(t)$), the inflow rate (denoted as $u_{as}(t)$), and the exit flow rate (denoted as $v_{as}(t)$) of some link a with respect to destination s at time t. The constraints can be expressed as follows:

$$\frac{\mathrm{d}x_{as}(t)}{\mathrm{d}t} = u_{as}(t) - v_{as}(t) \quad \forall a, s, t.$$

The above equation implies that $x_{as}(t) = \int_0^t [u_{as}(w) - v_{as}(w)]\mathrm{d}w + x_{as}(0) \;\; \forall a, s, t$. We further assume the initial condition as $x_{as}(0) = 0 \; \forall a, s$. Then, the mass balance constraints can be rewritten as

$$x_{as}(t) = \int_0^t [u_{as}(w) - v_{as}(w)]\mathrm{d}w \quad \forall a, s, t. \tag{2}$$

2.2.2. Flow conservation constraints

The flow conservation constraints require that for any given time instant, flow entering any node, together with the demand generated at this node, must all exit from this node unless the node is a destination. Mathematically, they can be expressed as

$$\sum_{a \in A(i)} u_{as}(t) = d_{is}(t) + \sum_{a \in B(i)} v_{as}(t) \quad \forall i, s, i \neq s, t. \tag{3}$$

Here $A(i)$ is the set of links whose tail nodes (i.e., starting nodes) are i; $B(i)$ is the set of links whose head nodes (i.e., ending nodes) are i.

2.2.3. FIFO constraints

FIFO constraints were first introduced into the DTA study by Carey (1987). Since then, it has been assumed to be a "discipline" respected by any dynamic traffic flow. FIFO requires that any vehicle entering into a link earlier must also exit from the link earlier. Mathematically, FIFO constraints can be represented as

$$t_1 + \tau_a(t_1) < t_2 + \tau_a(t_2) \quad \forall t_1 < t_2. \tag{4}$$

Ran and Boyce (1996) showed that in order for FIFO to hold, an extra restriction should be imposed on $\tau_a(t)$ as

$$\mathrm{d}\tau_a(t)/\mathrm{d}t > -1. \tag{5}$$

However, explicitly imposing such a constraint will dramatically increase the complexity of the resulting model. Therefore, most DUE models tend to implicitly guarantee FIFO by choosing a proper link performance function. How to design such a function, however, is beyond the scope of this paper. Here we conventionally assume $\tau_a(t)$ is a function of the link flow at time t on link a

$$\tau_a(t) = g(x_a(t)) \quad \forall a, t. \tag{6}$$

Here $x_a(t)$ is the total link flow on link a at time t (see Eq. (10) below). The function in (6) satisfies FIFO when g is linear, or if the gradient of g with respect to x is bounded from above when g is non-linear. For more discussions, one can refer to Nie and Zhang (2005), Carey et al. (2003), Xu et al. (1999), and Daganzo (1995b).

2.2.4. Flow propagation constraints

These constraints describe the consistent evolvement of traffic flows in both temporal and spatial domains. It has been proved that in a continuous-time fashion, the (exact) flow propagation constraints can be represented as (Astarita, 1996):

$$v_{as}[t + \tau_a(t)] = \frac{u_{as}(t)}{1 + \mathrm{d}\tau_a(t)/\mathrm{d}t} \quad \forall a, s, t. \tag{7}$$

In this paper, we further introduce an "inverse link travel time function" $p_a(t)$, which denotes the travel time of link a for vehicles *exiting* the link at time t (notice that $\tau_a(t)$ is the link travel time for vehicles *entering* link a at time t). According to this definition, $t - p_a(t)$ is the time when those vehicles entered link a. Therefore, we have

$$p_a(t) = \tau_a(t - p_a(t)). \tag{8}$$

Clearly, $p_a(t)$ is not close formed. However, under FIFO, $p_a(t)$ is a one-to-one mapping (i.e., a well-defined function).[4] Based on $p_a(t)$, the flow propagation constraint (7) can be expressed as

$$v_{as}(t) = \frac{u_{as}(t - p_a(t))}{1 + \dot{\tau}_a(t - p_a(t))} \quad \forall a, s, t.$$

Here $\dot{\tau}_a(t - p_a(t)) \equiv \mathrm{d}\tau_a(t - p_a(t))/\mathrm{d}(t - p_a(t)) = \mathrm{d}p_a(t)/(\mathrm{d}t - \mathrm{d}p_a(t))$. Therefore, (7) can be rewritten as

$$v_{as}(t) = u_{as}(t - p_a(t))\left(1 - \frac{\mathrm{d}p_a(t)}{\mathrm{d}t}\right) \quad \forall a, s, t. \tag{9}$$

Notice that from (8), we can obtain

$$\frac{\mathrm{d}p_a(t)}{\mathrm{d}t} = \frac{\mathrm{d}\tau_a(t - p_a(t))}{\mathrm{d}t} = \frac{\mathrm{d}\tau_a(t - p_a(t))}{\mathrm{d}(t - p_a(t))}\frac{\mathrm{d}(t - p_a(t))}{\mathrm{d}t} = \dot{\tau}_a(t - p_a(t))\left(1 - \frac{\mathrm{d}p_a(t)}{\mathrm{d}t}\right).$$

Denote $\rho = \dot{\tau}_a(t - p_a(t))$. From the above equation, we have $\frac{\mathrm{d}p_a(t)}{\mathrm{d}t} = \frac{\rho}{\rho+1}$. If FIFO holds, we must have $\rho > -1$ according to Eq. (5), which indicates that $\frac{\mathrm{d}p_a(t)}{\mathrm{d}t} < 1$. In other words, Eq. (9) implies that exit flow is always non-negative if FIFO holds.

Note that the inverse link travel time function defined here is similar to the inverse arc exit time function in Friesz et al. (1993). The main differences are (a) $p_a(t)$ in (8) is defined on an individual link and therefore there is no need to track the entire path and (b) $p_a(t)$ actually represents link travel time (from a previous time interval), but in Friesz et al. (1993), the inverse arc exit time function was defined as the departure time of vehicles entering a path. Since we assume vehicles to different destinations experience the same travel time as long as they enter the link at the same time, $p_a(t)$ is independent of individual destinations. In Bliemer and Bovy (2003), exact flow propagation was presented using the so-called "dynamic effective flow rate factor." The formulation (9) here is link based (i.e., no need to keep track of an entire path) and thus much simpler compared with that in Bliemer and Bovy (2003).

2.2.5. Other definitional constraints

Other definitional constraints are listed in Eqs. (10) and (11) as follows:

$$\begin{cases} u_a(t) = \sum_{s\in S} u_{as}(t), \\ v_a(t) = \sum_{s\in S} v_{as}(t), \quad \forall a, t, \\ x_a(t) = \sum_{s\in S} x_{as}(t), \end{cases} \tag{10}$$

$$\begin{cases} u_{as}(t) \geqslant 0, \quad v_{as}(t) \geqslant 0, \quad x_{as}(t) \geqslant 0 \quad \forall a, s, t, \\ \pi_{is}(t) \geqslant 0 \quad \forall i, s, i \neq s, t. \end{cases} \tag{11}$$

Here $u_a(t)$ and $v_a(t)$ denote, respectively, the total inflow rate and exit flow rate of link a at time t. Together with $x_a(t)$, they are referred as "aggregated" variables. On the other hand, $u_{as}(t)$, $v_{as}(t)$ and $x_{as}(t)$ are called "disaggregated" variables. For aggregated and disaggregated variables that satisfy (10), we call them "corresponding" to each other.

[4] This can be seen as follows. Under the FIFO condition (4), if $p_a(t)$ is not uniquely determined for a given t, we can assume $p_a(t) = p_1$, $p_a(t) = p_2$, and $p_1 \neq p_2$. Without loss of generality, we assume $p_1 > p_2$. We will then have $t = t - p_1 + \tau_a(t - p_1) < t - p_2 + \tau_a(t - p_2) = t$ since $t - p_1 < t - p_2$. This is a contradiction. Therefore, we must have $p_1 = p_2$, i.e., $p_a(t)$ is a one-to-one mapping.

2.3. MiCP formulation

Eqs. (1)–(3), (9) and (11) constitute the DUE model defined on disaggregated variables $x = (x_{as}(t))_{\forall a,s,t}$, $u = (u_{as}(t))_{\forall a,s,t}$, $v = (v_{as}(t))_{\forall a,s,t}$, and $\pi = (\pi_{is}(t))_{\forall i,s,t;i \neq s}$; whereas, $\tau = (\tau_a(t))_{\forall a,t}$, $p = (p_a(t))_{\forall a,t}$, and the aggregated variables can be treated as functions of these defining variables, as shown in Eqs. (6), (8), and (10). Denote this model as *DUEORI* (short for the original DUE problem). It turns out that *DUEORI* can be further simplified; in particular, we have the following Theorem:

Theorem 1. *The following two statements are true, provided FIFO holds:*

(a) *Assume* $x = (x_{as}(t))_{\forall a,s,t}$, $u = (u_{as}(t))_{\forall a,s,t}$, $v = (v_{as}(t))_{\forall a,s,t}$, *and* $\pi = (\pi_{is}(t))_{\forall i,s,t;i \neq s}$ *solves DUEORI with* $\tau = (\tau_a(t))_{\forall a,t}$, $p = (p_a(t))_{\forall a,t}$, *and the aggregated variables are defined in Eqs.* (6), (8) *and* (10), *respectively. Then* (u, π) *must solve the following model, denoted as DUEMiCP:*

$$0 \leqslant u_{as}(t) \perp \{\tau_a(t) + \pi_{h_a s}[t + \tau_a(t)] - \pi_{l_a s}(t)\} \geqslant 0 \quad \forall a, s, t, \tag{12a}$$

$$\sum_{a \in A(i)} u_{as}(t) = d_{is}(t) + \sum_{a \in B(i)} u_{as}(t - p_a(t))\left(1 - \frac{\mathrm{d}p_a(t)}{\mathrm{d}t}\right) \quad \forall i, s, i \neq s, t, \tag{12b}$$

$$\pi_{is}(t) \geqslant 0 \quad \forall i, s, i \neq s, t. \tag{12c}$$

(b) *If* (u, π) *solves DUEMiCP* (12), *with* p *defined in* (8), *and* τ *defined as*

$$\tau_a(t) = g\left(\int_0^t \sum_{s \in S}\left\{\left[u_{as}(w) - u_{as}(t - p_a(t))\left(1 - \frac{\mathrm{d}p_a(t)}{\mathrm{d}t}\right)\right]\right\}\mathrm{d}w\right) \quad \forall a, t. \tag{i}$$

Then there must exist vectors x and v such that (x, u, v, π) solves *DUEORI*. Also, τ and the aggregated variables satisfy Eqs. (6) and (10), respectively.

