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P_c-Matrices and the Linear Complementarity Problem

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ABSTRACT

We introduce a new matrix class P_c, which consists of those matrices M for which the solution set of the corresponding linear complementarity problem is connected for every $q \in \mathbb{R}^n$. We consider Lemke's pivotal method from the perspective of piecewise linear homotopies and normal maps and show that Lemke's method processes all matrices in $P_c \cap Q_0$. We further investigate the relationship of the class P_c to other known matrix classes and show that column sufficient matrices are a subclass of P_c, as are 2×2 P_0-matrices.

1. INTRODUCTION

The linear complementarity problem is a classical problem from optimization theory of finding $x \in \mathbb{R}^n$ with

$$z \geqslant 0, \qquad Mz + q \geqslant 0, \qquad z^{\top}(Mz + q) = 0.$$

Here $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ are given data, and the resulting problem will be denoted by LCP(q, M). We also define the set of feasible points of

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LCP(q, M) by

$$\mathrm{FEA}(q, M) := \{z \mid z \geqslant 0,\ Mz + q \geqslant 0\}.$$

In this paper, we investigate a new class of matrices, P_c, which is defined by

$$M \in \mathrm{P}_c \quad \Leftrightarrow \quad \mathrm{SOL}(q, M) \text{ is connected for all } q \in \mathbb{R}^n,$$

where SOL(q, M) is the set of solutions of LCP(q, M). The most widely used algorithm for solving LCP(q, M) is the pivotal algorithm of Lemke [10]. In [1], it is shown that Lemke's method processes all matrices $M \in \mathrm{P}_0 \cap \mathrm{Q}_0$, that is, it either finds a solution, or determines that FEA(q, M) $= \varnothing$. Here P_0 is the class of matrices having nonnegative principal minors. The principal result of this paper, given in Section 2, is that if $M \in \mathrm{P}_c \cap \mathrm{Q}_0$, then Lemke's method processes LCP(q, M). Note that Q_0 is the set of matrices for which feasibility of LCP(q, M) implies its solvability, that is,

$$M \in \mathrm{Q}_0 \quad \Leftrightarrow \quad [\mathrm{FEA}(q, M) \neq \varnothing \ \Rightarrow\ \mathrm{SOL}(q, M) \neq \varnothing].$$

Before proving this result, let us explain our motivation. An $n \times n$ matrix is a member of the matrix class P if all its principal minors are positive. It is well known that an equivalent definition is that SOL(q, M) is a singleton for every $q \in \mathbb{R}^n$. Therefore, a natural extension of the class P is the class of column sufficient matrices S_c, characterized by

$$M \in \mathrm{S}_c \quad \Leftrightarrow \quad \mathrm{SOL}(q, M) \text{ is convex for all } q \in \mathbb{R}^n.$$

Although there are other extensions of the class S_c, the most natural geometric extension would seem to be to P_c. Note that it is clear that

$$\mathrm{P} \subset \mathrm{S}_c \subset \mathrm{P}_c.$$

In order to relate our result to others found in the literature, we explore the class P_c further in Section 3. It is known [4, Theorems 3.3.4, 3.4.2] that

$$\mathrm{P} \subset \mathrm{S}_c \subset \mathrm{P}_0 \subset \mathrm{E}_0, \tag{1}$$

where E_0 is the class of matrices for which SOL(q, M) is a singleton for all $q > 0$. We know of no geometric properties of LCP(q, M) that characterize P_0 or E_0. In this paper, the geometrically defined class of matrices P_c is shown to be closely related to the algebraically defined class P_0. The interplay between algebraic and geometric characterizations of matrix classes is of paramount importance to a complete understanding of such classes. We show that within the class of 2×2 matrices

$$\mathrm{P}_0 \subset \mathrm{P}_c \subset \mathrm{E}_0, \tag{2}$$

and that these inclusions are strict. We conjecture that (2) holds for $n \times n$ matrices and hence that our main result extends the class of matrices that Lemke's method is known to process. However, in the 2×2 case, we also show that

