Complementarity Problems and Applications

CDGO 2005

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Mixture of equalities and inequalities

$$0 \leq x \quad \perp \quad F(x) \geq 0$$

$$x'F(x) = 0$$

$$x_i F_i(x) = 0, \quad i = 1, 2, \ldots, n$$

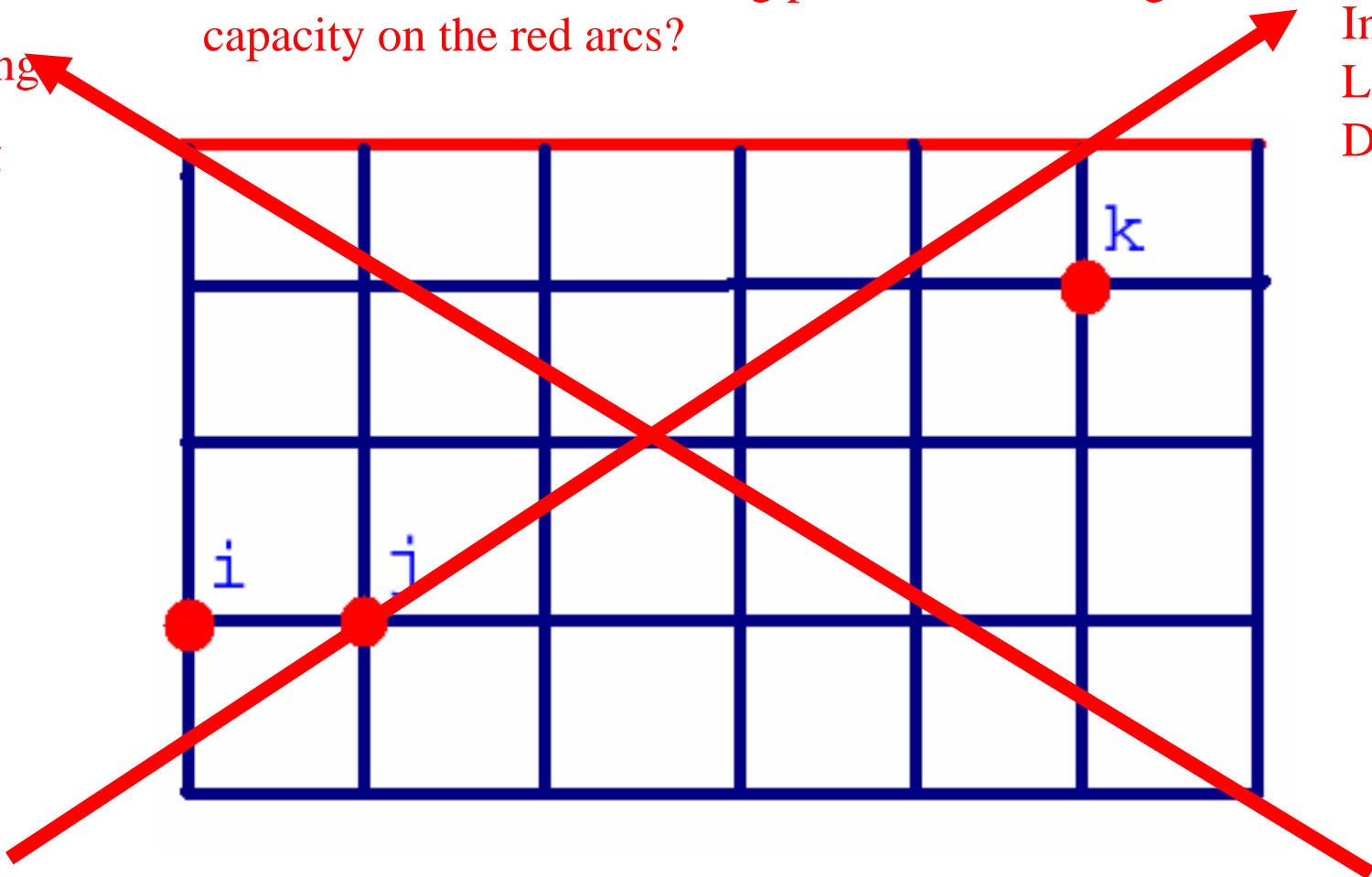either $x_i = 0$ or $F_i(x) = 0$

Thanks to…

- Todd Munson, Steve Dirkse
- Jong-Shi Pang
- Danny Ralph, Christian Kanzow

- Tom Rutherford, Alex Meeraus
- Andy Philpott, Jarrad Wallace
- Qian Li

Walras meets Wardrop

What is the effect on housing prices of increasing capacity on the red arcs?