Proof

(a) If $x = (x_{as}(t))_{\forall a,s,t}$, $u = (u_{as}(t))_{\forall a,s,t}$, $v = (v_{as}(t))_{\forall a,s,t}$, and $\pi = (\pi_{is}(t))_{\forall i,s,t;i \neq s}$ solves *DUEORI*, (12a) will hold. We can then substitute (9) into (3), resulting in (12b). Also, (12c) comes from (11).
(b) If (u, π) solves (12) and p and τ are defined in (8) and (i), respectively, (1) holds directly. We can then define

$$v_{as}(t) = u_{as}(t - p_a(t))\left(1 - \frac{\mathrm{d}p_a(t)}{\mathrm{d}t}\right) \quad \forall a, s, t, \tag{ii}$$

$$x_{as}(t) = \int_0^t [u_{as}(w) - v_{as}(w)]\mathrm{d}w \quad \forall a, s, t, \tag{iii}$$

which coincide with (9) and (2), respectively. This also implies τ defined in (i) above coincides with (6). In addition, (3) will be satisfied due to (12b) and (ii) above. Further, (10) and (11) are definitional. Especially, the non-negativity of v is guaranteed by the non-negativity of u and the fact that $\frac{\mathrm{d}p_a(t)}{\mathrm{d}t} < 1$ through (ii) above; the non-negativity of x is ensured by FIFO. □

Note that given the disaggregated inflow vector u, determining vectors p and τ requires a link based dynamic network loading (see Nie and Zhang, 2005). In Bliemer (2001), it is referred as "solving a fixed point problem" due to the mutual inclusion of τ and p in their definitions in Eqs. (6) and (8). With Theorem 1, *DUEORI* is simplified into *DUEMiCP* (12) that is defined on u and π only. Aggregated variables, p, and τ are functions of u. Note also that Eqs. (12a) and (12b) define an infinite dimensional MiCP (Ulbrich, 2002), while (12c) imposes a side constraint that requires the minimum travel time from node i to destination s at time t must be non-negative. Solving the infinite dimensional MiCP (12) with side constraints is generally difficult and we thus study the discretized problem starting from the next section.

3. Discrete-time DUE model

3.1. Discrete-time DUE with exact flow propagation

In order to obtain the discrete-time model, we can evenly divide the entire study period into K' time intervals by introducing the length of each time interval Δ such that $K'\Delta = T'$. We use index k to denote the kth time interval $[(k-1)\Delta, k\Delta)$ for $1 \leqslant k \leqslant K'$. The notation for the discrete-time model is first listed as follows:

$u_{as}^k = u_{as}((k-1)\Delta)$: the inflow rate to link a towards destination s at the beginning of time interval k, which is also assumed to be constant during the entire time interval k, $u = (u_{as}^k)_{\forall a,s,k}$
$v_{as}^k = v_{as}((k-1)\Delta)$: the exit flow rate from link a towards destination s at the beginning of time interval k, which is also assumed to be constant during the entire time interval k, $v = (v_{as}^k)_{\forall a,s,k}$
$u_{as}^k\Delta$: the inflows to link a towards destination s during time interval k
$v_{as}^k\Delta$: the exit flows from link a towards destination s during time interval k
$x_{as}^k = x_{as}((k-1)\Delta)$: the flows of link a towards destination s at the beginning of time interval k, which is assumed to be constant during the entire interval k, $x = (x_{as}^k)_{\forall a,s,k}$
$u_a^k = \sum_{s\in S} u_{as}^k$: the aggregated inflow rate to link a at the beginning of time interval k, which is assumed to be constant during the entire interval k, $u^A = (u_a^k)_{\forall a,k}$
$v_a^k = \sum_{s\in S} v_{as}^k$: the aggregated exit flow rate from link a at the beginning of time interval k, which is assumed to be constant during the entire interval k, $v^A = (v_a^k)_{\forall a,k}$
$x_a^k = \sum_{s\in S} x_{as}^k$: the aggregated flows of link a towards destination s at the beginning of time interval k, which is assumed to be constant during the entire interval k, $x^A = (x_a^k)_{\forall a,k}$
$d_{is}^k = d_{is}((k-1)\Delta)$: the demand rate generated from node i to destination s at the beginning of time interval k, which is assumed to be constant during the entire interval k, $d = (d_{is}^k)_{\forall i,s,k,i\neq s}$
$d_{is}^k\Delta$: the demands generated from node i to destination s during time interval k
$\tau_a^k(u) = \tau_a(k\Delta)$: the travel time of link a at the end of time interval k, a function of u, $\tau = (\tau_a^k)_{\forall a,k}$
$\pi_{is}^k = \pi_{is}(k\Delta)$: the minimum travel time from node i to destination s at the end of time interval k, $\pi = (\pi_{is}^k)_{\forall i,s,k,i\neq s}$
$e_a^k(u) \equiv (k-1)\Delta + \tau_a^{k-1}(u)$: the exit time for vehicles entering a at the beginning of time interval k, a function of $u, e = (e_a^k)_{\forall a,k}$

Note that in the above notation, u, v, x, and d are defined at the beginning of a time interval, while τ and π are defined at the end of the time interval. This is to associate the inflow to a link at the beginning of any time interval k with the cost (travel time) at the end of the time interval in DUE route choice. As already discussed in Heydecker and Verlander (1999), this way of associating inflow and cost will lead to predictive DUE which has more plausible solution properties compared with reactive DUE. For more details of the concepts of predictive and reactive DUE, one can refer to Heydecker and Verlander (1999), Han (2000), and Han (2003).

Using this discretization scheme, one should be cautious about how to discretize (12a) and (12b). First of all, (12b) is related to the flow conservation (3) and flow propagation (7). For any time interval k, (3) can be easily discretized as (note that the constant Δ at both sides of (13) is omitted)

$$\sum_{a\in A(i)} u_{as}^k = d_{is}^k + \sum_{a\in B(i)} v_{as}^k \quad \forall i,s,i\neq s,k. \tag{13}$$

According to (7), the exit flow rate v_{as}^k is related to inflow rate $u_{as}^{k'}$ if $e_a^{k'}(u) = (k'-1)\Delta + \tau_a^{k'-1} = (k-1)\Delta$ (note that both u and v are defined at the beginning of time intervals). However, such an integer k' may not exist since travel time $\tau_a^{k'-1}$ is real valued. In this paper, we adopt the discretization method in Astarita (1996) to address this problem. The method first constructs the pair of $(e_a^{k'}, v_{as}(e_a^{k'}))$ for any inflow at k'. This can be obtained by discretizing (7) as

$$v_{as}(e_a^{k'}) = u_{as}^{k'} \cdot \lambda_a^{1,k'}(u) \quad \forall a,s,k' \tag{14}$$

and

$$\lambda_a^{1,k'}(u) = \frac{\Delta}{\tau_a^{k'}(u) - \tau_a^{k'-1}(u) + \Delta}. \tag{15}$$

In the denominator of the right hand side of (15), $\tau_a^{k'}$ and $\tau_a^{k'-1}$ are, respectively, the travel times of link a at the beginning of time interval $k'+1$ and k' (or the end of time intervals k' and $k'-1$). Further, $\lambda_a^{1,k'}$ is a function of u since both $\tau_a^{k'}$ and $\tau_a^{k'-1}$ are so.

Give all $(e_a^{k'}, v_{as}(e_a^{k'}))$ pairs, we can compute v_{as}^k for any k as follows. First, we try to find k' such that $e_a^{k'} \leqslant (k-1)\Delta < e_a^{k'+1}$, as shown in Fig. 1.

In Fig. 1, the solid lines represent trajectories of vehicles entering link a at the beginning of time intervals k' and $k'+1$. The exit flow rate at time interval k can then be computed using linear interpolation of $v_{as}(e_a^{k'})$ and $v_{as}(e_a^{k'+1})$ as

$$v_{as}^k = \sum_{k': e_a^{k'} \leqslant (k-1)\Delta < e_a^{k'+1}} \lambda_a^{2,k',k}(u) \cdot v_{as}(e_a^{k'}) + (1 - \lambda_a^{2,k',k}(u)) \cdot v_{as}(e_a^{k'+1}) \quad \forall a, s, k, \tag{16}$$

where $\lambda_a^{2,k',k}(u)$ is defined as

$$\lambda_a^{2,k',k}(u) = \frac{e_a^{k'+1} - (k-1)\Delta}{e_a^{k'+1} - e_a^{k'}} = \frac{\tau_a^{k'}(u) + (k'+1-k)\Delta}{\tau_a^{k'}(u) - \tau_a^{k'-1}(u) + \Delta} \quad \forall a, k', k; e_a^{k'} \leqslant (k-1)\Delta < e_a^{k'+1}. \tag{17}$$

Substitute Eqs. (14)–(17) to (13), we obtain the discretized version of (12b) as

$$\sum_{a \in A(i)} u_{as}^k = d_{is}^k + \sum_{a \in B(i)} \sum_{k': e_a^{k'} \leqslant (k-1)\Delta < e_a^{k'+1}} [\lambda_a^{1,k'}(u) \cdot \lambda_a^{2,k',k}(u) \cdot u_{as}^{k'} + \lambda_a^{1,k'+1}(u) \cdot (1 - \lambda_a^{2,k',k}(u)) \cdot u_{as}^{k'+1}] \quad \forall i, s, i \neq s, k. \tag{18}$$