$$\mathrm{P}_0 \cap \mathrm{Q}_0 = \mathrm{P}_c \cap \mathrm{Q}_0.$$

2. TERMINATION OF LEMKE'S METHOD

Although the basic step of Lemke's method is a pivot (as in the simplex method for linear programming), the choice of pivot step is fundamentally different and is motivated by a path following or homotopy approach. An equivalent formulation of LCP(q, M) is to find a zero of the nonsmooth mapping

$$x \mapsto Mx_+ + q + x - x_+,$$

where $(x_+)_i := \max\{x_i, 0\}$ is the projection of x onto the nonnegative orthant. This map is sometimes referred to as the "normal map" [12]; the earliest known reference is [7]. The equivalence is established by noting that if z solves LCP(q, M), then $x = z - Mz - q$ is a zero of the normal map, and if x is a zero of the normal map, then $z = x_+$ is a solution of LCP(q, M). It is easy to see that the normal map is an affine map on each of the orthants of $\mathbb{R}^n$ and is continuous on $\mathbb{R}^n$. The normal mapping is thus an example of a piecewise affine map and is intimately related to the manifold defined by the collection of the faces of the set $\mathbb{R}^n_+$, called the normal manifold [12]. Lemke's method can be viewed as a clever way of traversing this manifold, as each pivot step corresponds to changing the affine map that currently represents the normal map. In fact, Lemke's method is an instance of a more general algorithm for solving equations with piecewise linear homotopies due to Eaves [8]. The analysis in this paper uses many of the ideas contained in [8] without further proof. We will also use the fact that the general algorithm applied to LCP(q, M) is in fact Lemke's algorithm; this is shown elsewhere [2, 8].

Let $\mathscr{N}$ be the piecewise-linear manifold in $\mathbb{R}^{n+1}$ constructed by forming the Cartesian product of each orthant of $\mathbb{R}^n$ with $\mathbb{R}_+$, the nonnegative half line in $\mathbb{R}$. We abuse notation slightly and let $\mathscr{N}$ represent both the collection of cells of the manifold and the union of this collection. $\mathscr{N}$ is a piecewise linear $(n+1)$-manifold in $\mathbb{R}^{n+1}$, as can easily be verified (see [8, Example 4.3]). Now let $e > 0$ and consider the piecewise linear map $F: \mathscr{N} \to \mathbb{R}^n$ defined by

$$F(x, \mu) := Mx_+ + q + x - x_+ + \mu e.$$

Clearly any x satisfying $F(x, 0) = 0$ solves LCP(q, M). Let $w(\mu) := -q - \mu e$, and note that since

$$w(\mu) = -\mu\left[e + \mu^{-1} q\right], \tag{3}$$

$w(\mu)$ lies interior to the orthant $\mathbb{R}^n_-$ for large positive μ. Therefore $(w(\mu), \mu)$ lies interior to the cell $\mathbb{R}^n_- \times \mathbb{R}_+$ of $\mathscr{N}$, and so it is a regular point of $\mathscr{N}$ (see the proof of Theorem 2). Further, for such μ we have $(w(\mu))_+ = 0$, so that

$$F(w(\mu), \mu) = -q - \mu e - (q + \mu e) = 0.$$

Therefore, for some $\mu_0 \geqslant 0$, $F^{-1}(0)$ contains the ray $\{(w(\mu), \mu) \mid \mu \geqslant \mu_0\}$.

Now the algorithm of [8] is applied to the PL equation $F(x, \mu) = 0$, using a ray start at $(w(\mu_1), \mu_1)$ for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e, -1)$. As the manifold $\mathscr{N}$ is finite, according to [8, Theorem 15.13] the algorithm generates, in finitely many steps, either a point (x_*, μ_*) in the boundary of $\mathscr{N}$ with $F(x_*, \mu_*) = 0$, or a secondary ray in $F^{-1}(0)$ different from the starting ray. In the first case $\mu_* = 0$ and, by our earlier remarks, x_* solves LCP(q, M). Many of the results pertaining to Lemke's method processing different classes of matrices just show that secondary ray termination cannot occur, or that ray termination guarantees FEA$(q, M) = \varnothing$.

