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On Cournot-Nash-Walras equilibria and their computation

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Abstract This paper concerns a model of Cournot-Nash-Walras (CNW) equilibrium where the Cournot-Nash concept is used to capture equilibrium of an oligopolistic market with non-cooperative players/firms who share a certain amount of a so-called rare resource needed for their production, and the Walras equilibrium determines the price of that rare resource. We prove the existence of CNW equilibria under reasonable conditions and examine their local stability with respect to small perturbations of problem data. In this way we show the uniqueness of CNW equilibria under mild additional requirements. Finally, we suggest some efficient numerical approaches and compute several instances of an illustrative test example.

Keywords Cournot-Nash-Walras equilibrium · Existence · Stationarity Conditions · Stability · MOPEC

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1 Introduction and preliminaries

Recently, S.D. Flåm investigated markets where the players/firms behave noncooperatively and some of their inputs are limited but transferable. These so-called rare resources are controlled by some national or international authority that typically provides each agent with some initial endowment of these resources. Examples of such rare resources include fish quotas or rights to water usage. In the same way one can also handle production allowances or pollution permits. Since these rare resources are transferable, after the initial allocation they may be bought or sold in a market. This eventually leads to a Walras equilibrium specifying the equilibrium price of a unit of the rare resources. This price is either nil, in the case when the available amount of rare resources exceeds the interests of the market, or it is nonnegative provided that the demand amounts exactly to the available quantity. In either case, the initial endowments can be reallocated, which leads to a joint improvement. In [11] the author speaks about Nash-Walras equilibria and divides the process of their finding into two phases. In the first one, the agents compute a Nash equilibrium corresponding to their initial endowments. In the second phase, the agents approach a Nash-Walras equilibrium step by step by bilateral exchanges of their shares of rare resources so that the overall amount of them remains unchanged. In this way the author attempts to model real processes leading to an equilibrium price of the rare resources.

In contrast to this approach, in this paper we look at this problem from a slightly different perspective. The authority controlling the rare resources might, in reality, be interested in computing a Nash-Walras equilibrium in one step in order to get a feedback about the influence of the initial allocation on the overall production and the price of the rare resources. Likewise a firm might wish to learn how a change in technology (leading to a different rate of consumption of the rare resource) or a change of other production costs would influence his profit. So, in this paper, we suggest a procedure for computing a Nash-Walras equilibrium in one step, without any phases and evolutionary processes. Since our agents are firms and behave according to the Cournot-Nash concept, we prefer to use the terminology Cournot-Nash-Walras (CNW) equilibrium in the sequel.

The plan of the paper is as follows: In Section 2 we formulate the problem, collect the standing assumptions and analyze some elementary properties. Section 3 proves the existence of a CNW equilibrium. Our proof differs from the existence proof in [11] because it does not use the notion of a normalized equilibrium and the associated existence results from [22], [13]. Moreover, this approach enables us to weaken the convexity assumptions from [11]. Instead of the existence of a CNW equilibrium one has then, however, only the existence of a CNW stationary point. Section 4 is devoted to local stability of CNW equilibria. It turns out that under relatively mild assumptions CNW equilibria are unique and depend on the problem data in a Lipschizian way. The computation of CNW equilibria amounts to solving a specially structured variational inequality with a polyhedral constraint set. Apart from many universal numerical methods, one can thus make use of non-smooth Newton methods such as PATH [4,10], based on successively solving affine variational inequalities using techniques outlined in [3]. This is explained in Section 5, where one finds also an illustrative example, based on an adaptation of the five-firm oligopolistic market from [16].

Our notation is basically standard. For a closed cone K with vertex at 0, K^0 denote its negative polar and for a set A, $\mathrm{dist}_A(x)$ stands for the distance of x to A. Given a multifunction $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m], \mathrm{Gr} F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in F(x)\}$ is the graph of F and $\mathbb{B}$ denotes the unit ball.

We conclude the introductory section with the definitions of some basic notions from modern variational analysis which will be extensively used in this paper.

Consider a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$. We define the *contingent (Bouligand) cone* to A at $\bar{x}$ as the cone

$$\begin{aligned} T_A(\bar{x}) :=& \operatorname*{Lim\,sup}_{\tau \downarrow 0} \frac{A - \bar{x}}{\tau} \\ =& \{h \in \mathbb{R}^n | \exists h_k \to h, \lambda_k \searrow 0 \text{ such that } \bar{x} + \lambda_k h_k \in A \text{ for all } k\} \end{aligned}$$

and the *regular (Fréchet) normal cone* to A at $\bar{x}$ as $\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^0$. Moreover, the *limiting (Mordukhovich) normal* cone to A at $\bar{x}$ is defined by

$$\begin{aligned} N_A(\bar{x}) :=& \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}(x) \\ =& \{x^* \in \mathbb{R}^n | \exists\, x_k \xrightarrow{A} \bar{x}, x_k^* \to x^* \text{ such that } x_k^* \in \widehat{N}_A(x_k) \text{ for all } k\}. \end{aligned}$$

We say that A is *(normally) regular* at $\bar{x}$ provided $N_A(\bar{x}) = \widehat{N}_A(\bar{x})$. Convex sets are regular at all points. Now consider a closed-graph multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ and a point $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, \Phi$.

The multifunction $D^*\Phi(\bar{x},\bar{y})[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ defined by

$$D^*\Phi(\bar{x},\bar{y})(y^*) := \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N_{\mathrm{Gr}\Phi}(\bar{x},\bar{y})\}$$

is the *limiting (Mordukhovich) coderivative* of Φ at $(\bar{x},\bar{y})$. If Φ is single-valued and continuously differentiable then $\bar{y} = \Phi(\bar{x})$ and $D^*\Phi(\bar{x},\bar{y})$ amounts to $(\nabla \Phi(\bar{x}))^\top$.

An interested reader may find a full account of properties of the above notions for example in the monographs [21] and [15].

