Error bounds and strong upper semicontinuity for monotone affine variational inequalities*

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Global error bounds for possibly degenerate or nondegenerate monotone affine variational inequality problems are given. The error bounds are on an arbitrary point and are in terms of the distance between the given point and a solution to a convex quadratic program. For the monotone linear complementarity problem the convex program is that of minimizing a quadratic function on the nonnegative orthant. These bounds may form the basis of an iterative quadratic programming procedure for solving affine variational inequality problems. A strong upper semicontinuity result is also obtained which may be useful for finitely terminating any convergent algorithm by periodically solving a linear program.

Keywords: Error bounds, upper semicontinuity, variational inequalities, linear complementarity problem.

1. Introduction

We consider here the monotone affine variational inequality problem [1, 3] of finding an $\bar{x}$ in X such that

$$(x-\bar{x})^{\mathrm{T}}(M\bar{x}+q) \geq 0, \quad \forall x \in X := \{x | Ax \geq b, x \geq 0\}, \tag{1}$$

where M is an $n \times n$ real positive semidefinite matrix (not necessarily symmetric) and A is an $m \times n$ real matrix. When X is the nonnegative orthant, the problem becomes the classical monotone linear complementarity problem (LCP) [2, 14]. Our principal concern here is: Given an arbitrary point x in $\mathbb{R}^n$, how far is it from the closed convex solution set $\bar{X}$ of (1), assuming that $\bar{X}$ is nonempty? Global error bounds for (1) have been given by [15, lemma 2] for positive definite M in terms of a "gradient projection" residual as well as by Luo and Tseng

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[7, theorem 2.1] locally for a general monotone affine variation inequality. See also [16] for results concerning the nonlinear case, where a strongly monotone Lipschitz continuous function f replaces the affine function $Mx+q$ in (1) and [12] for other related error bounds. In [11], global error bounds were also given for the case of positive semidefinite M in terms of a gradient projection residual multiplied by a term that involves the norm of the point x. In this work the error bound is motivated by the following formulation of (1) of finding an $\bar{x} \in X$ such that

$$(x-\bar{x})^{\mathrm{T}}(Mx+q) \geq 0, \quad \forall x \in X = \{x \mid Ax \geq b, x \geq 0\}. \tag{2}$$

This is equivalent to (1) when M is positive semidefinite. Thus, given any point x in $\mathbb{R}^n$, we consider the following proximal convex quadratic program of finding a $y(x)$ such that

$$y(x) \in \arg\min_{y \in X} (y-x)^{\mathrm{T}}(My+q) + \frac{\gamma}{2}\|y-x\|^2, \tag{3}$$

where $\gamma \geq 0$. Note that when $\gamma > 0$, problem (3) is a strongly convex quadratic program and hence has a unique solution $y(x)$. Furthermore, when $x \in \bar{X}$, it follows from the fact that M is positive semidefinite that

$$(y-x)^{\mathrm{T}}(My+q) \geq (y-x)^{\mathrm{T}}(Mx+q) \geq 0.$$

Hence, the minimum value of zero can be achieved by setting $y = x$. Thus when $x \in \bar{X}$, and $\gamma > 0$, the unique solution of (3) is $y(x) = x$. The error bounds we propose here are based on the quantity $y(x) - x$ which is zero when $\bar{x} \in \bar{X}$ and $y(\bar{x}) = \bar{x}$. This prompts us to propose the following iterative procedure for solving (1):

$$x^{i+1} \in \arg\min_{x \in X} (x-x^i)^{\mathrm{T}}(Mx+q) + \frac{\gamma}{2}\|x-x^i\|^2, \tag{4}$$

for which we have no convergence results at the present time. One advantage of (4) over the original problem (1) is the induced symmetry of the quadratic term which permits the potential use of successive overrelaxation (SOR) methods such as those of [4] especially for the monotone linear complementarity problem, that is, when X is the nonnegative orthant. In addition, (4) is strongly convex when $\gamma > 0$.