Such results are plentiful, and Lemke's algorithm is known to process many classes of matrices; see, for example [3, 4, 11]. There are, in fact, two large but distinct classes that contain most of these classes of matrices, namely L-matrices [6] and the class $\mathrm{P}_0 \cap \mathrm{Q}_0$ [1]. This paper is concerned with extending the algebraically defined class $\mathrm{P}_0 \cap \mathrm{Q}_0$ using geometric ideas. To this purpose, we introduce the class P_c. In the remainder of this section we will show that Lemke's method processes matrices from $\mathrm{P}_c \cap \mathrm{Q}_0$. We shall explore the class P_c more fully in the following section.

The set

$$K(M) := \{q \in \mathbb{R}^n \mid \mathrm{SOL}(q, M) \neq \varnothing\}$$

is the set of all right hand side vectors for which LCP(q, M) is solvable. This set is intimately related to the class Q_0, as the following theorem shows.

THEOREM 1 [6]. *For an $n \times n$ matrix M, the following are equivalent:*

1. $M \in \mathrm{Q}_0$.
2. $K(M)$ *is convex*
3. $K(M) = \mathrm{pos}(I, -M)$.

Here $\mathrm{pos}(I, -M)$ *represents the cone generated by the columns of the matrix* $(I, -M)$ *and the origin.*

Note that $\mathrm{pos}(I, -M)$ is a polyhedral convex cone. Our main result is summarized in the following theorem. The two key geometric facts that we use in the proof are

1. the connectedness of SOL(q, M) for all q,
2. the convexity of $K(M)$.

THEOREM 2. *Suppose M is in $\mathrm{P}_c \cap \mathrm{Q}_0$. Then Lemke's algorithm terminates at a solution of* LCP(q, M) *or determines that* FEA(q, M) $= \varnothing$. *Furthermore, the parameter μ in Lemke's algorithm is nonincreasing.*

Proof. Since 0 may not be a regular value of F, we use the pivotal algorithm from [8] which generates a solution of the original problem by solving the perturbed system

$$F(x, \mu) = -[\epsilon],$$

where $[\epsilon] = (\epsilon, \epsilon^2, \dots, \epsilon^n)^{\mathsf{T}}$, with $\epsilon > 0$.

Let $w(\mu) := -q - \mu e = -\mu[e + \mu^{-1}q]$, so that $w(\mu)$ lies interior to the orthant $\mathbb{R}^n_-$, for large positive μ. Therefore $(w(\mu) - [\epsilon], \mu)$ lies interior to the cell $\mathbb{R}^n_- \times \mathbb{R}_+$ of $\mathscr{N}$ for μ sufficiently large and ϵ sufficiently small. It is a regular point of $\mathscr{N}$, since $F(\mathbb{R}^n_- \times \mathbb{R}_+)$ has a nonempty interior. Further, for large μ we have $(w(\mu) - [\epsilon])_+ = 0$, so that

$$\begin{aligned} F(w(\mu) - [\epsilon], \mu) &= M(w(\mu) - [\epsilon])_+ + q + w(\mu) \\ &\quad -[\epsilon] - (w(\mu) - [\epsilon])_+ + \mu e = -[\epsilon]. \end{aligned}$$

Hence, $F^{-1}(-[\epsilon])$ contains the ray $\{(w(\mu) - [\epsilon], \mu) \mid \mu \geqslant \mu_0\}$ for some $\mu_0 \geqslant 0$.

Now the algorithm of [8] is applied to the PL equation $F(x, \mu) = -[\epsilon]$, using a ray start at $(w(\mu_1), \mu_1)$ for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e, -1)$.