We can use the similar approach to discretized $\pi_{h_a s}[t + \tau_a(t)]$ in (12a). Since the concept of predictive DUE is adopted in this paper, we need to associate the link inflow at the beginning of time interval k, u_{as}^k, to the link travel time at the end of this time interval, τ_{as}^k, as well as $\pi_{l_a s}^k$ and $\pi_{h_a s}(e_a^{k+1})$. Here $\pi_{l_a s}^k$ is the minimum travel time from the starting node (i.e., tail node) of link a to destination s at the end of time interval k. $\pi_{h_a s}(e_a^{k+1})$ is the minimum travel time from the ending node (i.e., head node) of link a to destination s at the exit time $e_a^{k+1} = k\Delta + \tau_a^k$. Since e_a^{k+1} is real valued, we assume it resides in the lth time interval, i.e., $(l-1)\Delta \leqslant e_a^{k+1} < l\Delta$. We then obtain $\pi_{h_a s}(e_a^{k+1})$ by linearly interpolating travel times at the beginning and end of the lth time interval. In other words,

$$\pi_{h_a s}[t + \tau_a(t)] = \pi_{h_a s}(e_a^{k+1}) = \lambda_a^{3,k,l}(u) \cdot \pi_{h_a s}^{l-1} + [1 - \lambda_a^{3,k,l}(u)] \cdot \pi_{h_a s}^l. \tag{19}$$

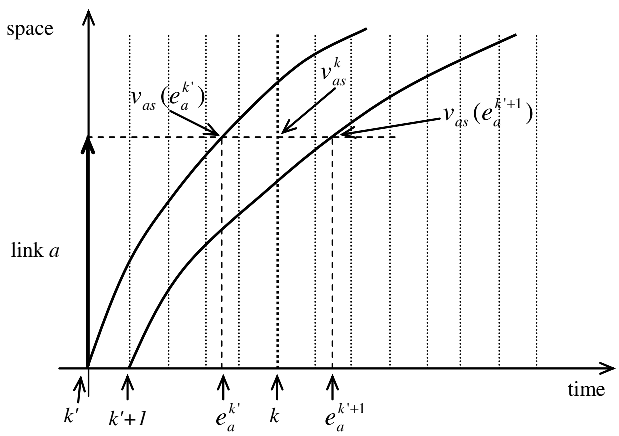

Fig. 1. Discretization of flow propagation constraints.

Here we define

$$\lambda_a^{3,k,l}(u) = \frac{l\Delta - e_a^{k+1}(u)}{\Delta} = \left(l - k - \frac{\tau_a^k(u)}{\Delta}\right) \quad \forall a,k,l;\ (l-1) \leqslant e_a^{k+1}(u)/\Delta < l. \tag{20}$$

Some observations thus follow. First, all $\lambda^1 = (\lambda_a^{1,k'})_{\forall a,k'}, \lambda^2 = (\lambda_a^{2,k',k})_{\forall a,k',k;e_a^{k'} \leqslant (k-1)\Delta < e_a^{k'+1}}, \lambda^3 = (\lambda_a^{3,k,l})_{a,k,l;(l-1)\leqslant e_a^{k+1}/\Delta < l}$ and $e = (e_a^k)_{\forall a,k}$ are functions of τ which is itself a function of the disaggregated link inflow vector u. λ^1 and λ^2 are defined to discretize the exact flow propagation constraint, while λ^3 is used to discretize the minimum travel time from a node to a destination node. Furthermore, since we assume the link travel time defined in (6) satisfies FIFO, $\lambda_a^{1,k'} > 0\ \forall a,k'$. From the definition of λ^2 and λ^3, we can also observe that $0 < \lambda_a^{2,k',k} \leqslant 1\ \forall a,k',k;\ e_a^{k'} \leqslant (k-1)\Delta < e_a^{k'+1}$ and $0 < \lambda_a^{3,k,l} \leqslant 1\ \forall a,k,l;\ l-1 \leqslant e_a^{k+1}/\Delta < l$.

Finally substituting (19) to (12a), we have the following discretized DUE model with exact flow propagations:

$$0 \leqslant u_{as}^k \perp \{\tau_a^k(u) + \sum_{l-1\leqslant e_a^{k+1}(u)/\Delta<l} \lambda_a^{3,k,l}(u)\cdot\pi_{h_as}^{l-1} + [1-\lambda_a^{3,k,l}(u)]\cdot\pi_{h_as}^{l} - \pi_{l_as}^k\} \geqslant 0 \quad \forall a,s,k, \tag{21a}$$

$$\sum_{a\in A(i)} u_{as}^k = d_{is}^k + \sum_{a\in B(i)} \sum_{k':e_a^{k'}(u)\leqslant(k-1)\Delta<e_a^{k'+1}(u)} [\lambda_a^{1,k'}(u)\cdot\lambda_a^{2,k',k}(u)\cdot u_{as}^{k'} + \lambda_a^{1,k'+1}(u)\cdot(1-\lambda_a^{2,k',k}(u))\cdot u_{as}^{k'+1}] \quad \forall i,s,\ i\neq s,k, \tag{21b}$$

$$\pi_{is}^k \geqslant 0 \quad \forall i,s,\ i\neq s,k. \tag{21c}$$

Note that (21) is an MiCP with side constraints (21c). Also since u is the defining variable, $e_a^k(u)$ will change as u does. This implies that the summation terms in the right hand sides of both (21a) and (21b) are not fixed; rather, they change as u does. In this sense, model (21) is not in a close-form. From the above discretization process, one can also see that the flow propagation constraint is discretized following the scheme in Astarita (1996) and thus satisfies the exact flow propagation constraints. On the other hand, the route choice condition (21a) is discretized by associating link inflows at the beginning of time intervals to costs (travel times) at the end of time intervals. Therefore, the discretized formulation (21) is a predictive DUE model with exact flow propagations.

We next show that (21) has an equivalent NCP formulation.

Theorem 2. *If the link travel time function* $\tau_a^k(u)$ *is positive for* $\forall a,k$ *and* $u \geqslant 0$, *then model* (21) *is equivalent to the following NCP model*: *find*(u,π) *such that*

$$0 \leqslant u_{as}^k \perp \left\{\tau_a^k(u) + \sum_{l-1\leqslant e_a^{k+1}(u)/\Delta<l} \lambda_a^{3,k,l}(u)\cdot\pi_{h_as}^{l-1} + [1-\lambda_a^{3,k,l}(u)]\cdot\pi_{h_as}^{l} - \pi_{l_as}^k\right\} \geqslant 0 \quad \forall a,s,k', \tag{22a}$$

$$0 \leqslant \pi_{is}^k \perp \left(\sum_{a\in A(i)} u_{as}^k - d_{is}^k - \sum_{a\in B(i)} \sum_{k':e_a^{k'}(u)\leqslant(k-1)\Delta<e_a^{k'+1}(u)} [\lambda_a^{1,k'}(u)\cdot\lambda_a^{2,k',k}(u)\cdot u_{as}^{k'} + \lambda_a^{1,k'+1}(u)\cdot(1-\lambda_a^{2,k',k}(u))\cdot u_{as}^{k'+1}]\right) \geqslant 0 \quad \forall i,s,\ i\neq s,k'. \tag{22b}$$

Proof. We need to prove (a) if (u,π) solves (21), then it is also a solution to (22); and (b) if (u,π) solves (22), then it must also solves (21). □

Firstly, (***a***) is straightforward. If (u,π) solves (21), then it is obvious that (u,π) also solves (22).

In order to prove (***b***), suppose (u,π) solves (22). Since (u,π) is given here, $e_a^k(u)$ is fixed for $\forall a,k$. This implies that both (21) and (22) become close-formed. For the sake of contradiction, assume (u,π) is not a solution of (21). Then we must have $i,s,i\neq s$ at some time interval k, such that $\sum_{a\in A(i)} u_{as}^k - d_{is}^k - \sum_{a\in B(i)}\sum_{k':e_a^{k'}(u)\leqslant(k-1)\Delta<e_a^{k'+1}(u)}[\lambda_a^{1,k'}(u)\cdot\lambda_a^{2,k',k}(u)\cdot u_{as}^{k'} + \lambda_a^{1,k'+1}(u)\cdot(1-\lambda_a^{2,k',k}(u))\cdot u_{as}^{k'+1}] > 0$. Due to (22b), we have $\pi_{is}^{k}=0$. This implies $\pi_{l_as}^{k}=\pi_{is}^{k}=0$ for any link $a\in A(i)$. Meanwhile, since $\lambda^1>0$, $0<\lambda^2\leqslant 1$, $u_{as}^{k'}\geqslant 0$, $u_{as}^{k'+1}\geqslant 0$, and $d_{is}^{k}\geqslant 0$, we have $\sum_{a\in A(i)}u_{as}^{k}>d_{is}^{k}\geqslant 0$. Thus, we must have at least one link $a\in A(i)$ at time interval k so that $u_{as}^{k}>0$. Then, we have $\tau_a^k(u)+\sum_{l-1\leqslant e_a^{k+1}(u)/\Delta<l}\lambda_a^{3,k,l}(u)\cdot\pi_{h_as}^{l-1}+[1-\lambda_a^{3,k,l}(u)]\cdot\pi_{h_as}^{l}-\pi_{l_as}^{k}=0$ due to (22a). This means $\tau_a^k(u)+\sum_{l-1\leqslant e_a^{k+1}(u)/\Delta<l}\lambda_a^{3,k,l}(u)\cdot\pi_{h_as}^{l-1}+[1-\lambda_a^{3,k,l}(u)]\cdot\pi_{h_as}^{l}=0$ since we have proved $\pi_{l_as}^{k}=0$. Nevertheless, $\tau_a^k(u)>0$, $\pi_{h_as}^{l-1}\geqslant 0$, $\pi_{h_as}^{l}\geqslant 0$, and $0<\lambda^3\leqslant 1$, this is a contradiction.