2 Problem formulation

Consider an oligopolistic market with m firms, each of which produces a homogeneous commodity. As mentioned in the introduction, they each need a certain amount of a *rare resource* for this production, that is dependent on the technology that is used. It follows that each firm optimizes his profit by using two strategies: his production and the amount of the rare resource that he intends to purchase or to sell. Consequently, the ith firm solves the *profit maximization problem*

$$\begin{array}{ll} \text{maximize} & p(T)y_i - c_i(y_i) - \pi x_i \\ \text{subject to} & \\ & (y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i, \end{array} \tag{1}$$

where y_i is the production, x_i is the amount of the rare resource that is purchased (or sold), $c_i[\mathbb{R}_+ \to \mathbb{R}_+]$ specifies the production costs, π is the price of the rare resource, $T = \sum_{i=1}^m y_i$ signifies the overall amount of the produced commodity in the market and $A_i = [a_i, b_i]$ specifies the production bounds. The function $p[\text{int}\, \mathbb{R}_+ \to \mathbb{R}_+]$ assigns each amount T the price at which (price-taking) consumers are willing to demand. It is usually called the *inverse demand curve*. The relationship between y_i and the required amount of the rare resource is reflected via the set

$$\mathcal{B}_i = \{(y_i, x_i) | q_i(y_i) \leq x_i + e_i\},$$

where e_i is the initial endowment of the rare resource and $q_i[\mathbb{R}_+ \to \mathbb{R}_+]$ is a (technological) function assigning to each production value the corresponding amount required of the rare resource. Denote by Ξ the overall available amount of the rare resource so that

$$\Xi \geq \sum_{i=1}^m e_i. \tag{2}$$

Observe that in problem (1) the variables $y_j, j \neq i$, and π play the role of parameters.

Unless stated otherwise, throughout the whole paper we will impose the following assumptions:

A1: All functions c_i can be extended to open intervals containing the sets A_i. These extensions are convex and twice continuously differentiable.
A2: p is strictly convex and twice continuously differentiable on int $\mathbb{R}_+$.
A3: $\alpha p(\alpha)$ is a concave function of α.
A4: For all i one has $0 \leq a_i < b_i$ and there is an index i_0 such that $a_{i_0} > 0$.
A5: All functions q_i satisfy $q_i(0) = 0$ and can be extended to open intervals containing the sets A_i. These extensions are convex, increasing and twice continuously differentiable.
A6: One has

$$\sum_{i=1}^m q_i(a_i) < \Xi.$$

A7: $\pi \geq 0$.

The assumptions A1 - A3 are not too restrictive and arise in a similar form in various treatments of oligopolistic markets, cf. [16], [18], [19]. They ensure in particular that the objective in (1) is concave for all i. Assumption A4 ensures that $p(T)$ is well-defined. Assumptions A5 and A6 are related to the rare resource and play an important role in the existence proof in the next section. The economic interpretation of A6 says that the overall amount of the rare resource is sufficient for all firms to run their productions at their lower bounds. Finally, A7 is natural.

Since in the sequel we will extensively employ various tools of modern variational analysis, tailored to minimization problems, from now on we will replace profit maximization problems (1) by the corresponding minimization problems with the objectives

$$J_i(\pi, y, x_i) := c_i(y_i) + \pi x_i - p(T)y_i,\ i = 1, \dots, m.$$

Further, to simplify the notation, $y = (y_1, y_2, \dots, y_m)$ and $x = (x_1, x_2, \dots, x_m)$ stand for the vectors of cumulative strategies y_i, x_i of all firms. To introduce the CNW equilibrium, we define first the Cournot-Nash equilibrium generated by problems (1).

Definition 1 The strategy pair $(\bar{y}, \bar{x})$ is a *Cournot-Nash equilibrium* in the considered market for a given $\pi \geq 0$ provided for all i one has

$$J_i(\pi, \bar{y}, \bar{x}_i) = \min_{(y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i} J_i(\pi, \bar{y}_i, \bar{y}_2, \dots, \bar{y}_{i-1}, y_i, \bar{y}_{i+1}, \dots, \bar{y}_m, x_i). \quad (3)$$

Remark 1 If the constraint (2) is neglected and all endowments e_i vanish (so that we do not consider a “rare” resource), then we may put $x_i = q(y_i)$, the production costs become $c_i(y_i) + \pi q_i(y_i)$ and the constraint set in (1) can be simplified to $y_i \in A_i$. Definition 1 then amounts to the classical notion of Cournot (or Cournot-Nash) equilibrium from 1838, cf. [16]. For this reason we use the terminology Cournot-Nash equilibrium also in our slightly more complex case reflecting the above described mechanism of trading with the rare resource.

Remark 2 In [11] the author assumes that the production cost functions c_i also depend on x_i.

Definition 2 *(Flåm)* The triple $(\bar{\pi}, \bar{y}, \bar{x})$ is a *Cournot-Nash-Walras* (CNW) equilibrium in the considered market provided that

(i) $(\bar{y}, \bar{x})$ is a Cournot-Nash equilibrium for $\pi = \bar{\pi}$, and
(ii) one has
$$\bar{\pi} \geq 0,\ \Xi - \sum_{i=1}^{m}(e_i + \bar{x}_i) \geq 0,\ \bar{\pi} \cdot (\Xi - \sum_{i=1}^{m}(e_i + \bar{x}_i)) = 0.$$

Clearly, the conditions in (ii) characterize a Walras equilibrium with respect to the rare resource which determines a price $\bar{\pi}$ under which the (secondary) market with the rare resource is cleared. From the point of view of the firms, the computation of $\bar{\pi}$ is a dynamical process starting after the initial allocation has been conducted. From the point of view of the authority controlling the rare resource, however, the whole problem can be solved in one step. The results provide the authority with information about the influence of the initial allocation on the CNW equilibrium.