Increasing Housing

Increasing Labor Demand

k

i   j

$$T_{ik} \leq c_{ij}(F_{ij}) + T_{jk} \perp x_{ij}^k \geq 0$$

$$F_{ij} = \sum_k x_{ij}^k$$

$$N_{jk} + \sum_i x_{ij}^k = \sum_i x_{ji}^k$$

x_{ij}^k = people travel (i,j) work at k

T_{ik} = (shortest) time i to k

N_{ik} = number live i work k

$$D_k(w_k) \leq N_{kk} + \sum_i x_{ik}^k \perp w_k \geq 0$$

$$V_{ik}(w_k, p_i, T_{ik}) \leq U \perp N_{ik} \geq 0$$

$$\bar{N} = \sum_{i,k} N_{ik}$$

w_k = labor cost at k

p_i = housing price at i

N_{ik} = number live i work k

$$T_{ik} \leq c_{ij}(F_{ij}) + T_{jk} \perp x_{ij}^k \geq 0$$

$$F_{ij} = \sum_k x_{ij}^k \qquad \perp F_{ij}$$

- Square
- Prices found
- Apply shocks

$$N_{jk} + \sum_i x_{ij}^k = \sum_i x_{ji}^k \quad \perp T_{jk}$$

$$D_k(w_k) \leq N_{kk} + \sum_i x_{ik}^k \perp w_k \geq 0$$

$$V_{ik}(w_k, p_i, T_{ik}) \leq U \perp N_{ik} \geq 0$$

- Tolling

$$\bar{N} = \sum_{i,k} N_{ik} \qquad \perp U$$

- Different housing types
- Different transportation types

$$\sum_k N_{ik} H_{ik}(w_k, p_i) \leq \bar{H}_i \perp p_i \geq 0$$

Equivalent Nonsmooth Map
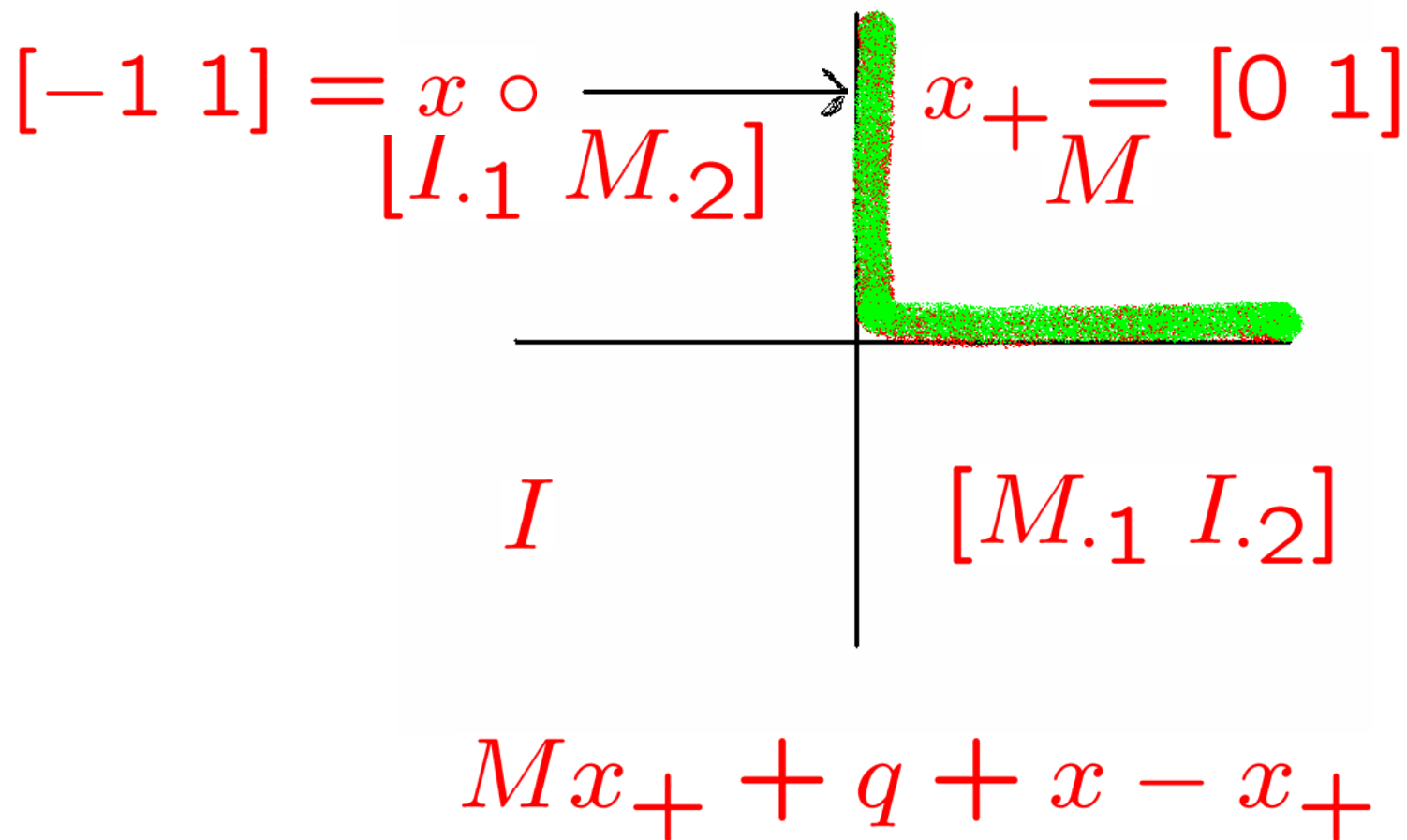
$$0 = F(x_+) + x - x_+$$