Note that in NCP (22), u is the defining variable, and λ^1, λ^2, λ^3, e and τ are functions defined on u. However, (22) is not in a close-form due to the same reason for model (21). We further define two vector functions

$$F_u(u,\pi)=\left(\tau_a^k(u)+\sum_{l-1\leqslant e_a^{k+1}(u)/\Delta<l}\lambda_a^{3,k,l}(u)\cdot\pi_{h_as}^{l-1}+[1-\lambda_a^{3,k,l}(u)]\cdot\pi_{h_as}^{l}-\pi_{l_as}^{k}\right)_{\forall a,s,k}, \tag{23}$$

$$F_\pi(u,\pi)=\left(\sum_{a\in A(i)}u_{as}^{k}-d_{is}^{k}-\sum_{a\in B(i)}\sum_{k':e_a^{k'}(u)\leqslant(k-1)\Delta<e_a^{k'+1}(u)}[\lambda_a^{1,k'}(u)\cdot\lambda_a^{2,k',k}(u)\cdot u_{as}^{k'}\right.$$
$$\left.+\lambda_a^{1,k'+1}(u)\cdot(1-\lambda_a^{2,k',k}(u))\cdot u_{as}^{k'+1}]\right)_{\forall i,s,i\neq s,k}. \tag{24}$$

We can then compactly express (22) as follows:

$$\begin{cases}0\leqslant u\perp F_u(u,\pi)\geqslant 0,\\ 0\leqslant\pi\perp F_\pi(u,\pi)\geqslant 0.\end{cases} \tag{25}$$

We denote model (22) or (25) as *DUENCP* thereafter in this paper.

3.2. Solution existence and compactness condition

This section establishes the solution existence condition for *DUENCP*. For this purpose, we need a solution existence condition for VIs, as stated in Lemma 1. This lemma can be found in Facchinei and Pang (2003) as Corollary 2.2.5. So we list it here without proof.

Lemma 1. *Let set $K\subseteq R^n$ be compact (closed and bounded) and convex and let function $F:K\to R^n$ be continuous. Then the solution set of VI defined by K and F is non-empty and compact.*

The solution existence condition of *DUENCP* can be summarized in Theorem 3, which extends the solution existence result for the NCP based static UE model in Facchinei and Pang (2003).

Theorem 3. *Suppose (a) the link travel time function $\tau_a^k(u)$ is positive for $\forall a,k$ and finite for any finite u*[5]*, (b) the given OD demand d_{is}^k is non-negative and bounded from above for $\forall i,s,i\neq s,k$, (c) λ^1 is bounded from above, and (d) $F_u(u,\pi)$ and $F_\pi(u,\pi)$ are continuous with respect to (u,π). Then the solution set of DUENCP is non-empty and compact.*

Proof. First of all, choose three scalars as follows:

$$\alpha_d>\max_{\forall s}\max_{\forall i,i\neq s}\max_{\forall k}d_{is}^{k}, \tag{26a}$$

$$\alpha_u>\max_{\forall s}\max_{\forall i,i\neq s}\max_{\forall k}$$
$$\max_{\forall u}\left(d_{is}^{k}+\sum_{a\in B(i)}\sum_{k':e_a^{k'}(u)\leqslant(k-1)\Delta<e_a^{k'+1}(u)}[\lambda_a^{1,k'}(u)\cdot\lambda_a^{2,k',k}(u)\cdot u_{as}^{k'}+\lambda_a^{1,k'+1}(u)\cdot(1-\lambda_a^{2,k',k}(u))\cdot u_{as}^{k'+1}]\right), \tag{26b}$$

[5] Another way to state condition (a) is to require that τ is coercive with respect to u, i.e., as $u\to+\infty$, we have $\tau\to+\infty$ as well.

$$\alpha_\pi > \max_{\forall s} \max_{\forall a} \max_{\forall k} \max_{\forall u,\pi} \left(\tau_a^k(u) + \sum_{l-1 \leqslant e_a^{k+1}(u)/\Delta < l} \lambda_a^{3,k,l}(u) \cdot \pi_{h_a s}^{l-1} + [1 - \lambda_a^{3,k,l}(u)] \cdot \pi_{h_a s}^{l} \right) \tag{26c}$$

α_d exists due to assumption (b), which implies that the inflow vector u is finite. Based on assumption (a), $\tau_a^k(u)$ is finite for $\forall a,k$, which implies that the vector π is finite as well. Therefore, α_u and α_π also exist because of assumptions (c) and the fact that $0 < \lambda_a^{2,k',k} \leqslant 1 \quad \forall a, k', k; e_a^{k'}(u) \leqslant (k-1)\Delta < e_a^{k'+1}(u)$ and $0 < \lambda_a^{3,k,l} \leqslant 1 \; \forall a, k, l; l-1 \leqslant e_a^{k+1}(u)/\Delta < l$. For the same reason, they are all positive. We then define a set $E = \{y = (u, \pi) \geqslant 0 | u \leqslant \alpha_u \mathbf{1}, \pi \leqslant \alpha_\pi \mathbf{1}\}$ and a function $F = (F_u(u,\pi), F_\pi(u,\pi))$. Here **1** is a vector with all components being 1 with the proper dimension. Set E and function F constitute a VI which tries to *find* $y^* = (u^*, \pi^*) \in E$ *such that*

$$F(y^*)^{\mathrm{T}}(y - y^*) \geqslant 0 \quad \forall y \in E. \tag{27}$$

Since E is compact and convex and further F is continuous with respect to (u, π), according to Lemma 1, the above VI has a non-empty and compact solution set, denoted as $\Gamma = \{y^* = (u^*, \pi^*)\}$. We next show that the solution set of *DUENCP*, denoted as Ξ, coincides with Γ.

First of all, for each solution $y^* = (u^*, \pi^*) \in \Gamma$, since E is a polyhedral set, there must exist multipliers γ and η such that (Facchinei and Pang, 2003):

$$\begin{cases} 0 \leqslant u^* \perp F_u(u^*, \pi^*) + \gamma \geqslant 0 & \text{(a)} \\ 0 \leqslant \pi^* \perp F_\pi(u^*, \pi^*) + \eta \geqslant 0 & \text{(b)} \\ 0 \leqslant \gamma \perp \alpha_u \mathbf{1} - u^* \geqslant 0 & \text{(c)} \\ 0 \leqslant \eta \perp \alpha_\pi \mathbf{1} - \pi^* \geqslant 0 & \text{(d)} \end{cases} \tag{28}$$

In order to prove $y^* = (u^*, \pi^*)$ solves *DUENCP*, we need to show that $\gamma = 0$ and $\eta = 0$. Suppose $\gamma_{bs}^k > 0$ for some b, s, k. We then must have $(u^*)_{bs}^k = \alpha_u > 0$ from (28c). This implies $\rho_1 = \tau_b^k(u^*) + \sum_{l-1 \leqslant e_b^{k+1}(u^*)/\Delta < l} \lambda_b^{3,k,l}(u^*) \cdot (\pi^*)_{h_b s}^{l-1} + [1 - \lambda_b^{3,k,l}(u^*)] \cdot (\pi^*)_{h_b s}^{l} - (\pi^*)_{l_b s}^k + \gamma_{bs}^k = 0$ due to (28a). This also means that for the tail node of link b, denoted as node i, $\sum_{a \in A(i)} (u^*)_{as}^k - d_{is}^k - \sum_{a \in B(i)} \sum_{k': e_a^{k'}(u^*) \leqslant (k-1)\Delta < e_a^{k'+1}(u^*)} [\lambda_a^{1,k'}(u^*) \cdot \lambda_a^{2,k',k}(u^*) \cdot (u^*)_{as}^{k'} + \lambda_a^{1,k'+1}(u^*) \cdot (1 - \lambda_a^{2,k',k}(u^*)) \cdot (u^*)_{as}^{k'+1}] + \eta_{is}^k \geqslant \alpha_u - (d_{is}^k + \sum_{a \in B(i)} \sum_{k': e_a^{k'}(u^*) \leqslant (k-1)\Delta < e_a^{k'+1}(u^*)} [\lambda_a^{1,k'}(u^*) \cdot \lambda_a^{2,k',k}(u^*) \cdot (u^*)_{as}^{k'} + \lambda_a^{1,k'+1}(u^*) \cdot (1 - \lambda_a^{2,k',k}(u^*)) \cdot (u^*)_{as}^{k'+1}]) + \eta_{is}^k > 0$ based on (26b) and the fact that $\sum_{a \in A(i)} (u^*)_{as}^k \geqslant (u^*)_{bs}^k = \alpha_u$. Hence $(\pi^*)_{is}^k = (\pi^*)_{l_b s}^k = 0$ due to (28b). We have $0 < \tau_b^k(u^*) \leqslant \rho_1 = \tau_b^k(u^*) + \sum_{l-1 \leqslant e_b^{k+1}(u^*)/\Delta < l} \lambda_b^{3,k,l}(u^*) \cdot (\pi^*)_{h_b s}^{l-1} + [1 - \lambda_b^{3,k,l}(u^*)] \cdot (\pi^*)_{h_b s}^{l} + \gamma_{bs}^k = 0$ which is a contradiction.

Similarly if $\eta_{is}^k > 0$ for some i, s, k, we have $(\pi^*)_{is}^k = \alpha_\pi > 0$ based on (28d). According to (28a), we have $\rho_2 = \tau_a^k(u^*) + \sum_{l-1 \leqslant e_a^{k+1}(u^*)/\Delta < l} \lambda_a^{3,k,l}(u^*) \cdot (\pi^*)_{h_a s}^{l-1} + [1 - \lambda_a^{3,k,l}(u^*)] \cdot (\pi^*)_{h_a s}^{l} - (\pi^*)_{l_a s}^k + \gamma_{as}^k \geqslant 0$ for any link $a \in A(i)$. However, since $(\pi^*)_{l_a s}^k = (\pi^*)_{is}^k = \alpha_\pi$ and $\gamma_{as}^k = 0$, $\rho_2 = \tau_a^k(u^*) + \sum_{l-1 \leqslant e_a^{k+1}(u^*)/\Delta < l} \lambda_a^{3,k,l}(u^*) \cdot (\pi^*)_{h_a s}^{l-1} + [1 - \lambda_a^{3,k,l}(u^*)] \cdot (\pi^*)_{h_a s}^{l} - \alpha_\pi < 0$ due to the definition in (26c). This is a contradiction.