Since $-[\epsilon] \in F(\mathscr{N})$ for all sufficiently small ϵ, it follows from [8, Lemma 14.2], that $-[\epsilon]$ is a regular value of F for each small positive ϵ. It then follows by [8, Theorem 9.1] that for such ϵ, $F^{-1}(-[\epsilon])$ is a 1-manifold neat in $\mathscr{N}$. This means that $F^{-1}(-[\epsilon])$ is closed in $\mathscr{N}$ and its boundary is its intersection with the boundary of $\mathscr{N}$. It is subdivided by $\sigma \cap F^{-1}(-[\epsilon])$, where σ is an n-cell of $\mathscr{N}$. Furthermore, we have $(w(\mu) - [\epsilon], \mu) \in F^{-1}(-[\epsilon])$ for sufficiently large μ.

Now, assume that the algorithm generates a sequence of points (x_1, μ_1), $(x_2, \mu_2), \dots, (x_k, \mu_k)$ with $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k$ and either terminates at step k with a ray different from the starting one or generates a point (x_{k+1}, μ_{k+1}) with $\mu_{k+1} > \mu_k$. Let $W(\epsilon)$ be the set of chords traversed by the algorithm up to this point. Then, due to the ray start, $W(\epsilon)$ cannot be PA homeomorphic to a circle, and therefore it is homeomorphic to an interval [8, Lemma 5.1].

Upon ray termination, μ is nondecreasing on the terminating ray. Thus, the set

$$\Xi = \{ \mu \mid (x, \mu) \in W(\epsilon)\}$$

admits a minimum $\bar{\mu} = \inf\{ \mu \in \Xi\} > 0$, which is achieved on (x_j, μ_j) for some $1 \leqslant j \leqslant k$. Let

$$S = \{x \mid (x, \bar{\mu}) \in W(\epsilon)\};$$

then $F(x, \bar{\mu}) = -[\epsilon]$ for $x \in S$. Hence

$$S \subset \mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M).$$

But $\mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M)$ cannot contain any other point z_1 such that $(z_1, \bar{\mu}) \notin W(\epsilon)$; otherwise, by our hypothesis on the connectedness of the solution set, there is a continuous path $z : [0, 1] \to \mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M)$ with $z(1) = z_1$ and $z(0) = z_0$ for any $z_0 \in S$. Thus

$$\{(z(t), \bar{\mu}) \mid 0 \leqslant t \leqslant 1\} \subset F^{-1}(-[\epsilon]).$$

But this contradicts the fact that $F^{-1}(-[\epsilon])$ is a 1-manifold, since $(z_0, \bar{\mu})$ contains a neighborhood not homeomorphic to an interval (see Figure 1).

Thus $S = \mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M)$ is a connected set. It is either a single point, or the union of finite number of consecutive chords in $W(\epsilon)$. In particular, S is closed.

We now show that if $q \in K(M)$, then $\bar{\mu} = 0$. The cone $K(M)$ contains the positive orthant in its interior, and it is convex because $M \in Q_0$. Since $q + [\epsilon] \in K(M)$ and $q + [\epsilon] + \bar{\mu}e \in K(M)$, it follows that

$$q + [\epsilon] + \lambda\mu e \in K(M)$$

for every $\mu \in [0, \bar{\mu}]$. Hence

$$\mathrm{SOL}(q + [\epsilon] + \mu e, M) \neq \varnothing$$

for all $\mu \in [0, \bar{\mu}]$. Consider a strictly increasing sequence $\{\mu_j \mid j = 1, 2, \dots\}$ with $\mu_1 < \bar{\mu}$ and $\lim_{j\to\infty} \mu_j = \bar{\mu}$. Assume that $x(\mu_j) \in \mathrm{SOL}(q + [\epsilon] + \mu_j e, M)$. Then $(x(\mu_j), \mu_j) \in F^{-1}(-[\epsilon])$; hence each $(x(\mu_j), \mu_j) \in F^{-1}(-[\epsilon])$ is contained in a 1-chord of $F^{-1}(-[\epsilon])$. Since the 1-manifold