This model covers also the possibility that some agents in the considered market do not intend to produce anything. Consequently, for them both lower and upper production bounds vanish and so assumption A4 is not fulfilled. Nevertheless, their presence does not cause any problems: It suffices to put Ξ to be the sum of endowments of the true oligopolists plus the sum of endowments of the non-producing agents, while in all remaining parts of the model only the true oligopolists are considered. This is due to the fact that the non-producing agents do not perform any optimization.

The Cournot-Nash equilibrium from Definition 1 can easily be characterized via standard stationarity/optimality conditions. For the readers' convenience we state this result here with a proof.

Proposition 1 *Given a price $\pi \geq 0$, under the posed assumptions, a pair $(\bar{y}, \bar{x})$ is a Cournot-Nash equilibrium in the sense of Definition 1 if and only if it fulfills the relations*

$$0 \in \begin{bmatrix} \nabla c_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \pi \begin{bmatrix} \nabla q_1(y_1) \\ \vdots \\ \nabla q_m(y_m) \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i}(y_i) \tag{4}$$

$$\begin{aligned} &\pi \cdot (q_i(y_i) - x_i - e_i) = 0, \quad i = 1, 2, \ldots, m. \\ &q_i(y_i) \leq e_i + x_i, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{5}$$

Proof The constraints in (3) satisfy the linear independence constraint qualification (LICQ) due to A4. Moreover, the standing assumptions ensure that the functions J_i are jointly convex in (y_i, x_i). The Cournot-Nash equilibria are henceforth characterized by the standard first-order optimality conditions for the single optimization problems (3). Putting them together, we obtain the *generalized equation* (GE)

$$0 \in \begin{bmatrix} \nabla c_1(y_1) - y_1 \nabla p(T) - p(T) \\ \pi \\ \vdots \\ \nabla c_m(y_m) - y_m \nabla p(T) - p(T) \\ \pi \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i \times \mathbb{R}}(y_i, x_i) + \begin{bmatrix} \lambda_1 \nabla q_1(y_1) \\ -\lambda_1 \\ \vdots \\ \lambda_m \nabla q_m(y_m) \\ -\lambda_m \end{bmatrix}, \tag{6}$$

where $\lambda_1, \ldots, \lambda_m$ are nonnegative Lagrange multipliers associated with the inequalities defining the sets $\mathcal{B}_i$. They must fulfill the complementarity conditions

$$\lambda_i(q_i(y_i) - x_i - e_i) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{7}$$

Since $N_{\mathbb{R}}(x_i) = \{0\}$ for all i, we immediately conclude that

$$\pi = \lambda_1 = \lambda_2 = \ldots = \lambda_m. \tag{8}$$

In this way we arrive at the simplified (but equivalent) conditions (4), (5) in which only the partial derivatives $\nabla_{y_i} J_i$ arise.

In numerous applications the technological functions q_i may not fulfill the convexity requirement in A5 because, e.g., $\nabla q_i(\cdot)$ is a decreasing function. In this case, conditions (4) and (5) are only necessary for a pair $(\bar{y}, \bar{x})$ to be a Cournot-Nash equilibrium for a given π.

3 Existence of CNW equilibria

To simplify the proof, let us associate with the ith firm, instead of (1), a different problem, namely

$$\begin{array}{ll} \text{minimize} & c_i(y_i) + \pi(q_i(y_i) - e_i) - p(T)y_i \\ \text{subject to} & \\ & y_i \in A_i \end{array} \tag{9}$$

solely in the variable y_i. It corresponds to replacing the inequality

$$q_i(y_i) \leq x_i + e_i$$

by an equality so that variable x_i can be completely eliminated. If we replace the functions J_i in Definition 1 by the objectives from (9), we obtain a different non-cooperative equilibrium characterized by the GE

$$0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \mathop{\times}_{i=1}^{m} N_{A_i}(y_i). \tag{10}$$

Lemma 1 *Let $\bar{y}$ satisfy condition (10). Then the pair $(\bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i$ for all i is a Cournot-Nash equilibrium in the sense of Definition 1. Conversely, for each solution $(\bar{y}, \bar{x})$ of system (4), (5), the component $\bar{y}$ fulfills GE (10) whenever $\pi > 0$.*

The proof follows immediately from the comparison of GE (10) with the conditions (4), (5).

Denote by $S[\mathbb{R}_+ \rightrightarrows \mathbb{R}^m]$ the mapping which assigns each $\pi \geq 0$ the set of solutions to GE (10). The statement of Lemma 1 can then be written down as follows:

(i) For any $\pi \geq 0$ one has the implication

$$y \in S(\pi), x_i = q_i(y_i) - e_i \text{ for all } i \;\Rightarrow\; (y, x) \text{ fulfills conditions (4), (5).}$$

(ii) For $\pi > 0$ the above implication becomes equivalence.

Remark 3 It follows from Lemma 1 that for $\pi \geq 0$ the initial endowment $e_1, \ldots, e_m$ does not influence the component y of the Cournot-Nash equilibrium pair (y, x).

Lemma 2 *There is a positive real L such that in all CNW equilibria one has $\pi \leq L$.*

Proof Assume that $\bar{\pi} > 0$ is so large that

$$\min_{i=1,\dots,m} \left\{ \nabla c_i(a_i) - a_i \nabla p \left(\sum_{i=1}^m a_i \right) - p \left(\sum_{i=1}^m a_i \right) + \bar{\pi} \nabla q_i(a_i) \right\} > 0. \tag{11}$$

By virtue of (11) it follows that the stationarity condition (4) can be fulfilled only in the case when $y_i = a_i$ for all i. Indeed, since the functions J_i are convex in variables (y_i, x_i), their partial derivatives with respect to y_i are nondecreasing, and so for $y_i \geq a_i$ the quantities $\nabla c_i(y_i) - y_i \nabla p(T) - p(T) + \bar{\pi} \nabla q_i(y_i)$ are positive as well. It follows that $y_i = a_i$ for all i in order to bring the normal cones to A_i into play. This means that the respective values of x_i are given by $x_i = q_i(a_i) - e_i$ and thus, thanks to assumption A6, the corresponding excess demand $\sum_{i=1}^m (e_i + x_i) - \varXi$ is negative, which contradicts the complementarity condition of the Walras equilibrium. As L we can thus choose any positive real satisfying inequality (11) with $\bar{\pi}$ replaced by L.