So far we have proved that $\Gamma \subseteq \Xi$. In order to prove $\Xi \subseteq \Gamma$, we notice that for any solution $y^* = (u^*, \pi^*) \in \Xi$, (28) has to be satisfied with the two set of multipliers $\gamma = 0$ and $\eta = 0$. Due to the equivalence between (28) and VI (27), y^* must also be a solution to VI (27). This implies $y^* \in \Gamma$ and further $\Xi \subseteq \Gamma$. Therefore, $\Xi = \Gamma$. Since Γ is nonempty and compact, so is Ξ, the solution set of *DUENCP*. □

4. Solution algorithm

The solution algorithm in this section is based on the fact that for a given inflow vector $\bar{u}$ (denoted as the base inflow), the link travel time vector τ will be fixed by a dynamic network loading procedure. Hence, all λ^1, λ^2, λ^3 and e will be fixed as well. The original non-close-form NCP model (22) will then become close-formed as in (29):

$$0 \leqslant u_{as}^k \perp \left\{ \tau_a^k(u) + \sum_{l-1 \leqslant e_a^{k+1}(\bar{u})/\Delta < l} \lambda_a^{3,k,l}(\bar{u}) \cdot \pi_{h_a s}^{l-1} + [1 - \lambda_a^{3,k,l}(\bar{u})] \cdot \pi_{h_a s}^{l} - \pi_{l_a s}^{k} \right\} \geqslant 0 \quad \forall a, s, k, \tag{29a}$$

$$0 \leqslant \pi_{is}^k \perp \left(\sum_{a \in A(i)} u_{as}^k - d_{is}^k - \sum_{a \in B(i)} \sum_{k': e_a^{k'}(\bar{u}) \leqslant (k-1)\Delta < e_a^{k'+1}(\bar{u})} [\lambda_a^{1,k'}(\bar{u}) \cdot \lambda_a^{2,k',k}(\bar{u}) \cdot u_{as}^{k'} + \lambda_a^{1,k'+1}(\bar{u}) \cdot (1 - \lambda_a^{2,k',k}(\bar{u})) \cdot u_{as}^{k'+1}] \right) \geqslant 0 \quad \forall i, s,\ i \neq s, k'. \tag{29b}$$

Note the only difference between model (22) and (29) is that λ^1, λ^2, λ^3 and e in (29) are evaluated at the base inflow $\bar{u}$, while in (22) they are functions of u. In other words, (29) is a close-form NCP which is much easier to solve. We denote (29) as the "relaxed" NCP of (22).

The above observation outlines an iterative algorithm to solve model (22). It is heuristic in the sense that the convergence cannot be established using regularity conditions. This is mainly due to the fact that (22) cannot be expressed in a close-form. The algorithm is listed as below.

Algorithm DUE

Step 1. Initialization. Assign an initial feasible base inflow $(\bar{u})^0$.
Step 2. Main Loop. Set $n = 0$.
Step 2.1. Construct current relaxed NCP at $(\bar{u})^n$ by a dynamic network loading procedure (see Section 4.1).
Step 2.2. Solve the relaxed NCP and denote its solution $(\tilde{u})^n, (\tilde{\pi})^n$ as a "candidate" solution (see Section 4.2).
Step 2.3. Convergence Test. If certain convergence criterion is satisfied at the candidate solution, go to Step 3; else, go to Step 2.4.
Step 2.4. Update and Move. Set $(\bar{u})^{n+1} = (\bar{u})^n + \theta \cdot ((\tilde{u})^n - (\bar{u})^n)$, $n = n + 1$ and go to Step 2.1.
Step 3. Find an optimal solution $(\tilde{u})^n, (\tilde{\pi})^n$.

In the above algorithm, $0 < \theta \leqslant 1$ is a pre-defined step size. In addition, there are several gap functions that can be used for the convergence test in Step 2.3. The most commonly used one is the difference of link inflows between two consecutive iterations, i.e.,

$$Gap\_U = |u^n - u^{n+1}|_2 \leqslant \varepsilon_1. \tag{30}$$

Here, $|\cdot|_2$ denotes the 2-norm. A more rigorous way is to check whether the complementarity condition in Eq. (1) holds

$$Gap\_DUE = u^{\mathrm{T}} F_u(u, \pi) \leqslant \varepsilon_2. \tag{31}$$

In (30) and (31), ε_1 and ε_2 are chosen as small positive scalars.

The next two sections will further discuss the dynamic network loading procedure and the method for solving the relaxed NCP model (29).

4.1. Link based dynamic network loading

The dynamic network loading procedure in *Algorithm DUE* is link based. For a given base inflow $\bar{u}$, it is to load/propagate $\bar{u}_{as}^k$ to every link a at any time interval k towards a given destination s. This will generate other traffic measurements such as $\bar{v}$, $\bar{x}$, $\tau(\bar{u})$, $e(\bar{u})$. In turn, $\lambda^1(\bar{u})$, $\lambda^2(\bar{u})$, $\lambda^3(\bar{u})$ can also be determined. In Carey and Ge (2004) and Nie and Zhang (2005), this loading procedure is called the "solution algorithm for link travel time model." In this paper, we intend to call it a "loading procedure" since it indeed generates not only travel times but also link flows and exit flows.

To be consistent with the discretization scheme in Section 3.1, we need to adopt the loading process in Astarita (1996). The original algorithm by Astarita (1996), however, may have numerical problems (Nie and Zhang, 2005). In this paper, we apply the improved algorithm by Nie and Zhang (2005) which performs the loading based on cumulative departure curves. Furthermore, in order to construct the relaxed NCP model (29), one needs to track the relation between $\bar{u}_{as}^{k'}$ and $\bar{v}_{as}^{k}$ for any a and s and appropriate (k',k) pairs. Due to the fact that inflows to any destination s will experience the same travel time at each entrance time k, we can first use a three dimensional matrix $V(a,k',k)$ to represent this relation. In particular, $V(a,k',k)=\rho$ means that a proportion of ρ ($0\leqslant\rho\leqslant 1$) of the inflows $\bar{u}_{as}^{k'}\Delta$ will exit the link a at time k (i.e., become part of $\bar{v}_{as}^{k}\Delta$). Note that the matrix V represents λ^1 and λ^2 in (29), and λ^3 can be computed using Eq. (20). Consequently, NCP (29) can be constructed using V and λ^3.

It turns out that $V(a,k',k)$ can be obtained via the *Algorithm D2* in Nie and Zhang (2005) after minor modifications. Details are omitted here and one can refer to Nie and Zhang (2005) for more discussions.

4.2. Solving the relaxed NCP

Since the relaxed model (29) is a well-defined NCP with continuous and close-form defining functions, it can be readily solved using existing solution techniques. Facchinei and Pang (2003) provided a comprehensive review of solution methods for NCPs. In particular, projection based methods play a central role in solving NCPs because calculating the projection on the non-negative orthant, the defining set of an NCP, is much more efficient compared with that on a general convex set. Based on this observation, Dirkse and Ferris (1995) developed a path search algorithm. Under certain regularity conditions, the algorithm was proved to be globally convergent with quadratic convergence rate (near the solution). The algorithm was later evolved to the PATH solver which is now available in GAMS (general algebraic modeling system, see Brooke et al., 1998). In this paper, we directly adopt the PATH solver which we found is effective to solve NCP (29). For detailed descriptions of the solver, one can refer to Ferris and Munson (1998). Using the PATH solver requires developing GAMS codes for the relaxed NCP (29), and the details are omitted here.

5. Numerical examples

In this section, numerical examples are provided to demonstrate the model and solution algorithm proposed in the paper. We start with the link travel time function that we actually used.

5.1. Link travel time function

In this paper, we choose the following linear form for the link travel time function:

$$\tau_a^k=\alpha_a(1+\beta_a^x x_a^{k+1}), \tag{32}$$

where α_a is the free flow travel time for link a, and $\beta_a^x>0$ is a constant. We can easily observe that (32) satisfies condition (a) in Theorem 3. This is because for a finite vector u, the total link flow vector x^Δ is also finite, which guarantees that $\tau=(\tau_a^k)_{\forall a,k}$ calculated using (32) is finite as well.

Note that since τ_a^k is the travel time of link a at the end of time interval k (or at the beginning of time interval $k+1$), it is defined as a linear function of the link flow at the beginning of time interval $k+1$, x_a^{k+1}. Since x_a^{k+1} includes u_a^k (the inflow to link a at time interval k), in the route choice condition (22a), how to assign inflow to link a at some time interval will then be impacted by the cost incurred by the inflow itself. This further explains why the DUE model proposed in this paper is *predictive*. Note that this inflow and cost association only applies to the route choice condition. For the flow propagation constraint, τ_a^{k-1}, i.e., the link travel time at the beginning of a time interval is used instead as shown in equation (15). Therefore, FIFO is not violated.

Heydecker and Verlander (1999) proposed the predictive DUE concept and provided rational from a route choice perspective why it is superior to the reactive DUE model. Comparisons were also conducted to illustrate that reactive DUE models tend to produce solutions that have abrupt fluctuations, while predictive DUE solutions are much smoother and thus more desirable. In this section, we further show using the NCP model developed in this paper why algorithmically the predictive DUE is more preferable. In fact, the predictive model makes the solution process for the relaxed NCP (29) numerically easier and more stable. To see this, we first rewrite NCP (29) in a matrix notation as

$$\begin{cases} 0 \leqslant u_s \perp [\tau(u) + \Omega_s \pi_s] \geqslant 0, \\ 0 \leqslant \pi_s \perp [\Lambda_s u_s - d_s] \geqslant 0, \end{cases} \quad \forall s \in S \tag{33}$$

with Ω_s and Λ_s are fixed matrices computed using the base inflow. Here $u_s = (u^k_{as})_{\forall a,k}$, $\pi_s = (\pi^k_{is})_{\forall i,k;i \neq s}$, and $d_s = (d^k_{is})_{\forall i,k;i \neq s}$ denote destination based vectors. If we compute the Jacobian matrix of (33), denoted as M, we will have

$$M = \begin{array}{c} \begin{array}{cccccc} \pi_1 & \cdots & \pi_{|S|} & u_1 & \cdots & u_{|S|} \end{array} \\ \left[\begin{array}{cccccc} 0 & \cdots & 0 & \Lambda_1 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \Lambda_{|S|} \\ \Omega_1 & 0 & 0 & \partial\tau/\partial u_1 & \cdots & \partial\tau/\partial u_{|S|} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \Omega_{|S|} & \partial\tau/\partial u_1 & \cdots & \partial\tau/\partial u_{|S|} \end{array} \right] \end{array} \begin{array}{c} \pi_1 \\ \cdots \\ \pi_{|S|} \\ u_1 \\ \cdots \\ u_{|S|} \end{array}. \tag{34}$$

In (34), each variable on the right side of the matrix indicates that the corresponding (block) row is computed by taking the partial derivative of the function perpendicular to the variable; each variable on the top, whereas, indicates that the (block) column is computed by taking the partial derivative of each function to the particular variable.