Our principal tool in deriving our bounds here is the error bound that was obtained in [13, theorem 2.7] for the monotone linear complementarity problem. In section 2 of this paper we give a simplified version of this error bound, and in section 3 we derive our error bounds for (1), both for a possibly degenerate problem (theorem 3) and a nondegenerate problem (theorem 6). We also establish (theorem 7) a strong upper semicontinuity result for the nondegenerate monotone affine variational inequality problem in terms of solutions of linear programs ((34) below) obtained by linearizing around any sequence $\{x^i\}$ which converges to a solution of the problem. From a certain $\bar{\imath}$ onward, all solutions of the linear programs yield a solution of the variational inequality problem. This may be a useful result for finitely terminating any convergent algorithm for variational inequalities.

A word about our notation. For a vector x in the n-dimensional space $\mathbb{R}^n$, x_+ will denote the orthogonal projection on the nonnegative orthant $\mathbb{R}^n_+$, that is, $(x_+)_i := \max\{x_i, 0\}$, $i = 1, \ldots, n$. The norm $\|\cdot\|$ will denote the Euclidean norm, while other norms will be appropriately subscripted. The transpose of a matrix M will be denoted by M^{T}. For an $m \times n$ real matrix A, A_i will denote the ith row, while A_J will denote the set of rows A_i for $i \in J \subset \{1, \ldots, m\}$. Similar notation, b_i and b_J is used for a vector b in $\mathbb{R}^m$. The identity matrix of arbitrary dimension will be denoted by I.

2. Preliminary background

We begin by giving a simplification of the error bound of [13, theorem 2.7] for the monotone linear complementarity problem

$$Nz + p \geq 0, \quad z \geq 0, \quad z^{\mathrm{T}}(Nz + p) = 0, \tag{5}$$

where N is a $k \times k$ real positive semidefinite matrix, not necessarily symmetric. We shall assume that the solution set of (5), $SOL(N,p)$, is nonempty. By slight modification of the proofs of lemma 2.5 and theorem 2.7 of [13] and making use of theorem 2.6 of [10] we obtain the following simplified error bounds in terms of another "natural" residual $s(z)$ (see (8) below).

THEOREM 1

(*Error bound for monotone linear complementarity problems*)

Let N be positive semidefinite, let the solution set $SOL(N,p)$ of (5) contain $\bar{z}$ and let

$$\hat{N} := \frac{1}{2}(N + N^{\mathrm{T}}) \quad \text{and} \quad d := 2\hat{N}\bar{z} + p. \tag{6}$$

For any point z in $\mathbb{R}^k$ there exists a $\bar{z}(z)$ in $SOL(N,p)$ such that [13, theorem 2.7]

$$\|z - \bar{z}(z)\| \leq \sigma(N,p)(s(z) + s(z)^{1/2}), \tag{7}$$

where $s(z)$ is the residual

$$s(z) := \|(-Nz - p, -z, z^{\mathrm{T}}(Nz + p))_{+}\|, \tag{8}$$

and

$$\sigma(N,p) := \sqrt{k\nu(N,p)}\,\tau(N,p), \tag{9}$$

$$\nu(N,p) := \min_{\bar{z} \in SOL(N,p)} \|1, \bar{z}, N\bar{z} + p\|, \tag{10}$$

$$\tau(N,p) := \sup_{(u,v,\xi,z)} \left\{ \|u, v, \xi, z\| \;\middle|\; \begin{array}{c} \|N^{\mathrm{T}}u + v - d\xi + \hat{N}^{1/2}z\|_1 = 1 \\ (u,v,\xi) \geq 0 \\ \text{Columns of } [N^{\mathrm{T}}\, I\, d\, \hat{N}^{1/2}], \\ \text{corresponding to nonzero} \\ \text{elements of } (u,v,z,\xi), \\ \text{are linearly independent} \end{array} \right\}. \tag{11}$$

If the LCP (5) has some nondegenerate solution $\hat{z}$, that is, $\hat{z} + N\hat{z} + p > 0$, then the bound (7) simplifies to [10, theorem 2.6]

$$\|z - \bar{z}(z)\| \leq \sigma(N,p)s(z), \tag{12}$$

with the term $\nu(N,p)$ deleted from (9), and the terms $\hat{N}^{1/2}z$ and $\hat{N}^{1/2}$ deleted from (11).