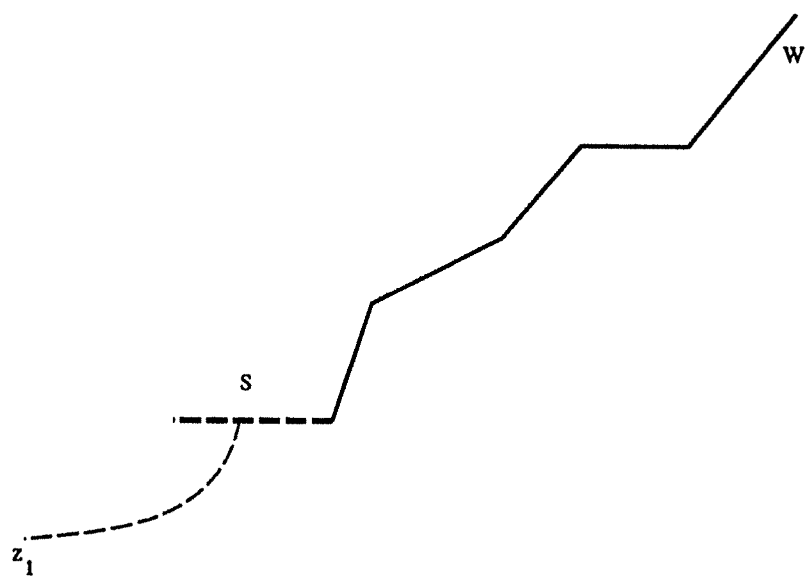

FIG. 1. The path connecting z_1 to S forms a branch of W.

$F^{-1}(-[\epsilon])$ is finite, there exists a chord l (which is a line segment) such that $(x(\mu_j), \mu_j) \in l$ for infinitely many j, and without loss of generality we can assume that $(x(\mu_j), \mu_j) \in l$ for all j. Therefore l contains the set

$$\{(x(\mu), \mu) \in F^{-1}(-[\epsilon]) \mid \bar{\mu} - \delta \leq \mu < \bar{\mu}\}$$

for some $\delta > 0$. Thus l contains a point $(w(\bar{\mu}), \bar{\mu})$ with $w(\bar{\mu}) \in S$.

On the other hand, by definition of $\bar{\mu}$,

$$(w(\mu), \mu) \notin W(\epsilon)$$

for any $\mu < \bar{\mu}$. Hence l is not a subset of $W(\epsilon)$, and l forms a branch from $S \times \{\bar{\mu}\}$ (see Figure 2). This is in contradiction to the fact that $F^{-1}(-[\epsilon])$ is a 1-manifold.

So if $q \in K(M)$, the algorithm terminates at a point in the boundary, that is, a solution of $F(x, 0) = -[\epsilon]$.

If the algorithm also terminates in the boundary when $q \notin K(M)$, this leads immediately to a contradiction. Thus in this case, ray termination must occur. ∎

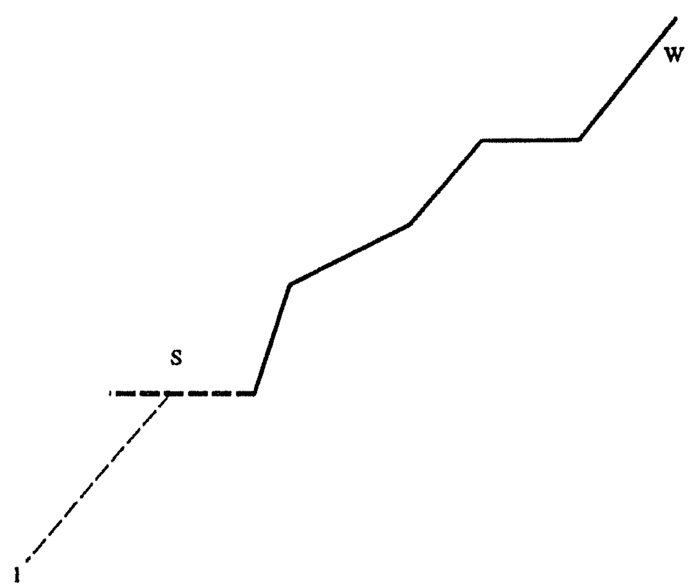

FIG. 2. The chord l forms a branch of W.