For the predictive DUE model in this paper, we have $\partial\tau^k_a/\partial u^k_{as} > 0$ since according to (32), τ^k_a (linearly) depends on u^k_{as}. This means the diagonal elements of $\partial\tau/\partial u_s$ is positive. For reactive DUE models, however, τ^k_a will be the travel time of link a at the *beginning* of time interval k. In this case, we need to replace x^{k+1}_a in (32) with x^k_a. We will then have $\partial\tau^k_a/\partial u^k_{as} = 0$ since x^k_a includes inflows up to time $(k-1)$ but u^k_a. This means the diagonal elements of $\partial\tau/\partial u_s$ will be all zero, implying that the diagonal elements of M are all zero for reactive DUE. Such a matrix can be proved to be not positive definite and thus may cause problems to solve the relaxed NCP.

It is easy to check that the Jacobian matrix of the proposed predictive DUE model is also not positive definite due to the zero block on the top-left of M in (34). However, by associating link inflows at the beginning of time intervals to travel times at the end of time intervals, predictive DUE models actually "add" positive values to the diagonal of the Jacobian matrix M. This is generally helpful to stabilize the solution process of the relaxed NCP model. In the mathematical programming literature, reinforcing the diagonal matrix is called *regularization* that has been shown to be effective for solving NCPs and VIs (see Facchinei and Pang, 2003, Chapter 12).

It is also worth noting, from (33) and (34), that the link-node NCP model proposed in this paper has a very special structure such that it can be easily decomposed to individual destinations. The interactions between variables of different destinations only exist in the link travel time vector $\tau(u)$. This illustrates the potential advantages of the link-node NCP model from an algorithmic point of view. Namely, the special structure of the link-node model makes it easier to apply the destination based decomposition scheme for solving DUE problems. Such decomposition schemes have been shown previously to be effective for obtaining static user equilibria (Bar-Gera, 1999; Ban et al., 2006a).

5.2. Test problem and result discussion

Numerical examples are provided on a hypothetical network shown in Fig. 2, denoted as the D3 network. In the DUE literature, this network was also used by other researchers (Chen and Hsuen, 1998; Chen, 1999). In this paper, we use slightly different specifications for the D3 network, as shown in Table 1. The network has two origins (Nodes 1 and 2) and one destination (Node 3). Further, the length of each time interval is set as $\Delta = 0.25$ min (15 s).

To simulate the fluctuation of traffic demands during peak hours, we employ a parabolic-shaped curve to represent the OD demand between each OD pair. In particular, the demand rate between any OD pair rs is assumed to be calculated through Eq. (35):

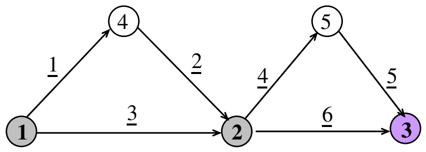

Fig. 2. Test network.

Table 1
Configuration of the test network

Link	α_a (min)	β_a^x (min/veh)
1	1.2	0.01
2	1.2	0.01
3	2.16	0.0056
4	1.2	0.01
5	1.2	0.01
6	2.4	0.005

$$d^{rs}(k) = 40 + 120 * \left(1 - \left(\frac{k - K/2}{K/2}\right)^2\right) \quad \forall 1 \leqslant k \leqslant K, \ \forall r \in R, \ s \in S, \tag{35}$$

where K denotes the total number of intervals during which OD trips will be generated.

For the case study, we set $K = 120$ which is equivalent to 30 minutes. *Algorithm DUE* can solve successfully the proposed model. Fig. 3 first depicts the convergence performance of the algorithm. In this figure, although the units for the two gaps are different as indicated, we plot them together in order to illustrate their slightly different performances. We can easily observe that both gaps decrease monotonically. *Gap_DUE* decreases faster at the very beginning, but stabilizes when close to 10^{-4} vehicle; however, *Gap_U* decreases approaching 10^{-6} veh/min. After 21 iterations, the absolute difference of link inflows between two consecutive iterations is close to 10^{-6} veh/min. This should be accurate enough for most transportation related applications. Note that *Gap_DUE* and *Gap_U* are represented by the absolute value of number of vehicles and veh/min, respectively. If certain criteria based on relative values are used, the performance of *Algorithm DUE* may look more appealing. For example, we can define a "relative gap" using *Gap_DUE* similarly as that in Bar-Gera (1999):

$$Rel\_Gap\_DUE = \frac{Gap\_DUE}{u^{\mathrm{T}}\tau} = \frac{u^{\mathrm{T}}F_u(u,\pi)}{u^{\mathrm{T}}\tau}. \tag{36}$$

In Eq. (36), $u^T\tau$ is used as the denominator since it is related to the total system travel time.[6] For this particular example, *Rel_Gap_DUE* is close to 10^{-9} after 21 iterations. It is also worth noting that due to the exact solve of each relaxed NCP by the PATH solver (usually to 10^{-6}), *Algorithm DUE* requires much fewer number of major iterations than previous DUE solution methods based on FW.

Before displaying more results of the test problem, we first show the difference of the predictive and reactive DUE solutions. In this paper, we implemented the reactive DUE model by associating link inflows with travel times at the beginning of each time interval. We found that it takes more iterations (25 for the test example as shown in Ban et al., 2006b) for the reactive DUE to converge to the same level of convergence as aforementioned for the predictive DUE. Fig. 4 further depicts the inflow rates to Link 1 ($1 \rightarrow 4$) and 3 ($1 \rightarrow 2$) for both reactive and predictive DUE models. Note that since we have only one destination, the disaggregated and aggregated variables coincide with each other. It is clear from the figure that reactive DUE produces solutions that have abrupt fluctuations, which is not desirable as first noticed in Heydecker and Verlander (1999). The predictive DUE solutions, on the other hand, are much smoother. We can also see that the long-term trends of the solutions from predictive and reactive DUE are very close to each other. Similar observation can be found

[6] Actually $(\Delta u)^T\tau$ is the total system travel time. But in order to make *Rel_Gap_DUE* in (36) unit-less, $u^T\tau$ is used instead since Δ is a constant.

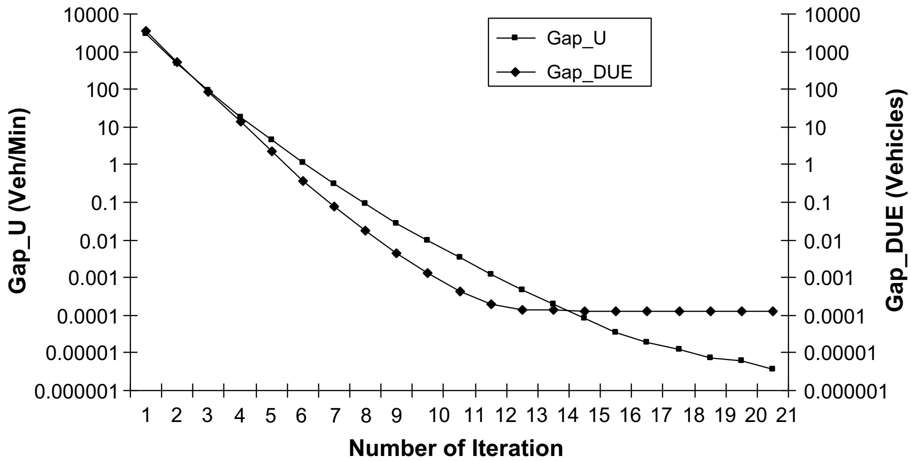

Fig. 3. Convergence of the algorithm.

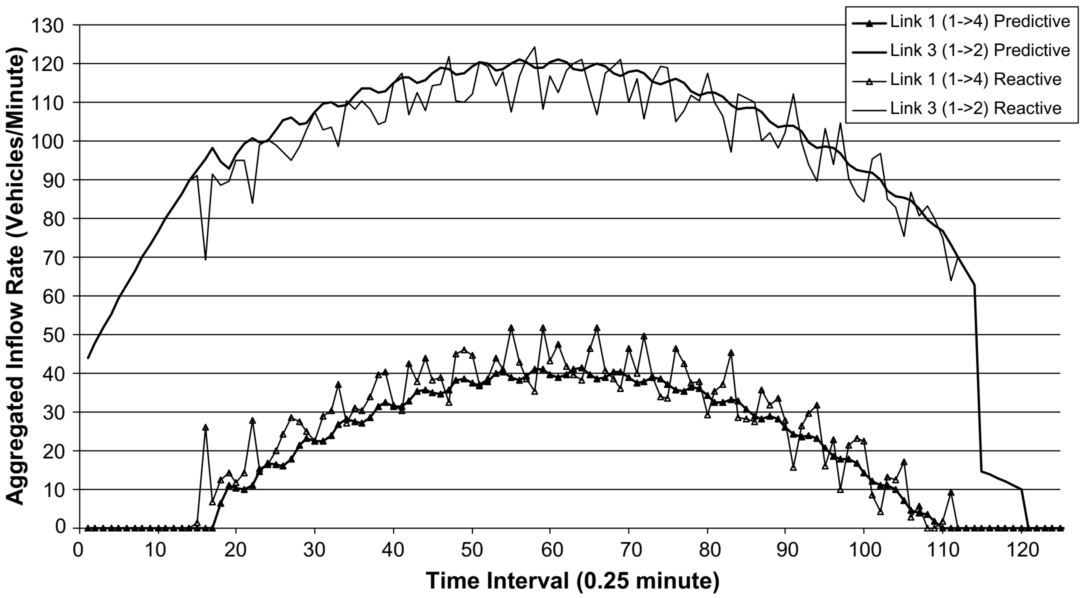

Fig. 4. Predictive and reactive DUE solutions.

for inflows to other links of the network as well. This implies that solutions from predictive DUE are more preferable.

Fig. 5 shows the inflow rate to each link for predictive DUE. We can see that the inflow rates are smooth, except a few discontinuous points. In particular, the inflow rates to Link 1 and 3 have more fluctuations than those to Link 4 and 6. This may be due to the fact that the choice of links at Node 1 will be impacted by the inflows at Node 2, but not vice versa. In Fig. 6, we further display the total link flow (i.e., x^A). We can see that the total link flows are almost continuous everywhere. This is due to the summation effect of total link flows over inflows.