It was noted in [13, example 2.9] that the square root term in the bound (7) is essential and cannot be dispensed with even locally.

In order to apply the above bounds to the monotone affine variational inequality problem (1) we state an equivalent characterization of (1) as a linear complementarity problem in $\mathbb{R}^{n+m}$ as follows.

PROPOSITION 2

A point $\bar{x}$ in $\mathbb{R}^n$ solves the affine variational inequality (1) if and only if $\bar{x}$ and some $\bar{u}$ in $\mathbb{R}^m$ solve the following linear complementarity problem:

$$w = \begin{bmatrix} M & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} q \\ -b \end{bmatrix} \geq 0, \quad \begin{bmatrix} x \\ u \end{bmatrix} \geq 0, \quad w^{\mathrm{T}} \begin{bmatrix} x \\ u \end{bmatrix} = 0. \tag{13}$$

Proof

An $\bar{x}$ is a solution of (1) if and only if $\bar{x} \in \arg\min_{x \in X} x^T(M\bar{x} + q)$. This is so if and only if $\bar{x}$ and some $\bar{u}$ satisfy the Karush–Kuhn–Tucker conditions (13) for the linear program $\min_{x \in X} x^T(M\bar{x} + q)$. □

Note that the LCP (13) was precisely the same problem used in [11] to obtain an error bound for the monotone affine variational inequality (1) in terms of a gradient projection residual. Note also that the error bound (7) of theorem 1 could be applied directly to (13) to obtain an error bound on both (x, u) and hence on x. However, the whole thrust of our approach is to generate an error bound on a given point x using a *computable* residual in terms of x only. Using a residual for the LCP (13) directly requires knowledge of a multiplier u associated with x which we do not have. To avoid an explicit computation of u, a gradient projection residual was used in [11], and in this paper a residual based on a "proximal minimum principle" (3) is employed.

3. Error bounds for the monotone affine variational inequality

We first derive the global error bounds (14) and (32) below, for the monotone variational inequality (1), by using the error bounds (7) and (12) on the equivalent LCP formulation (13) of (1).

THEOREM 3

(*Global error bound for monotone affine variational inequalities*)

Let M be positive semidefinite, $\gamma \geq 0$, let (1) be solvable and let x be any given point in $\mathbb{R}^n$. Suppose one of the following three conditions holds:

(i) $\gamma > 0$, or

(ii) $Mx - A^T u + q \geq 0$ for some $u \geq 0$, or

(iii)

$$Ar \geq 0, \quad r \geq 0, \quad (M + M^T)r = 0\rangle \Rightarrow r = 0.$$

Then there exists a solution $\bar{x}(x)$ of (1) such that

$$\|x - \bar{x}(x)\| \leq \tau(M, A, q, b)(t(x) + t(x)^{1/2}) + \|y(x) - x\|, \tag{14}$$

where

$$y(x) \in \arg\min_{y \in X} (y - x)^T(My + q) + \frac{\gamma}{2}\|y - x\|^2, \tag{15}$$

$$t(x) := \|([M+\gamma I]^{\mathrm{T}}(y(x)-x), -y(x)^{\mathrm{T}}[M+\gamma I]^{\mathrm{T}}(y(x)-x))_+\|, \tag{16}$$

$$\tau(M,A,q,b) := \sigma(N,p), \tag{17}$$

and σ is defined in (9) with

$$N = \begin{bmatrix} M & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}, \quad p = \begin{bmatrix} q \\ -b \end{bmatrix}, \quad k = n+m. \tag{18}$$

Remark

Conditions (i), (ii) and (iii) of the theorem are really imposed to ensure that $y(x)$ exists (and is unique under (iii)). Furthermore, a necessary and sufficient condition for an arbitrary quadratic function to be bounded below on a polyhedral region is given in [5, corollary 4].