In practice the algorithm does not actually use a positive ϵ, but only maintains the information necessary to compute $W(\epsilon)$ for all small positive ϵ, employing the lexicographic ordering to resolve possible ambiguities when $\epsilon = 0$. Therefore after finitely many steps it will actually have computed x_0 with $M_{\mathbb{R}^n_+}(x_0) + q = 0$, or prove that $\mathrm{FEA}(q, M) = \varnothing$.

3. RELATIONSHIP TO OTHER CLASSES

The aim of this section is to explore the relationship of P_c to other known classes of matrices. In particular, we consider its relationship to column sufficient matrices, P_0 and E_0.

We first show how column sufficient matrices are related to P_0, and also to P_c. A matrix M is said to be *column sufficient* if, given $z \in \mathbb{R}^n$,

$$z_i(Mz)_i \leqslant 0 \quad \text{for all } i \quad \Rightarrow \quad z_i(Mz)_i = 0 \quad \text{for all } i.$$

The class of such matrices is denoted as S_c, and it is shown in [4, Proposition 3.5.8] that this definition is equivalent to the one given in the introduction. M is *row sufficient* if its transpose is column sufficient, and M is *sufficient* if it is both column and row sufficient.

Clearly, $\mathrm{P} \subset \mathrm{S}_c \subset \mathrm{P}_c$, since the solutions sets are a singleton, convex, and connected, respectively. The following corollary, which also follows from [5, p. 239], is now immediate.

COROLLARY 3. *Suppose $M \in \mathrm{S}_c \cap \mathrm{Q}_0$. Then Lemke's algorithm terminates at a solution of* $\mathrm{LCP}(q, M)$ *or determines that* $\mathrm{FEA}(q, M) = \varnothing$.

Proof. Since M is column sufficient, $\mathrm{SOL}(q, M)$ is convex, and is hence connected for all q. The corollary now follows from Theorem 2. ∎

It is also known that Lemke's method processes row sufficient matrices, since these are contained in $\mathrm{P}_0 \cap \mathrm{Q}_0$ [4, 3.5.3 and 3.5.5].

The following example shows that P_c is not a subclass of P_0. The matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

does not belong to P_0. But for any $q \in \mathbb{R}^n$, we have

$$\mathrm{SOL}(q, M) = \begin{cases} \{(0,0)\} & \text{if } q_1 > 0, q_2 > 0 \\ \{(0, y) \mid y \geqslant 0\} & \text{if } q_1 > 0, q_2 = 0, \\ \varnothing & \text{if } q_1 > 0, q_2 < 0, \\ \{(x,0) \mid x \geqslant 0\} & \text{if } q_1 = 0, q_2 > 0, \\ \{(x,0) \mid x \geqslant 0\} \cup \{(0, y) \mid y \geqslant 0\} & \text{if } q_1 = 0, q_2 = 0, \\ \{(x,0) \mid x \geqslant -q_2\} & \text{if } q_1 = 0, q_2 < 0, \\ \varnothing & \text{if } q_1 < 0, q_2 > 0, \\ \{(0, y) \mid y \geqslant -q_1\} & \text{if } q_1 < 0, q_2 = 0, \\ \{(-q_2, -q_1)\} & \text{if } q_1 < 0, q_2 < 0. \end{cases}$$

We see that $\mathrm{SOL}(q, M)$ is connected for all q and hence $M \in P_c$. Clearly $M \notin Q_0$.

Note also that E_0 is not contained in P_c. The following example proves this fact:

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Here, $M \in E_0$, but the solution set for the given q is

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \right\},$$

which is not connected.

Now that P_0 does not contain P_c, does P_c contain P_0? When an $n \times n$ matrix M is in P_0, we can prove that for all q except those in a set $\kappa(M)$ of measure zero, the solution set is connected.