On the basis of Lemmas 1 and 2 we are now able to state our main existence result.

Theorem 1 *Under the posed assumptions there is a CNW equilibrium.*

Proof Define the mapping $Q[\mathbb{R}^m \to \mathbb{R}]$ by

$$Q(y) := \sum_{i=1}^m q_i(y_i).$$

By virtue of Lemma 1 it suffices to show the existence of a pair $(\bar{\pi}, \bar{y})$ which solves the (aggregated) GE

$$\left.\begin{aligned} &0 \in \varXi - Q(y) + N_{\mathbb{R}_+}(\pi) \\ &0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \mathop{\times}_{i=1}^m N_{A_i}(y_i) \end{aligned}\right\} \tag{12}$$

in variables (π, y). Thanks to Lemma 2, $\mathbb{R}_+$ in the first line of (12) can be replaced by a bounded interval $[0, L]$. In this way, one obtains a variational inequality with a bounded constraint set which possesses a solution $(\bar{\pi}, \bar{y})$ as a consequence of the Brouwer Fixed Point Theorem. It follows that $(\bar{\pi}, \bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i$ is a CNW equilibrium.

If the functions q_i are not convex, then the whole above argumentation remains valid provided that in Lemma 2 we replace the expression on the left-hand side of (11) by

$$\min_{i=1,\dots,m} \min_{y_i \in A_i} \left\{ \nabla c_i(y_i) - y_i \nabla p \left(\sum_{i=1}^m y_i \right) - p \left(\sum_{i=1}^m y_i \right) + \bar{\pi} \nabla q_i(y_i) \right\} > 0. \tag{13}$$

Note that the second minimum on the left-hand side of (13) is attained by the boundedness of intervals A_i and by assumptions A1 and A2. Moreover, by increasing π, the validity of inequality (13) can be ensured due to positivity of $\nabla q_i(y_i)$ for all i.

As mentioned above, in case of nonconvex functions q_i, GE (10) is not a characterization but only a stationarity condition for the Cournot-Nash equilibria generated by problems (9). In Theorem 1 we thus do not prove the existence of CNW equilibria, but only the existence of points satisfying a stationarity condition for CNW equilibria.

Given such a point $(\bar{\pi}, \bar{y})$ solving (12), one can check whether, e.g., the second-order sufficient conditions for the (nonconvex) optimization problems (9) are fulfilled. In that case, $(\bar{\pi}, \bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i, i = 1, \ldots, m$, is a CNW equilibrium.

In the above existence proof we have not fully employed assumptions A1-A3. In fact, they might be replaced by the (weaker) requirement that functions $\varphi_i[\mathbb{R}^m \to \mathbb{R}]$ defined by

$$\varphi_i(y) := c_i(y_i) - p(T)y_i, i = 1, \ldots, m,$$

can be extended to an open neighborhood of $\mathsf{X}_{i=1}^m A_i$ and these extensions are convex and continuously differentiable.

Thus, the optimization problems (3) could be, for instance, linear programs. On the other hand, assumptions A1-A4 play an important role in the next section devoted to analysis of the properties of CNW equilibria.

4 Properties of CNW equilibria

Throughout this section a crucial role is played by the strong monotonicity of the operator

$$G(y) := \begin{bmatrix} \nabla c_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix},$$

which has been proved in [18, Lemma 12.2] under A1-A4. This operator arises in the GE characterizing the standard Cournot-Nash equilibrium in markets without the rare resource.

Proposition 2 *The set of solutions to (12) is closed and convex.*

Proof By virtue of [21, Example 12.48] it suffices to prove that the single-valued part of GE (12) is a monotone operator relative to $\mathbb{R}_+ \times \mathsf{X}_{i=1}^m A_i$. By invoking [17, Theorem 5.4.3 (a)] this is ensured provided the symmetric matrix $\frac{1}{2}[D(\pi, y) + (D(\pi, y))^T]$ with

$$D(\pi, y) := \begin{bmatrix} 0 & -\nabla q_1(y_1) & \ldots & -\nabla q_m(y_m) \\ \nabla q_1(y_1) & & & \\ \vdots & & \nabla_y H(\pi, y) & \\ \nabla q_m(y_m) & & & \end{bmatrix} \quad (14)$$

is positive semidefinite over $\mathbb{R}_+ \times \bigtimes_{i=1}^m A_i$. In (14), H stands for the mapping defined by

$$H(\pi, y) := G(y) + \pi \begin{bmatrix} \nabla q_1(y_1) \\ \vdots \\ \nabla q_m(y_m) \end{bmatrix}. \tag{15}$$

Clearly,

$$\frac{1}{2}[D(\pi, y) + (D(\pi, y))^T] = \begin{bmatrix} 0 \; 0 & \cdots & 0 \\ 0 & & \\ \vdots & \frac{1}{2}[\nabla_y H(\pi, y) + (\nabla_y H(\pi, y))^T] & \\ 0 & & \end{bmatrix}.$$

The matrix $\frac{1}{2}[\nabla G(y) + (\nabla G(y))^T]$ is positive definite due to [18, Lemma 12.2]. Under A5 the second matrix in (15) is symmetric positive definite as well and so the proof is complete.

Note that in the above statement the convexity of functions $q_i, i = 1, \ldots, m$, is not needed.