Proof

Note first that (15) has a solution. This is trivial under assumption (i), by strong convexity and $X \neq \emptyset$. For the case when $\gamma = 0$, this follows from the facts that X is nonempty and the dual quadratic program [8], given by

$$\max_{y,u}\{-y^{\mathrm{T}}My - q^{\mathrm{T}}x + b^{\mathrm{T}}u \mid (M+M^{\mathrm{T}})y - M^{\mathrm{T}}x - A^{\mathrm{T}}u + q \geq 0, u \geq 0\}, \tag{19}$$

is feasible and hence the convex quadratic objective of (15) is bounded from below. That the dual problem (19) is feasible follows from assumption (ii) by taking $y = x$ in (19), or from assumption (iii) since this implies

$$(M+M^{\mathrm{T}})r = 0, \quad Ar \geq 0, \quad 0 \neq r \geq 0 \text{ has no solution } r,$$

and hence by Tucker's Theorem [8, theorem 2.4.3] there exist (y,u) such that

$$(M+M^{\mathrm{T}})y - A^{\mathrm{T}}u > 0, \quad u \geq 0.$$

By multiplying this (y,u) by sufficiently large $\lambda > 0$ it follows that

$$(M+M^{\mathrm{T}})\lambda y - Mx - A^{\mathrm{T}}\lambda u + q > 0, \quad \lambda u \geq 0,$$

and hence $(\lambda y, \lambda u)$ is feasible for (19).

It now follows from duality [8, theorem 8.2.4] that if $y(x)$ is a solution of the convex quadratic program (15), then $y(x)$ and some $u(x)$ must also solve the dual quadratic program

$$\begin{aligned} &\underset{y,u}{\text{maximize}} \; -y^{\mathrm{T}}My + \frac{\gamma}{2}\|y-x\|^2 - \gamma y^{\mathrm{T}}(y-x) - q^{\mathrm{T}}x + b^{\mathrm{T}}u \\ &\text{subject to } (M+M^{\mathrm{T}})y + \gamma(y-x) - M^{\mathrm{T}}x - A^{\mathrm{T}}u + q \geq 0, \\ &\qquad\qquad u \geq 0, \end{aligned} \tag{20}$$

and

$$(y(x)-x)^{\mathrm{T}}(My(x)+q) = -y(x)^{\mathrm{T}}My(x) - \gamma y(x)^{\mathrm{T}}(y(x)-x) - q^{\mathrm{T}}x + b^{\mathrm{T}}u(x). \tag{21}$$

We now apply the error bound, theorem 1, to the linear complementarity problem (13) (which is equivalent to (1)) at the point $z = [y(x), u(x)]^{\mathrm{T}}$ with N and p as defined in (18). We note first that $s(z)$ as defined by (8) is

$$\begin{aligned} s(z) = \Bigg\| \Bigg(&\begin{bmatrix} -My(x) + A^{\mathrm{T}}u(x) - q \\ -Ay(x) + b \end{bmatrix}, \begin{bmatrix} -y(x) \\ -u(x) \end{bmatrix}, \\ &y(x)^{\mathrm{T}}My(x) + q^{\mathrm{T}}y(x) - b^{\mathrm{T}}u(x) \Bigg)_+ \Bigg\|. \end{aligned} \tag{22}$$

From the equality of (21) we have that

$$y(x)^{\mathrm{T}}My(x) + q^{\mathrm{T}}y(x) - b^{\mathrm{T}}u(x) = -(y(x)-x)^{\mathrm{T}}[M+\gamma I]y(x). \tag{23}$$

From the feasibility of $y(x)$, $u(x)$ for (20) we get

$$-My(x) + A^{\mathrm{T}}u(x) - q \leq [M+\gamma I]^{\mathrm{T}}(y(x)-x). \tag{24}$$