$$0 \leq x_+, \; F(x_+) = x_+ - x \geq 0$$

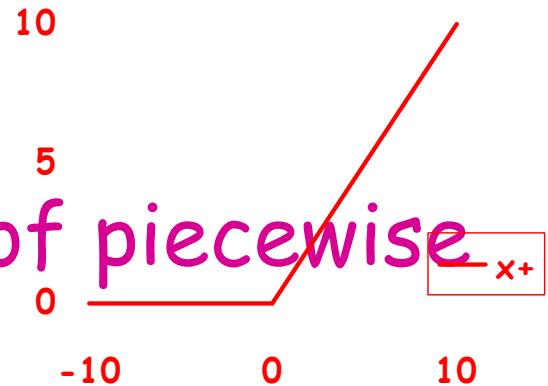$$0 = F(\pi_C(x)) + x - \pi_C(x)$$

$$x'_+ F(x_+) = x'_+(x_+ - x) = 0$$

Normal Manifold

$[-1\ 1] = x \circ \xrightarrow{} \quad x_+ = [0\ 1]$

$[I_{.1}\ M_{.2}]$

M

$I \qquad\qquad [M_{.1}\ I_{.2}]$

$Mx_+ + q + x - x_+$

The PATH Solver

- PATH: Newton method based on nonsmooth Normal map
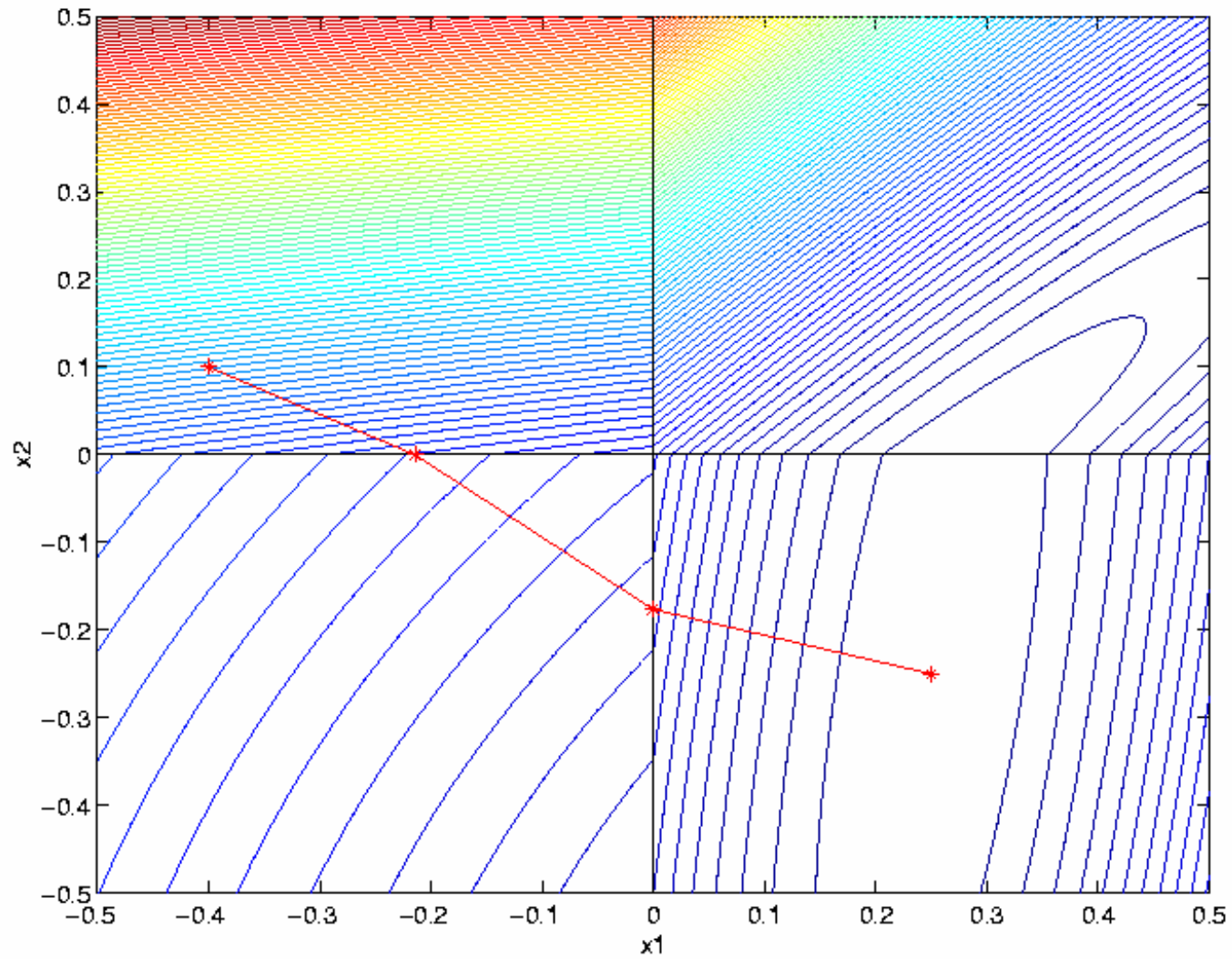
$$F(x_+) + x - x_+$$

- Newton point is solution of piecewise linearization

$$F(x_+^k) + \nabla F(x_+^k)(x_+ - x_+^k) + x - x_+ = 0$$

- Uses more general projection:

$$x_+ \Leftarrow \pi_{\mathcal{B}}(x)$$

10

5

0

-10   0   10

x+

The "Newton" Step

Key solver features

- Underlying robust theory
- Large scale linear algebra
- Ease of model generation/checking
- Globalization and merit functions
- Treat singularities/ill conditioning
- Crash methods and preprocessing

- Alternative: Semismooth based Newton approaches

Definition of MPEC (MPCC)

$$\min \quad f(x, y)$$
$$\text{s.t.} \quad g(x, y) \le 0$$

Add parameterization to definition of F; parameter y

$$0 \le x \perp F(x, y) \ge 0$$

Theory hard; no constraint qualification, specify in AMPL/GAMS

NCP functions

Definition: $\phi(a, b) = 0 \Leftrightarrow 0 \leq a \perp b \geq 0$

Example: $\phi_{\min}(a, b) := \min(a, b)$

Example: $\phi_{FB}(a, b) := \sqrt{a^2 + b^2} - a - b$

Componentwise: $\Phi_i(x) := \phi(x_i, F_i(x))$

$$\Phi(x) = 0 \Longleftrightarrow 0 \leq x \perp F(x) \geq 0$$

Aside: (semismooth) fact

$\Psi(x) := \Phi(x)'\Phi(x)$ cont. diffble.