The previous statement can very well be combined with the local stability results derived next. Assume that we are given a pair $(\bar{\pi}, \bar{y})$ solving GE (12) and consider the local behavior of the multifunction $\Psi[\mathbb{R} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{m+1}]$, defined by

$$\Psi(\pi, y) := \begin{bmatrix} \Xi - Q(y) \\ \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + N_{\mathbb{R}_+ \times \bigtimes_{i=1}^m A_i}(\pi, y), \tag{16}$$

around $(\bar{\pi}, \bar{y}, 0) \in \mathrm{Gr}\, \Psi$. Denoting by Σ the inverse of Ψ and picking a point $(u, v) \in \mathbb{R} \times \mathbb{R}^m, \Sigma(u, v)$ amounts to the set of solutions to GE (12), where $(0, 0) \in \mathbb{R} \times \mathbb{R}^m$ on the left-hand side is replaced by (u, v). One speaks about *canonical perturbations* of (12).

A multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^l]$ is called *strongly metrically regular* at $(\bar{b}, \bar{a}) \in \mathrm{Gr}\, \Phi$, provided Φ^{-1} has a Lipschitz single-valued localization s around $(\bar{a}, \bar{b})$, i.e., there are neighborhoods $\mathcal{U}$ of $\bar{a}$, $\mathcal{V}$ of $\bar{b}$ and a Lipschitz single valued mapping $s[\mathcal{U} \to \mathbb{R}^m]$ such that

$$\bar{b} = s(\bar{a}) \text{ and } \Phi^{-1}(a) \cap \mathcal{V} = \{s(a)\} \text{ for all } a \in \mathcal{U}.$$

It turns out that under relatively mild assumptions Ψ is strongly metrically regular at $(\bar{\pi}, \bar{y}, 0)$ whenever $\bar{\pi} > 0$.

Theorem 2 *Let $(\bar{\pi}, \bar{y})$ be a solution of GE (12) and assume that $\bar{\pi} > 0$ and $\bar{y}_i \in \mathrm{int} A_i$ for at least one $i \in \{1, 2, \ldots, m\}$. Then Ψ is strongly metrically regular at $(\bar{\pi}, \bar{y}, 0_{\mathbb{R}^{m+1}})$, i.e., Σ has a Lipschitz single-valued localization around $(0_{\mathbb{R}^{m+1}}, \bar{\pi}, \bar{y})$.*

Proof By combining the results in [6, Theorem 3G4] and [5, Theorem 1], and applying the Mordukhovich criterion to ensure the metric regularity of Ψ at $(\bar{\pi}, \bar{y}, 0)$ [15, Corollary 4.61], it suffices to prove that the GE

$$0 \in (D(\bar{\pi}, \bar{y}))^T \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_m \end{bmatrix} + \begin{bmatrix} D^* N_{\mathbb{R}_+}(\bar{\pi}, P_0(\bar{\pi}, \bar{y}))(z_0) \\ D^* N_{A_1}(\bar{y}_1, P_1(\bar{\pi}, \bar{y}))(z_1) \\ \vdots \\ D^* N_{A_m}(\bar{y}_m, P_m(\bar{\pi}, \bar{y}))(z_m) \end{bmatrix} \tag{17}$$

has only the trivial solution $(z_0, z_1, \ldots, z_m) = 0$. In (17), $P(\pi, y) := (P_0(\pi, y), P_1(\pi, y), \ldots, P_m(\pi, y))$ denotes the single-valued mapping on the right-hand side of (16). It follows from $\bar{\pi} > 0$ that the first component of the multi-valued part of (17) vanishes so that, with $\tilde{z} := (z_1, \ldots, z_m)$, GE (17) amounts to the system

$$0 = \langle \nabla Q(\bar{y}), \tilde{z} \rangle \tag{18}$$

$$0 \in \nabla Q(\bar{y}) z_0 + (\nabla_y H(\bar{\pi}, \bar{y}))^T \tilde{z} + D^* N_{\bigtimes_{i=1}^m A_i}(\bar{y}, -\tilde{P}(\bar{\pi}, \bar{y}))(\tilde{z}), \tag{19}$$

where $\tilde{P}(\pi, y) = (P_1(\pi, y), \ldots, P_m(\pi, y))$.

Premultiplying GE (19) by $\tilde{z}^T$, we obtain that

$$\begin{aligned} 0 &= \langle \tilde{z}, \nabla Q(\bar{y}) z_0 \rangle + \langle \nabla_y H(\bar{\pi}, \bar{y}) \tilde{z}, \tilde{z} \rangle + \langle \tilde{z}, d \rangle \\ d &\in D^* N_{\bigtimes_{i=1}^m A_i}(\bar{y}, -\tilde{P}(\bar{\pi}, \bar{y}))(\tilde{z}). \end{aligned} \tag{20}$$

The first term on the right-hand side of (20) amounts to zero due to (18). Further we note that $\langle \tilde{z}, d \rangle \geq 0$ which follows from the well-known result in [20, Theorem 2.1] because of the maximal monotonicity of the normal-cone mapping to a convex set. Since $\nabla_y H(\bar{\pi}, \bar{y})$ is positive definite by virtue of [18, Lemma 12.2] and by assumption A5, we conclude that $\tilde{z} = 0$ and (19) reduces thus to

$$0 = \nabla Q(\bar{y}) z_0 + \bigtimes_{i=1}^m D^* N_{A_i}(\bar{y}_i, -\tilde{P}_i(\bar{\pi}, \bar{y}))(0).$$

By the assumption there is an index $i_0 \in \{1, 2, \ldots, m\}$ such that $\bar{y}_{i_0} \in \mathrm{int} A_{i_0}$ and, consequently, $\tilde{P}_{i_0}(\bar{\pi}, \bar{y}) = 0$. It follows that $D^* N_{A_{i_0}}(\bar{y}_{i_0}, -\tilde{P}_i(\bar{\pi}, \bar{y}))(0) = \{0\}$ as well and, since $\nabla q_0(\bar{y}_{i_0}) > 0$, one has that $z_0 = 0$. The statement has been established.