Using (23), (24), $y(x) \geq 0$, $u(x) \geq 0$ and $Ay(x) \geq b$ in (22) we obtain (upon invoking the monotonicity of the Euclidean norm)

$$s(z) \leq \|([M+\gamma I]^{\mathrm{T}}(y(x)-x), -y(x)^{\mathrm{T}}[M+\gamma I]^{\mathrm{T}}(y(x)-x))_+\| = t(x), \tag{25}$$

where the last equality follows from the definition (16) of $t(x)$. Using expression (25) in (7) we obtain that there exists

$$\begin{bmatrix} \bar{x}(x) \\ \bar{u}(x) \end{bmatrix} := \begin{bmatrix} \bar{x}(y(x), u(x)) \\ \bar{u}(y(x), u(x)) \end{bmatrix},$$

which solves (13) such that

$$\|y(x)-\bar{x}(x)\| \le \left\| \begin{matrix} y(x)-\bar{x}(x) \\ u(x)-\bar{u}(x) \end{matrix} \right\| \le \tau(M,A,q,b)(t(x)+t(x)^{1/2}). \tag{26}$$

However, since

$$\|x-\bar{x}(x)\| \le \|x-y(x)\| + \|y(x)-\bar{x}(x)\|, \tag{27}$$

it follows that

$$\|x-\bar{x}(x)\| \le \tau(M,A,q,b)(t(x)+t(x)^{1/2}) + \|x-y(x)\|,$$

which gives the required result upon noting that $\bar{x}(x)$ solves (1). □

We point out the difference between the two cases $\gamma > 0$ and $\gamma = 0$. In the former case, the error bound (14) would seem to be weaker than the corresponding one for $\gamma = 0$. However, if $x \in \bar{X}$, it is easy to see that $y(x) = x$ when $\gamma > 0$, so the corresponding error bound is zero on the solution set. There is no such guarantee for the case $\gamma = 0$, since there may be other solutions to (15). The following lemma and example make this clearer.

LEMMA 4

Suppose that M is positive semidefinite and $\bar{x}$ is in the solution set $\bar{X}$ of (1). Let

$$\bar{Y} := \{x \in X \mid \exists u \ge 0, Mx + q - A^{\mathrm{T}}u \ge 0, (M+M^{\mathrm{T}})(x-\bar{x}) = 0, \\ (M\bar{x}+q)^{\mathrm{T}}(x-\bar{x}) = 0\},$$

$$\bar{Z} := \arg\min_{y \in X} (y-\bar{x})^{\mathrm{T}}(My+q)$$

and

$$\bar{N} := \{x \mid M(x-\bar{x}) = 0\}.$$

Then $\bar{X} \subset \bar{Y} \subset \bar{Z}$, $\bar{Z} \cap \bar{N} \subset \bar{X}$ and hence

$$\bar{X} \cap \bar{N} = \bar{Y} \cap \bar{N} = \bar{Z} \cap \bar{N}.$$

Proof

In order to show that $\bar{X} \subset \bar{Y}$, suppose $\bar{y} \in \bar{X}$. Then

$$(x - \bar{y})^{\mathrm{T}}(M\bar{y} + q) \geq 0, \quad \forall x \in X.$$

The fact that M is positive semidefinite and $\bar{x} \in X$ gives

$$(\bar{x} - \bar{y})^{\mathrm{T}}(M\bar{x} + q) \geq (\bar{x} - \bar{y})^{\mathrm{T}}(M\bar{y} + q) \geq 0.$$

Since $\bar{x} \in \bar{X}$, it follows that $(x - \bar{x})^{\mathrm{T}}(M\bar{x} + q) \geq 0$ for all $x \in X$, so $\bar{y} \in X$ implies $(\bar{y} - \bar{x})^{\mathrm{T}}(M\bar{x} + q) \geq 0$ and hence

$$(\bar{y} - \bar{x})^{\mathrm{T}}(M\bar{x} + q) = 0. \tag{28}$$

Furthermore,

$$\begin{aligned} 0 &\leq (\bar{y} - \bar{x})^{\mathrm{T}}M(\bar{y} - \bar{x}) \\ &= (\bar{y} - \bar{x})^{\mathrm{T}}(M\bar{y} + q) - (\bar{y} - \bar{x})^{\mathrm{T}}(M\bar{x} + q) \\ &= (\bar{y} - \bar{x})^{\mathrm{T}}(M\bar{y} + q) \\ &\leq 0, \end{aligned}$$