Note that inflows to Link 2 and 5 are exactly the exit flows from Link 1 and 4, respectively. We can observe that the shape of the exit flow is similar to its corresponding inflow with a time lag, but is smoother. For example, in Fig. 5, we indicate (in the left dashed circle) a relatively large change of inflow to Link 4 from $k = 120$ (about 65 veh/min) to $k = 121$ (about 46 veh/min); this change, nevertheless, is smoothed in the exit flow curve (i.e., the inflow curve to Link 5) in later time intervals (from 67 veh/min at $k = 125$ to 53 veh/min at $k = 126$, and then 48 veh/min at $k = 127$). This is indicated in the right dashed circle in Fig. 5. This observation is

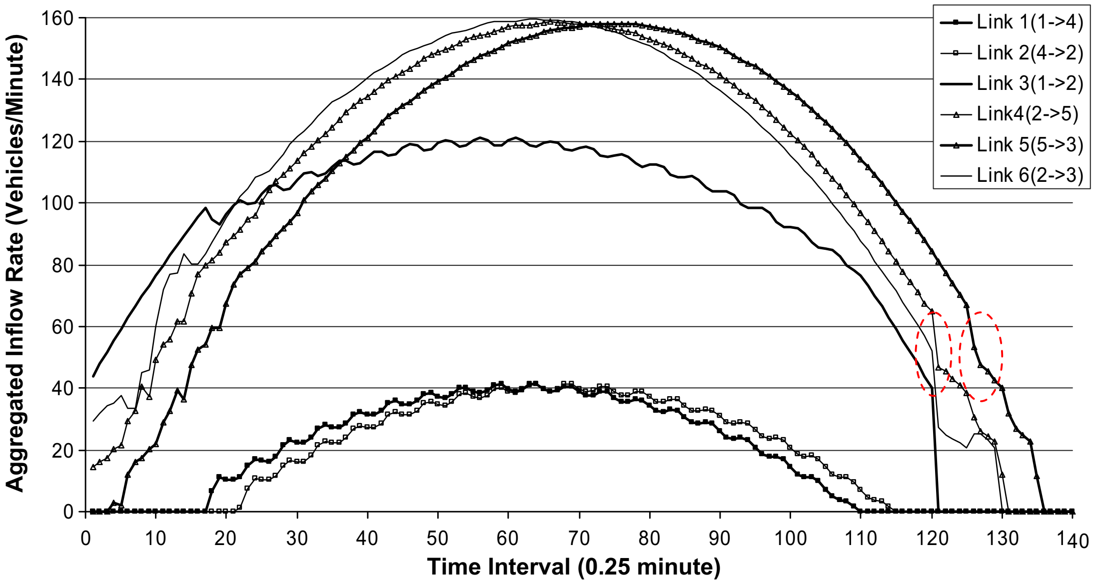

Fig. 5. Inflow rates.

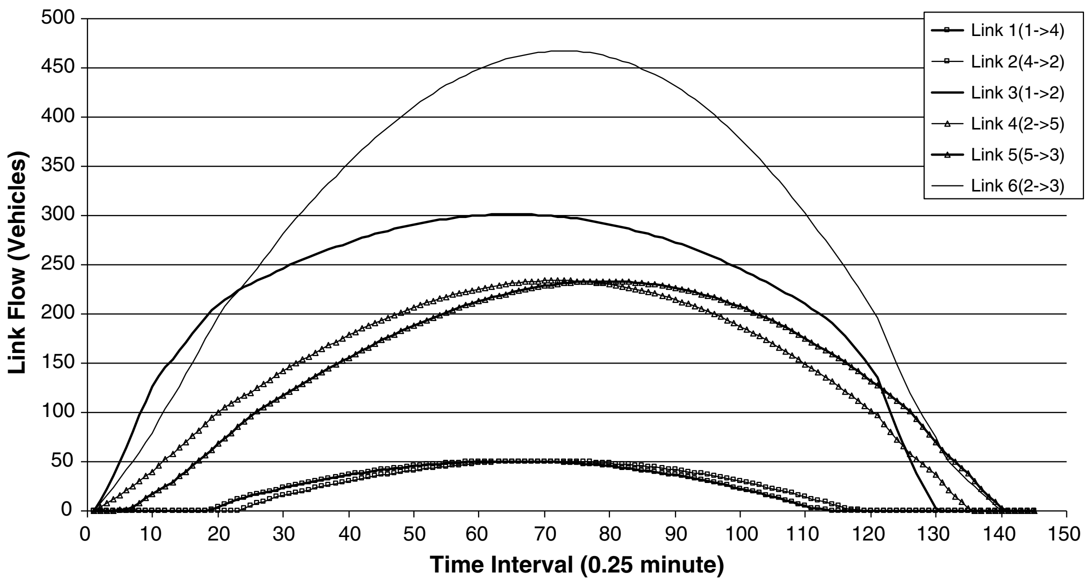

Fig. 6. Total link flows.

consistent with previous studies in terms of the relationship between inflow and exit flow rates of the same link (e.g., Carey and McCartney, 2002).

We depict in Fig. 7 the minimum travel times from Node 1 and 2, respectively, to destination 3 via different links. Here the minimum travel time from the tail node of link a to destination s via link a at (the end of) time k is $\tau_a^k + \pi_{h_a s}[k + \tau_a^k]$ as in the DUE condition (1). This figure shows that at the beginning (roughly $k = 1$–17), the travel time from Node 1 to 3 via Link 1 is higher than that via Link 3. Consequently, all vehicles choose Link 3 during this period, which can be seen from the link inflow curves in Fig. 5. This is also true for the period of $k \geqslant 110$. While from Node 2 to 3, choosing either Link 4 or Link 6 will have equal travel times until $k = 130$. Hence, both Link 4 and 6 are selected up to $k = 130$, after which only Link 4 is selected.

To further illustrate that the obtained results satisfy the DUE condition, we show in Fig. 8 both inflow rates to Link 1 and 3, together with travel times via these two links to the destination (Node 3). Notice that since predictive DUE is adopted in this paper, inflow at any time interval in the figure represents the inflow rate at

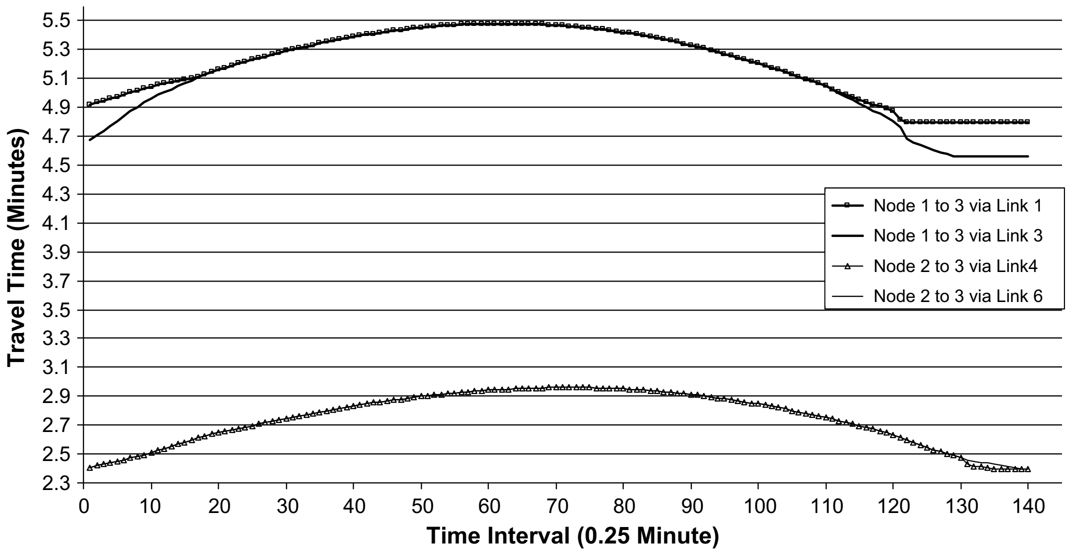

Fig. 7. Minimum travel times.

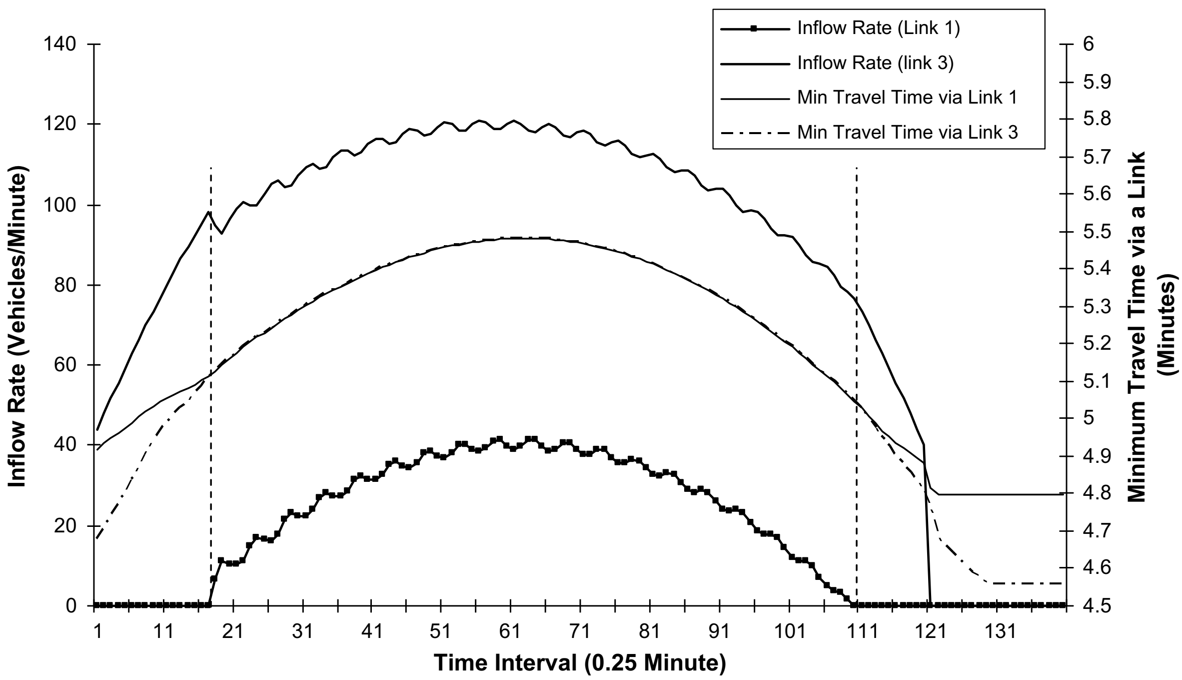

Fig. 8. Illustration of DUE condition.

the beginning of the interval, while travel time curves represent travel times at the end of time intervals. Clearly, the travel time via Link 3 represents the minimum travel time from Node 1 to 3. The two dashed lines indicate the time points where the travel time via Link 1 becomes the same (left dashed line) or larger (right dashed line) than the minimum travel times from Node 1 to 3. The figure depicts that the inflow to a link will be zero whenever the travel time via this link is higher than the minimum travel time from the tail node of the link to the destination. This exactly matches the required DUE condition stated in Section 2.1.