If $\bar{\pi} = 0$, then Ψ enjoys a somewhat weaker stability property. A multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^l]$ is called *strongly metrically subregular* at $(\bar{b}, \bar{a}) \in \mathrm{Gr}\, \Phi$, provided Φ^{-1} has the *isolated calmness property* at $(\bar{a}, \bar{b})$, i.e., there are neighborhoods $\mathcal{U}$ of $\bar{a}$, $\mathcal{V}$ of $\bar{b}$ and a modulus $\ell \geq 0$ such that

$$\Phi^{-1}(a) \cap \mathcal{V} \subset \{\bar{b}\} + \ell ||a - \bar{a}|| \mathbb{B}_{\mathbb{R}^n} \text{ for all } a \in \mathcal{U}.$$

Theorem 3 *Let $(\bar{\pi}, \bar{y})$ be a solution of GE (12) and assume that $\bar{\pi} = 0$ and either $\Xi - Q(\bar{y}) > 0$ or $\bar{y}_i \in \mathrm{int} A_i$ for at least one $i \in \{1, 2, \ldots, m\}$. Then Ψ is strongly metrically subregular at $(\bar{\pi}, \bar{y}, 0_{\mathbb{R}^{m+1}})$, i.e., Σ has the isolated calmness property at $(0_{\mathbb{R}^{m+1}}, \bar{\pi}, \bar{y})$.*

Proof By applying the criterion from [5, Theorem 4E.1] it suffices to prove that the GE

$$0 \in D(\bar{\pi}, \bar{y}) \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_m \end{bmatrix} + \begin{bmatrix} N_{K_0}(z_0) \\ N_{K_1}(z_1) \\ \vdots \\ N_{K_m}(z_m) \end{bmatrix} \tag{21}$$

has only the trivial solution $(z_0, \tilde{z}) := (z_0, z_1, \ldots, z_m) = 0$. In (21), the critical cones $K_0, K_i, i = 1, \ldots, m$, are given by

$$K_0 = T_{\mathbb{R}_+}(0) \cap \{P_0(\bar{\pi}, \bar{y})\}^{\perp}, K_i = T_{A_i}(\bar{y}_i) \cap \{P_i(\bar{\pi}, \bar{y})\}^{\perp}, i = 1, \ldots, m.$$

If $\Xi - Q(\bar{y}) > 0$, then $K_0 = \{0\}$ and so (21) amounts to the GE

$$0 \in \nabla_y H(\bar{\pi}, \bar{y})\tilde{z} + \bigtimes_{i=1}^{m} N_{K_i}(z_i). \tag{22}$$

Premultiplying GE (22) by $\tilde{z}^{\top}$, we obtain that

$$\langle \tilde{z}, \nabla_y H(\bar{\pi}, \bar{y})\tilde{z}) \rangle = 0,$$

because for all z_i and $d_i \in N_{K_i}(z_i), i = 1, \ldots, m$, one has $\langle z_i, d_i \rangle = 0$. Since $\nabla_y H(\bar{\pi}, \bar{y}) = \nabla G(\bar{y})$ is positive definite ([18, Lemma 12.2]), we conclude that $\tilde{z} = 0$ and the statement holds true.

If $\Xi - Q(\bar{y}) = 0$, then $K_0 = \mathbb{R}_+$. If $z_0 = 0$, we can proceed exactly as in the preceding case. So, let us assume that $z_0 > 0$. GE (21) amounts then to the system

$$\begin{aligned} \langle \nabla Q(\bar{y}), \tilde{z} \rangle &= 0 \\ 0 \in \nabla Q(\bar{y}) z_0 + \nabla_y H(\bar{\pi}, \bar{y})\tilde{z} &+ \bigtimes_{i=1}^{m} N_{K_i}(z_i). \end{aligned} \tag{23}$$

By the same argumentation as in the proof of the preceding case we detect that $\tilde{z}$ must vanish so that the 2nd line in (23) reduces to

$$0 \in \nabla Q(\bar{y}) z_0 + \bigtimes_{i=1}^{m} N_{K_i}(0).$$

Now it follows from the posed assumption that $K_i = \mathbb{R}$ for some i and, consequently,

$$\nabla q_i(\bar{y}) z_0 = 0.$$

By virtue of A5 this contradicts the positivity of z_0 and so the statement has been established.

Note again that, as in Proposition 2, the convexity of functions $q_i, i = 1, \ldots, m$, is not needed in the proofs of Theorems 2 and 3.

A combination of Theorem 2 and Proposition 2 yields the following:

Corollary 1 *Let* $(\bar{\pi}, \bar{y})$ *be a solution of GE (12) with* $\bar{\pi} > 0$ *and* $\bar{y}_i \in \text{int}A_i$ *for some* $i \in \{1, \ldots, m\}$*. Then* Σ *is single-valued and Lipschitz around* $(0, 0_{\mathbb{R}^m})$*.*

Proof Indeed, under the posed assumptions, by Theorem 2 there exist neighborhoods $\mathcal{U}$ of $(0, 0_{\mathbb{R}^m})$, $\mathcal{Z}$ of $(\bar{\pi}, \bar{y})$ and a single-valued and Lipschitz map $\sigma[\mathcal{U} \to \mathbb{R} \times \mathbb{R}^m]$ such that $\sigma(0,0) = (\bar{\pi}, \bar{y})$ and

$$\Sigma(u, v) \cap \mathcal{Z} = \{\sigma(u, v)\} \text{ for } (u, v) \in \mathcal{U}. \tag{24}$$

Since the monotonicity argument from the proof of Proposition 2 remains valid also under canonical perturbations, the sets $\Sigma(u, v)$ are convex. By virtue of (24), however, this is possible only when $\Sigma(u, v) = \sigma(u, v)$ over $\mathcal{U}$ and we are done.