the second equality following from (28) and the final inequality following from the fact that $\bar{y} \in \bar{X}$. Hence, $(\bar{y} - \bar{x})^{\mathrm{T}}M(\bar{y} - \bar{x}) = 0$, giving

$$(M + M^{\mathrm{T}})(\bar{y} - \bar{x}) = 0.$$

The proof that $\bar{X} \subset \bar{Y}$ is complete if we can exhibit a $u \geq 0$ such that $M\bar{y} + q \geq A^{\mathrm{T}}u$. However, by proposition 2, $\bar{y} \in \bar{X}$ implies that $\bar{y}$ and some $\bar{u}$ solve (13), which by feasibility of $\bar{y}$ and $\bar{u}$ shows

$$M\bar{y} - A^{\mathrm{T}}\bar{u} + q \geq 0, \quad \bar{u} \geq 0,$$

as required.

We now show $\bar{Y} \subset \bar{Z}$. It is easy to see that $\bar{x} \in \bar{Z}$ and so it follows from the characterization of the solution set of a convex quadratic program in [9, corollary 1] that

$$\bar{Z} = \{x \in X | (M + M^{\mathrm{T}})(x - \bar{x}) = 0, \quad (M\bar{x} + q)^{\mathrm{T}}(x - \bar{x}) = 0\}. \tag{29}$$

The inclusion $\bar{Y} \subset \bar{Z}$ is now obvious.

To establish the last conclusions of the lemma, let $x \in \bar{Z}$. By the definition of $\bar{Z}$, x and some u must satisfy the following Karush–Kuhn–Tucker optimality conditions

$$\begin{aligned} (M+M^{\mathrm{T}})x+q-M^{\mathrm{T}}\bar{x}-A^{\mathrm{T}}u &\geq 0, \quad Ax-b\geq 0, \\ x^{\mathrm{T}}((M+M^{\mathrm{T}})x+q-M^{\mathrm{T}}\bar{x}-A^{\mathrm{T}}u) &= 0, \quad u^{\mathrm{T}}(Ax-b)=0, \\ x &\geq 0, \quad u\geq 0. \end{aligned} \tag{30}$$

Hence, if $x \in \bar{Z} \cap \bar{N}$, it follows from (29) that $M^{\mathrm{T}}(x-\bar{x})=0$ and consequently (30) becomes (13) and so x solves (1), that is, $x \in \bar{X}$. This gives the desired inclusion $\bar{Z} \cap \bar{N} \subset \bar{X}$. Since we already have that $\bar{X} \subset \bar{Y} \subset \bar{Z}$, it follows that

$$\bar{X} \cap \bar{N} = \bar{Y} \cap \bar{N} = \bar{Z} \cap \bar{N}. \qquad \square$$

The following example from [13] shows that the inclusions in lemma 4 may be strict, even for the case when the linear complementarity problem (13) associated with (1) has a nondegenerate solution.

EXAMPLE 5

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad X = \mathbb{R}^2_+.$$

Then $\bar{X} = \{(1,1)\}$, $\bar{Y} = \{(x,1) \mid x \geq 1\}$ and $\bar{Z} = \{(x,1) \mid x \geq 0\}$.

If the monotone affine variational inequality (1) is nondegenerate in the sense that the corresponding LCP (13) has some nondegenerate solution, then by using the corresponding error bound (12) of theorem 1, the error bound (14) given above simplifies as follows.

THEOREM 6

(*Error bound for nondegenerate monotone affine variational inequality*)

Let M be positive semidefinite, $\gamma \geq 0$ and let (1) be solvable with $\hat{x}$ being *some* nondegenerate solution of (1), that is, $\hat{x}$ and a corresponding $\hat{u}$ solve (13) such that

$$\hat{x} + M\hat{x} - A^{\mathrm{T}}\hat{u} + q > 0, \quad \hat{u} + A\hat{x} - b > 0. \tag{31}$$

Suppose one of the following three conditions holds:

(i) $\gamma > 0$, or

(ii) $Mx - A^{\mathrm{T}}u + q \geq 0$ for some $u \geq 0$, or

(iii)

$$Ar \geq 0, \quad r \geq 0, \quad (M + M^{\mathrm{T}})r = 0\rangle \Rightarrow r = 0.$$