6. Conclusions and future research

We presented a link-node based discrete-time NCP model for the basic DUE problem. The proposed model explicitly captures the exact flow propagation and is also predictive DUE since inflow to a link at the beginning of a time interval is associated with the cost at the end of the interval in the DUE route choice. The solution existence and compactness condition was established under mild assumptions. We also developed an iterative solution algorithm for the proposed model by solving a relaxed NCP in each iteration. Since the relaxed NCP can be solved very accurately using existing solution techniques, the proposed solution algorithm only requires a fairly small number of iterations. The case study demonstrated that the proposed model and solution approach are effective for solving DUE problems. By investigating the Jacobian matrix of the proposed NCP model, we also showed that the predictive DUE model reinforces the diagonal of the Jacobian matrix, which makes it algorithmically more plausible than its reactive counterpart.

For future studies, the proposed NCP model, especially its solution approach, merits further investigations. Especially, certain solution convergence condition may need to be established. Also, the model and solution approach proposed in this paper need to be further tested on large scale DUE problems. For this purpose, the special structure of the proposed link-node model makes it possible to use certain decomposition scheme, e.g., the Gauss–Seidel decomposition, to solve large scale DUE problems. The effectiveness of such decomposition has been shown previously by Bar-Gera for symmetric UE problems (1999), by Ban et al. (2006a) for asymmetric UE problems, and by Ban (2005) for DUE with approximate flow propagation constraints. Applying decomposition techniques for solving large scale DUE problems is being investigated by the authors.

Acknowledgements

The authors would like to thank the associate editor and the two anonymous referees for their insightful comments and helpful suggestions on earlier versions of this paper. This research was partially supported by Air Force Office of Scientific Research Grant FA9550-07-1-0389, and National Science Foundation Grants DMI-0521953, DMS-0427689 and IIS-0511905. The authors remain solely responsible for all the materials presented in this paper.

References

Akamatsu, T., 2001. An efficient algorithm for dynamic traffic equilibrium assignment with queues. Transportation Science 35 (4), 389–404.

Astarita, V., 1996. A continuous time link model for dynamic network loading based on ravel time function. In: Lesort, J.-B. (Ed.), Transportation and Traffic Theory. Pergamon, Oxford, pp. 79–102.

Ban, X., 2005. Quasi-variational inequality formulations and solution approaches for dynamic user equilibria. Ph.D. Thesis, University of Wisconsin-Madison.

Ban, X., Liu, H., Ferris, M.C., 2006a. A link-node based complementarity model and its solution algorithm for asymmetric user equilibria. In: Proceedings of the 85th Transportation Research Board Annual Meeting (CD-ROM).

Ban, X., Liu, H., Ferris, M.C., Ran, B., 2006b. A link-node complementarity model and solution algorithm for dynamic user equilibria with exact flow propagations. In: Proceedings of DTA 2006: The First International Symposium on Dynamic Traffic Assignment, Leeds, UK.

Bar-Gera, H., 1999. Origin-based algorithm for transportation network modeling. Ph.D. Thesis, University of Illinois at Chicago, Chicago, IL.

Billups, S.C., Dirkse, S.P., Ferris, M.C., 1997. A comparison of large scale mixed complementarity problem solvers. Computational Optimization and Applications 7 (1), 3–25.

Bliemer, M.C.J., 2001. Analytical dynamic traffic assignment with interacting user-classes: Theoretical advances and applications using a variational inequality approach. Ph.D. Thesis, Delft University of Technology, The Netherlands.

Bliemer, M.C.J., Bovy, P.H.L., 2003. Quasi-variational inequality formulation of the multiclass dynamic traffic assignment problem. Transportation Research Part B 37 (6), 501–519.

Brooke, A., Kendrick, D., Meeraus, A., et al., 1998. GAMS, A User's Guide. GAMS Development Corporation.

Cao, M., Ferris, M.C., 1996. A pivotal method for affine variational inequalities. Mathematics of Operations Research 21 (1), 44–64.

Carey, M., 1987. Optimal time-varying flows on congested networks. Operations Research 35 (1), 58–69.

Carey, M., Ge, Y.E., 2004. Efficient discretisation for link travel time models. Networks and Spatial Economics 4 (3), 269–290.

Carey, M., McCartney, M., 2002. Behavior of a whole-link travel time model used in dynamic traffic assignment. Transportation Research Part B 36 (1), 83–95.

Carey, M., Ge, Y.E., McCartney, M., 2003. A whole-travel time model with desirable properties. Transportation Science 37 (1), 83–96.

Chen, H.K., 1999. Dynamic Travel Choice Models. Springer-Verlag, New York.

Chen, H.K., Hsuen, C.F., 1998. A model and an algorithm for the dynamic user-optimal route choice problem. Transportation Research Part B 32 (3), 219–234.

Dafermos, S.C., 1980. Traffic equilibrium and variational inequality. Transportation Science 14 (1), 42–54.

Daganzo, C.F., 1994. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. Transportation Research Part B 28 (4), 269–287.

Daganzo, C.F., 1995a. The cell transmission model part II: Network traffic. Transportation Research Part B 29 (2), 79–93.

Daganzo, C.F., 1995b. Properties of link travel time functions under dynamic loads. Transportation Research Part B 29 (2), 95–98.

Dirkse, S.P., Ferris, M.C., 1995. The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems. Optimization Methods and Software 5 (2), 123–156.

Facchinei, F., Pang, J.S., 2003. In: Finite-Dimensional Variational Inequalities and Complementarity Problems, vols. I and II. Springer-Verlag, New York.

Ferris, M.C., Munson, T.S., 1998. PATH 4.6 User Manual. University of Wisconsin-Madison.

Ferris, M.C., Fourer, R., Gay, D.M., 1999. Expressing complementarity problems and communicating them to solvers. SIAM Journal on Optimization 9 (4), 991–1009.

Friesz, T.L., Mookherjee, R., 2006. Solving the dynamic network user equilibrium problem with state-dependent time shifts. Transportation Research Part B 40 (3), 207–229.

Friesz, T.L., Bernstein, D., Smith, T.E., et al., 1993. A variational inequality formulation of the dynamic network user equilibrium problem. Operations Research 41 (1), 179–191.

Friesz, T.L., Bernstein, D., Suo, Z., Tobin, R.L., 2001. Dynamic network user equilibrium with time-dependent time lags. Networks and Spatial Economics 1 (3–4), 319–347.

Han, S., 2000. Dynamic traffic assignment techniques for general road networks. Ph.D. Thesis, University of College London.

Han, S., 2003. Dynamic traffic modeling and dynamic stochastic user equilibrium assignment for general road networks. Transportation Research Part B 37 (3), 225–249.

Han, S., Heydecker, B.G., 2006. Consistent objectives and solution of dynamic user equilibrium models. Transportation Research Part B 40 (1), 16–34.

Heydecker, B.G., Verlander, N., 1999. Calculation of dynamic traffic equilibrium assignments. In: Proceedings of the European Transport Conferences, Seminar F, pp. 79–91.

Lo, H., Szeto, W.Y., 2002. A cell-based variational inequality formulation of the dynamic user optimal assignment problem. Transportation Research Part B 36 (5), 421–443.

Lo, H.K., Ran, B., Hongola, B., 1996. Multiclass dynamic traffic assignment model: formulation and computational experiences. Transportation Research Record 1537, 74–82.

Nie, X.J., Zhang, H.M., 2005. Delay-function-based link models: their properties and computational issues. Transportation Research Part B 39 (8), 729–751.

Peeta, S., Ziliaskopoulos, A.K., 2001. Foundations of dynamic traffic assignment: the past, the present and the future. Networks and Spatial Economics 1 (3–4), 233–265.

Ran, B., Boyce, D.E., 1994. Dynamic urban transportation network models: theory and implications for intelligent vehicle highway systems. In: Lecture Notes in Economics and Mathematical Systems, vol. 417. Springer-Verlag, New York.

Ran, B., Boyce, D.E., 1996. Modeling Dynamic Transportation Networks: An Intelligent Transportation Systems Oriented Approach, second revised ed. Springer-Verlag, New York.

Ran, B., Lo, H., Boyce, D.E., 1996. A formulation and solution algorithm for a multi-class dynamic traffic assignment problem. In: Lesort, J.-B. (Ed.), Transportation and Traffic Theory. Pergamon, Oxford, pp. 195–216.

Smith, M.J., 1979. The existence, uniqueness and stability of traffic equilibria. Transportation Research Part B 13 (4), 295–304.

Ulbrich, M., 2002. Nonsmooth Newton-like methods for variational inequalities and constrained optimization problems in function spaces. Ph.D. Thesis, Technische Universitat Munchen, Germany.

Wardrop, J.G., 1952. Some theoretical aspects of road traffic research. In: Proceedings of the Institution of Civil Engineers Part II (1), pp. 325–378.

Wie, B.W., Tobin, R.L., Carey, M., 2002. The existence, uniqueness and computation of an arc-based dynamic network user equilibrium formulation. Transportation Research Part B 36 (10), 897–918.

Xu, Y.W., Wu, J.H., Florian, M., Marcotte, P., Zhu, D.L., 1999. Advances in the continuous dynamic network loading problem. Transportation Science 33 (4), 341–353.