From [6, Theorem 3G.4] we infer that the above property of Σ is inherited by all mappings which assign (π, y) to any scalar- or vector-valued parameter on which P depends in a continuously differentiable way (at the respective points). Likewise, under assumptions of Theorem 3, these mappings possess the isolated calmness property thanks to [6, Theorem 3I.12]. This could be, e.g., Ξ or any parameter arising in the functions p, c_i or q_i.

We conclude this section with the following uniqueness result.

Theorem 4 *(i) Consider the triple* $(\bar{\pi}, \bar{y}, \bar{x})$*, where* $(\bar{\pi}, \bar{y})$ *is a solution of GE (12) with* $\bar{\pi} > 0$*,* $\bar{y}_i \in \text{int}A_i$ *for some* $i \in \{1, \ldots, m\}$ *and* $\bar{x}_i = q_i(\bar{y}_i) - e_i$ *for all* $i = 1, \ldots, m$*. Then* $(\bar{\pi}, \bar{y})$ *is a unique solution of GE (12) and* $(\bar{\pi}, \bar{y}, \bar{x})$ *is a unique CNW equilibrium.*

(ii) Assume that $(0, \bar{y})$ *is a solution of GE (12), where* $\bar{y}_i \in \text{int}A_i$ *for some* $i \in \{1, \ldots, m\}$*. Then* $(0, \bar{y})$ *is a unique solution of GE (12).*

Proof The statement (i) follows easily from Corollary 1 and Lemma 1. To prove (ii), assume that $(\tilde{\pi}, \tilde{y})$ is a solution of GE (12) different from $(0, \bar{y})$. Assumptions A1-A4 imply, by virtue of [18, Lemma 12.2] that $\tilde{\pi} > 0$. Further, it follows that $\tilde{y}_i \in \text{bd } A_i$ for all i, because otherwise we had a contradiction with Corollary 1. Nevertheless, by Proposition 2, the pair $(\frac{\tilde{\pi}}{2}, \frac{\bar{y}}{2} + \frac{\tilde{y}}{2})$ is then also a solution of GE (12) and by the imposed assumptions

$$\left(\frac{\bar{y}_i}{2} + \frac{\tilde{y}_i}{2}\right) \in \text{int } A_i$$

for some $i \in \{1, \ldots, m\}$. This contradicts the statement (i) of this theorem and so the proof is complete.

5 Computation of CNW equilibria

This section is devoted to the numerical solution of GE (12), which provides us either directly with CNW equilibria (under A1-A7) or with stationary points in the sense explained in Section 3. Since the structure of the constraint set in GE (12) is relatively simple, there are a considerable number of efficient numerical methods that can be used for this purpose. Many of them can be found, e.g., in the monograph [7]. On the other hand, GE (12) amounts to a family of optimization problems (9) coupled with the complementarity constraint

$$0 \leq \Xi - Q(y) \perp \pi \geq 0,$$

and so it is an example of a MOPEC (multiple optimization problems coupled with equilibrium constraints), cf. [1].

One approach for solving this problem is to convert the MOPEC into a complementarity model, by replacing each optimization problem (9) by its first-order optimality conditions (10) and solving the resulting standard mixed complementarity using the PATH solver [4,10] for example. The PATH solver employs a non-smooth Newton method for the complementarity problem, solving a succession of piecewise linear approximations of the piecewise smooth complementarity system. In addition, a number of computational enhancements (to preprocess the model, to identify an active set and to perform the linear algebra in an efficient manner for large scale systems) are employed. In fact, the model (4) and (5) is also a MOPEC and can be directly processed by PATH. The results in both cases are identical, and are generated within the GAMS modeling system [2] using the extended mathematical programming tools [8].

Alternatively, instead of GE (12), one could consider the optimization problem

$$\begin{array}{ll} \text{minimize} & \pi \cdot (\Xi - Q(y)) \\ \text{subject to} & \\ & 0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i}(y_i), \\ & \pi \geq 0, \\ & \Xi - Q(y) \geq 0. \end{array} \tag{25}$$

It is easy to see that any solution of $(\bar{\pi}, \bar{y})$ of (25) such that the corresponding (optimal) objective value vanishes is a solution of GE (12). Problem (25) is a *mathematical program with equilibrium constraints* (MPEC), where π is the *control* and y is the *state variable*.

Note that the objective in (25) amounts to the so-called primal gap function which is frequently used in connection with complementarity problems, cf. [7]. For the numerical solution of (25) there is again a number of efficient universal techniques, see, e.g., [8], [14]. Apart from them, one can exploit the specific properties of the GE in (25) and apply the so-called *implicit programming approach* (ImP), cf. [18], [19], which amounts in this case to a decomposition of GE (12) with respect to variables π and y.

For numerical tests we have adopted an example of an oligopolistic market with 5 firms from [16], cf. also [18], [19]. Several instances of this example have been successfully solved in the MOPEC framework by using the PATH solver and via the MPEC reformulation by using a relaxation method [23], [14] and

Table 1 Parameter specification

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
q_i	1.36	1.5	1.48	1.5	1.4
c_i	10	8	6	4	2
K_i	5	5	5	5	5
β_i	1.2	1.1	1.0	0.9	0.8

a variant of the ImP approach from [19]. Thanks to the low dimensionality of this example ($m = 5$) all methods used worked well and reached the same solutions (with a sufficient accuracy) within seconds.

Example 1 Consider the oligopolistic market with five producers/firms supplying a quantity $y_i \in \mathbb{R}_+, i = 1, \ldots, 5$, of some homogeneous product on the market with the inverse demand function

$$p(T) = 5000^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}},$$

where γ is a positive parameter termed *demand elasticity*.