Then there exists a solution $\bar{x}(x)$ of (1) such that

$$\|x - \bar{x}(x)\| \leq \tau(M, A, q, b)t(x) + \|y(x) - x\|, \tag{32}$$

where

$$y(x) \in \arg\min_{y \in X} (y - x)^{\mathrm{T}}(My + q) + \frac{\gamma}{2}\|y - x\|^2,$$

$$t(x) := \|([M + \gamma I]^{\mathrm{T}}(y(x) - x), -y(x)^{\mathrm{T}}[M + \gamma I]^{\mathrm{T}}(y(x) - x))_+\|,$$

and

$$\tau(M, A, q, b) := \sqrt{k} \sup_{(u,v,\xi)} \left\{ \|u, v, \xi\| \;\middle|\; \begin{array}{c} \|N^{\mathrm{T}}u + v - d\xi\|_1 = 1 \\ (u, v, \xi) \geq 0 \\ \text{Columns of } [N^{\mathrm{T}}\, I\, d], \\ \text{corresponding to nonzero} \\ \text{elements of } (u, v, \xi), \\ \text{are linearly independent} \end{array} \right\},$$

with N, p and k defined by (18).

Remark

Theorems 3 and 6 can be specialized to the monotone linear complementarity problem by taking X to be the nonnegative orthant in $\mathbb{R}^n$. In this case, $y(x)$ as defined by (3) simplifies to

$$y(x) \in \arg\min_{y \geq 0} (y - x)^{\mathrm{T}}(My + q) + \frac{\gamma}{2}\|y - x\|^2.$$

We now derive a strong upper semicontinuity result for the nondegenerate monotone affine variational inequality problem based on a similar result for the monotone linear complementarity problem [6].

THEOREM 7

(*Strong upper semicontinuity of nondegenerate monotone affine variational inequalities*)

Let M be positive semidefinite and let (1) be solvable with $\hat{x}$ being *some* nondegenerate solution of (1), that is, $\hat{x}$ and a corresponding $\hat{u}$ solve (13) and such that (31) holds. Let $\{x^i\}$ be a sequence in X converging to an $\bar{x} \in \bar{X}$ such that there exist $\{u^i\}$ satisfying

$$Mx^i - A^T u^i + q \geq 0, \quad u^i \geq 0. \tag{33}$$

Then for sufficiently large i, $y^i \in \bar{X}$, where (y^i, u^i) solve

$$\min_{y,u}\{((M+M^T)x^i + q)^T y - b^T u | My - A^T u + q \geq 0, \quad Ay \geq b, \quad y, u \geq 0\}. \tag{34}$$

Proof

By corollary 14 of [6], there exists an $\bar{i}$, such that for all $i \geq \bar{i}$

$$\arg\min_{y,u}\{((M+M^T)x^i + q)^T y - b^T u | My - A^T u + q \geq 0,$$

$$Ay \geq b, \quad y, u \geq 0\} \subset SOL(N, p),$$

where $SOL(N,p)$ is the solution set of (13). That is, any solution of a linearization of the quadratic program

$$\min_{y,u}\{y^T My + q^T y - b^T u | My - A^T u + q \geq 0, \quad Ay \geq b, \quad y, u \geq 0\} \tag{35}$$

associated with the LCP (13) around (x^i, u^i) yields an exact solution of (13) and hence x^i is an exact solution of (1). □

Remark

It is interesting to note that since the quadratic program (35) is linear in u, the sequence $\{u^i\}$ is not needed in solving the linearization (34). From a practical point of view, this leads to finite termination of any convergent algorithm for solving (1) in the nondegenerate case by periodically solving the linear program of (34).

4. Conclusion

We have presented error bounds for the monotone affine variational inequality problem, both for the degenerate and nondegenerate cases. These bounds may potentially be useful in constructing an iterative quadratic programming algorithm for solving affine variational inequalities. We have also derived a strong upper semicontinuity results for the nondegenerate monotone affine variational inequality problem which may be useful for finitely terminating any convergent algorithm by periodically solving a linear program.

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