THEOREM 4. *Let $M \in P_0$, and $\kappa(M)$ denote the union of the facets of all the complementary cones of M. If $q \notin \kappa(M)$ or $q > 0$, the number of solutions of* LCP(q, M) *is zero or one, and hence the solution set is connected.*

Proof. Since $P_0 \subset E_0$ [Equation (1)], $\mathrm{SOL}(q, M)$ is a singleton for all $q > 0$ and hence connected.

According to a result in [9, Theorem 2], originally due to Cottle and Guu, $\mathrm{SOL}(q, M)$ contains either 0, 1, or infinitely many points whenever $M \in \mathrm{P}_0$. However, by [4, Theorem 6.1.8], $q \notin \kappa(M)$ implies that the local degree of q relative to M is well defined, which implies that $\mathrm{SOL}(q, M)$ is finite. Thus, $\mathrm{SOL}(q, M)$ has 0 or 1 elements for all $q \in \mathbb{R}^n$ except those that belong to a finite union of polyhedral convex cones of dimension less than n. ∎

The question whether $\mathrm{SOL}(q, M)$ is connected when it has infinitely many elements remains open. However, in the 2×2 case we can show the following results.

THEOREM 5. *Suppose $M \in \mathbb{R}^{2\times 2}$. Then*

1. $\mathrm{P}_0 \subset \mathrm{P}_c$,
2. $\mathrm{P}_c \subset \mathrm{E}_0$,
3. $\mathrm{P}_0 \cap \mathrm{Q}_0 = \mathrm{P}_c \cap \mathrm{Q}_0$.

Proof. The following proofs assume that

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and essentially consider all cases. Some details are omitted.

1: From Theorem 4, we only need to consider $q \in \kappa(M)$, that is (without loss of generality)

$$q = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \quad \text{or} \quad q = -\lambda \begin{bmatrix} a \\ c \end{bmatrix}$$

for $\lambda \geqslant 0$. Furthermore, since $M \in \mathrm{P}_0$, we have $a \geqslant 0$, $d \geqslant 0$, and $ad \geqslant bc$.

When $\lambda = 0$, using the set valued inverse operator

$$A^{-1}(S) := \{x \mid Ax \in S\},$$

we see that $\mathrm{SOL}(0, M)$ is given by

$$[M^{-1}(0) \cap \mathbb{R}^2_+] \cup [M^{-1}(0 \times \mathbb{R}_+) \cap \mathbb{R}_4 \times 0]$$
$$\cup [M^{-1}(\mathbb{R}_+ \times 0) \cap 0 \times \mathbb{R}_+].$$

Each of the three sets of the union is polyhedral and contains the origin; hence $\mathrm{SOL}(q, M)$ is connected.

For $\lambda > 0$ and $q = (\lambda, 0)^{\mathrm{T}}$,

$$\mathrm{SOL}(q, M) = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \cup (0,0),$$

where

$$\mathscr{A}_1 := M^{-1}(-q) \cap \mathbb{R}^2_+,$$

$$\mathscr{A}_2 := M_1^{-1}(-\lambda) \cap M_2^{-1}(\mathbb{R}_+) \cap \mathbb{R}_+ \times 0,$$

$$\mathscr{A}_3 := M_1^{-1}(-\lambda + \mathbb{R}_+) \cap M_2^{-1}(0) \cap 0 \times \mathbb{R}_+.$$

Note that $\mathscr{A}_2 = \emptyset$. Furthermore, $(0,0) \in \mathscr{A}_3$. It remains to show that $\mathscr{A}_1$ and $\mathscr{A}_3$ have nontrivial intersection if $\mathscr{A}_1$ is nonempty. If $\mathscr{A}_1$ is nonempty, then it is easy to show that it has a point in common with $\mathscr{A}_3$ by considering the cases when M is invertible and when M is not invertible.