Let the production cost functions be of the form

$$c_i(y_i) = c_i y_i + \frac{\beta_i}{1+\beta_i} K_i^{-\frac{1}{\beta_i}} (y_i)^{\frac{1+\beta_i}{\beta_i}},$$

where c_i, K_i and $\beta_i,\ i = 1, \ldots, 5$, are positive parameters. Suppose that $q_1(y_1) = q_1 y_1 + \sqrt{y_1 + 1} - 1$ and the technological functions $q_i, i = 2, \ldots, 5$, are linear and in the form $q_i(y_i) = q_i y_i$. Table 1 specifies values of parameters q_i, c_i, K_i and $\beta_i, i = 1, \ldots, 5$. Further, let the demand elasticity $\gamma = 1.3$, assume initial endowments of the rare resource $e_i = 25$ for each firm i, $i = 1, \ldots, 5$, put $\Xi = \sum_{i=1}^{5} e_i$ and consider production bounds $A_i = [0, 30], i = 1, \ldots, 4$ and $A_5 = [1, 30]$.

Each production cost function is convex and twice continuously differentiable on some open set containing the feasible set of strategies of a corresponding player. The inverse demand curve is twice continuously differentiable on int $\mathbb{R}_+$, strictly decreasing, and convex. Observe that the so-called *industry revenue curve*

$$Tp(T) = 5000^{\frac{1}{\gamma}} T^{\frac{\gamma-1}{\gamma}}$$

is concave on int $\mathbb{R}_+$ for $\gamma \geq 1$. Thus, all assumptions A1 - A7 are satisfied, except that $q_1(y_1)$ is not convex in A5.

This basic setting of problem data corresponds to the case A in Table 2, where the achieved numerical results are displayed. In the subsequent cases some of the above specified problem data have been changed in order to illustrate the behavior of CNW equilibria.

Let us comment briefly on the influence of the performed data changes on the corresponding CNW equilibria. In case A, $\bar{\pi}$ is positive, all production lies within the production intervals and the first two firms sell certain amounts of the rare resource to the remaining ones. In case B, there is a sixth player in the market who does not produce anything but is endowed with 10 units of

Table 2 Production, profits and purchased rare resources

		Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
case A	$\pi = 6.484$					
	production	8.016	13.597	18.218	21.009	23.732
	profit	176.647	216.959	264.905	309.177	372.600
	purchased rare resource	-12.096	-4.604	1.962	6.513	8.224
case B	$\pi = 5.529$					
	production	9.225	14.955	19.723	22.516	24.899
	profit	156.614	198.451	248.355	294.131	354.654
	purchased rare resource	-10.256	-2.568	4.191	8.774	9.859
case C	$\pi = 7.381$					
	production	19.579	10.094	15.024	18.173	21.572
	profit	282.729	215.059	254.569	294.105	356.919
	purchased rare resource	5.163	-9.858	-2.765	2.259	5.201
case D	$\pi = 0$					
	production	21.218	28.081	32.345	33.790	32.664
	profit	67.210	125.581	186.056	237.492	272.578
	purchased rare resource	-12.430	3.145	2.870	5.685	0.729
case E	$\pi = 5.764$					
	production	0	16.215	20.608	23.132	25.342
	profit	144.097	220.921	274.314	321.432	383.849
	purchased rare resource	-25.000	-0.677	5.500	9.699	10.479
case F	$\pi = 6.446$					
	production	8.236	13.770	18.372	21.143	23.000
	profit	176.652	217.429	265.769	310.248	369.377
	purchased rare resource	-11.760	-4.345	2.190	6.715	7.200

the rare resource. Hence, $\Xi = \sum_{i=1}^{5} e_i + 10$. As a result, π decreases and the additional non-producing agent earns 55.29 units by selling his endowment to the firms 3-5. In case C, Firm 1 decreases his production costs ($c_1 = 5$). As a result, his production increases and, instead of selling some rare resource, this firms buys it. Consequently, π increases. In case D, upper bounds on production are increased to 35 and initial endowments of the rare resource are increased to 45. Consequently, the secondary market in the rare resource is not necessarily cleared (even though the solution found by PATH does clear) and $\bar{\pi} = 0$. In case E, the consumption of the rare resource of Firm 1 has increased ($q_1 = 4$). As a consequence, Firm 1 has to completely stop production and his profit amounts just to the income from the rare resource. This situation again leads to a decrease of π. Finally, in case F the upper bounds on production are lowered to 23. Firm 5 has to decrease production and π slightly decreases from Case A as well. Note that in all considered instances, apart from D, one has unique CNW equilibria, while in case D the uniqueness concerns only the price of the rare resource and the production. Since $q_1(y_1)$ is not convex, we check the optimality conditions of each firm to guarantee that we have found a minimizer and not just a stationary point in all cases above.

6 Concluding remarks

The stability results of Section 4 enable the authority, controlling the rare resource, for instance, to optimize the choice of Ξ via the MPEC

$$\begin{array}{ll} \text{minimize} & \mathcal{J}(\Xi, \pi, y) \\ \text{subject to} & \text{GE (12)}, \end{array} \tag{26}$$

where $\mathcal{J}$ is a suitable objective which correlates the amount of Ξ with the corresponding price π and production y. Likewise a firm, knowing the data of his competitors and the policy of the authority, controlling the rare resource, may optimize his investments into the manufacturing process taking into account the cost of the technological improvements. In this way we again obtain an MPEC with GE (12) among the constraints. The controls are then the parameters of the respective functions c_i and q_i.

A similar model can be constructed also in the case of multiple outputs and/or multiple rare resources. With the ith firm we associate then the technological functions

$$q_i^j(y_i^1, \ldots, y_i^n), j = 1, \ldots, k,$$

which specify the amount of the jth rare resource needed to produce the output vector $(y_i^1, \ldots, y_i^n) \in \mathbb{R}^n$. The monotonicity and convexity requirements in A5 can be replaced by suitable conditions in terms of Jacobians

$$\nabla q_i(y_i) = \begin{bmatrix} \nabla q_i^1(y_i) \\ \vdots \\ \nabla q_i^k(y_i) \end{bmatrix}, \; y_i = (y_i^1, \ldots, y_i^n), \; i = 1, \ldots, m,$$

over the production intervals.

Both above mentioned goals go beyond the scope of the current paper and may be addressed in a future work.

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