MPEC approaches

- Implicit: $\min f(x(y),y)$
- Auxiliary variables: $s = F(x,y)$
- NCP functions: $\Phi(s,x) = 0$
- Smoothing: $\Phi_\mu(s,x) = 0$
- Penalization: $\min f(x,y) + \mu \{s'x\}$
- Relaxation: $s'x <= \mu$
- Different problem classes require different solution techniques

Parametric algorithm NLPEC

- Reftype mult
- Aggregate none
- Constraint inequ
- Initmu = 0.01
- Numsolves = 5
- Updatefac = 0.1
- Finalmu = 0

$$NLP(\mu) : \min f(x, y)$$
$$g(x, y) \leq 0$$
$$s = F(x, y)$$
$$x, s \geq 0$$
$$s_i x_i \leq \mu, \ \forall i$$

Reformulate problem and set up sequence of solves

Running NLPEC

- Create the GAMS model as an MPEC
- Setup nlpec.opt
- <u>Gams modelfile mpec=nlpec optfile=1</u>

- Reformulated automagically
- Results returned directly to GAMS
- Modeler tip: use "convert" to get AMPL
- Modeler tip: use "kestrel" to solve GAMS models using AMPL linked solvers

Optimal Yacht Rig Design

- Current mast design trends use a large primary spar that is supported laterally by a set of tension and compression members, generally termed the rig

- Complementarity determines member loads and deflections for given geometry and design variables

- Reduction in either the weight of the rig or the height of the VCG will improve performance

Complementarity Feature

- Stays are tension-only members (in practice) – Hookes Law

- When tensile load becomes zero, the stay goes slack (low material stiffness)
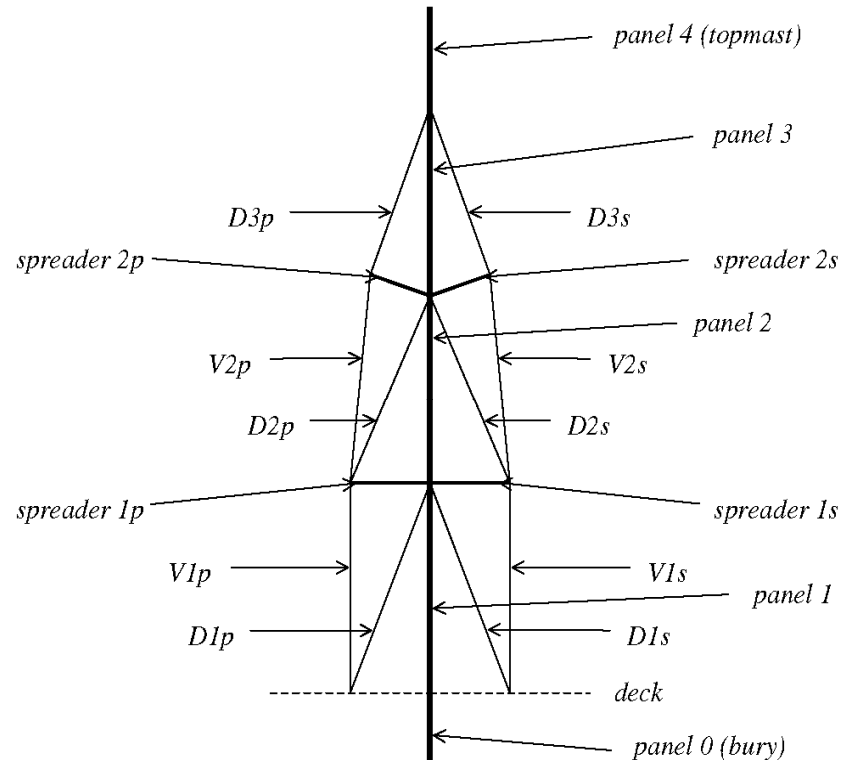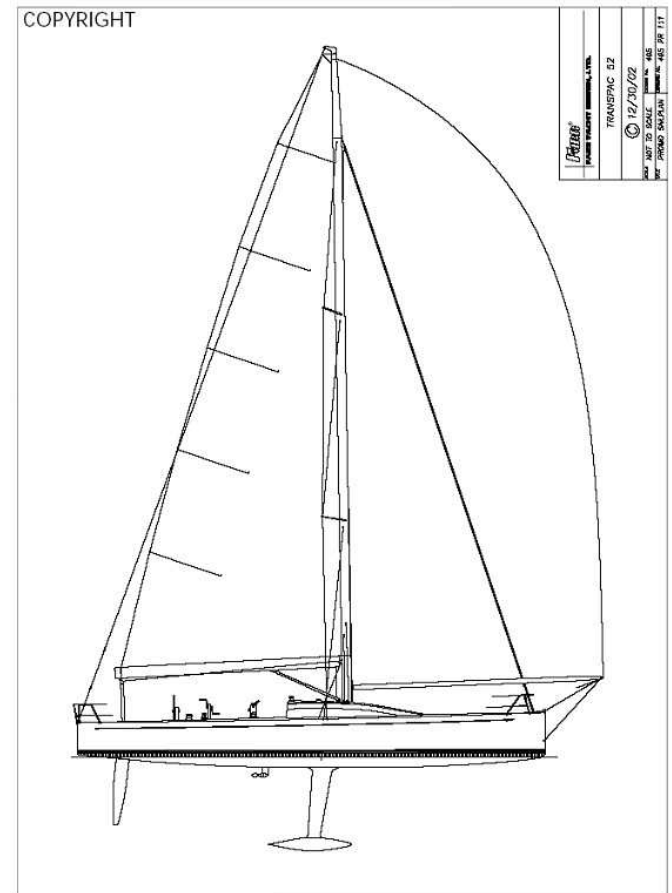