Now consider the case $\lambda > 0$ and $q = -\lambda(a, c)^{\mathrm{T}}$. We may assume without loss of generality that either a or c is nonzero. Then

$$\mathrm{SOL}(q, M) = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \cup \mathscr{A}_4,$$

where

$$\mathscr{A}_1 := M^{-1}(-q) \cap \mathbb{R}^2_+,$$

$$\mathscr{A}_2 := M_1^{-1}(\lambda a) \cap M_2^{-1}(\lambda c + \mathbb{R}_+) \cap \mathbb{R}_+ \times 0,$$

$$\mathscr{A}_3 := M_1^{-1}(\lambda a + \mathbb{R}_+) \cap M_2^{-1}(\lambda c) \cap 0 \times \mathbb{R}_+,$$

$$\mathscr{A}_4 := M_1^{-1}(\lambda a + \mathbb{R}_+) \cap M_2^{-1}(\lambda c + \mathbb{R}_+) \cap (0,0).$$

If $\mathscr{A}_4 \neq \emptyset$ then $a \leqslant 0$; hence $a = 0$ and thus $(0,0) \in \mathscr{A}_2$. If $\mathscr{A}_3 \neq \emptyset$, let $x \in \mathscr{A}_3$. If $x_2 = 0$, then $a = c = 0$, which is a contradiction. Thus $x_2 > 0$, and it then follows that $bc = ad$. It is easy to see that this implies $x \in \mathscr{A}_1$. The proof is completed by noting that $(\lambda, 0) \in \mathscr{A}_1 \cup \mathscr{A}_2$.

2: It is easy to see that the 2×2 matrix $M \in \mathrm{E}_0$ if and only if $a \geqslant 0$, $d \geqslant 0$, and either $ad \geqslant bc$, $b \geqslant 0$, or $c \geqslant 0$. Thus suppose that $M \in \mathrm{P}_c$ but $M \notin \mathrm{E}_0$. Then $M \notin \mathrm{P}_0$, and one of the above inequalities must be violated. It is easy to see that if $a < 0$ or $d < 0$, then for $q > 0$, $\mathrm{SOL}(q, M)$ is not connected, which is a contradiction. To complete the proof, we derive a contradiction in the case where $a \geqslant 0$, $d \geqslant 0$, $b < 0$, $c < 0$, and $bc > ad$. Again, let $q > 0$. Note that is $x_1 = 0$, then $x_2 = 0$, and conversely. Since M is invertible, it now follows that the only solutions are $(0, 0)$ and $-M^{-1}q$. Note that $-M^{-1}q > 0$, contradicting the connectedness of the solution set.

3: From the above, it is known that $P_0 \cap Q_0 \subset P_c \cap Q_0$. We now show the reverse inclusion. Let $M \in P_c \cap Q_0$. First note that from the above it follows that $M \in E_0 \cap Q_0$ and hence that $a \geqslant 0$, $d \geqslant 0$, and either $ad \geqslant bc$, $b \geqslant 0$, or $c \geqslant 0$. Suppose that $bc > ad$, so that $b \geqslant 0$ or $c \geqslant 0$ and thus both are strictly positive. It now follows that $K(M) = R^2$. Taking $q_1 > 0$, $q_2 < 0$ implies that $d > 0$; similarly $a > 0$. Now let $q < 0$. It then follows from the connectedness of $\mathrm{SOL}(q, M)$ that exactly one of the following must hold:

(a) $aq_2 - cq_1 \geqslant 0$,
(b) $dq_1 - bq_2 \geqslant 0$,
(c) $aq_2 - cq_1 > 0$ and $dq_1 - bq_2 > 0$.

A contradiction now follows by letting $q_1 = -a$, $q_2 = -c$. ■

Note that $E_0 \cap Q_0$ is strictly bigger than $P_c \cap Q_0$, as the example given above shows. In fact,

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is in $E_0 \cap Q_0$.

4. CONCLUSIONS

This paper has introduced a new class of matrices P_c and exhibited some of its properties. Some outstanding questions remain, which include determining an effective test for inclusion in the class P_c. An effective test of this sort will allow the conjecture relating P_0 and P_c to be verified or proven false and establish whether the solution set of $\mathrm{LCP}(q, M)$ is in fact connected when $M \in P_0 \cap Q_0$. Essentially, a key open question is to establish Theorem 5 or exhibit counterexamples in the general $n \times n$ case.

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