$$0 \geq s \perp s - k * dl \leq 0$$

s: axial load
k: member spring constant
dl: member length extension

MPEC extension for design

- TransPac 52' (TP52)
- Optimal rig design minimum weight problem using NLPEC
- One/two-spreader rig
- NLP starting value is a solution from CP
- Optimal val = 10.0873

Benefits/Drawbacks

- Easy to adapt existing models
- Large-scale potential
- Customizable solution to problem
- Available within GAMS right now
- Models hard to solve
- Local solutions found
- Alternatives: Filter, LOQO, KNITRO

What can we model via CP?

$$\min(G(x), H(x)) \leq y$$

$$\min(F^1(x), F^2(x), \ldots, F^m(x)) = 0$$

$$\text{kth-largest}(F^1(x), F^2(x), \ldots, F^m(x)) = 0$$

Switch off: $g(x)h(x) \leq 0, \ h(x) \geq 0$

Variational Inequality: $VI(F, C)\}$

Chemical Phase Equilibrium

$$f(\alpha) = \sum_i y_i - x_i$$

$$y_i = K_i x_i, \ x_i = \frac{z_i}{K_i \alpha + 1 - \alpha}$$

$$Vapor : f(\alpha) \geq 0, \ \alpha = 1$$

$$TwoPhase : f(\alpha) = 0, \ 0 \leq \alpha \leq 1$$

$$Liquid : f(\alpha) \leq 0, \ \alpha = 0$$

$$median\{\alpha, \alpha - 1, -f(\alpha)\} = 0$$

Other applications

- Option pricing (electricity market)
- Contact problems (with friction)
- Free boundary problems
- Optimal control (ELQP)
- Earthquake propogation
- Structure design
- Dynamic traffic assignment

Future Challenges

- MPEC/EPEC
  - theory and computation
- All solutions
  - Structure failure, Nash equilibria
- Large scale iterative solvers
  - Factors not available in RAM
- Complementarity Systems / Projected dynamical systems
- New application areas

Solver/Example Availability

- Student version downloadable (full license downloadable yearly)
- AMPL/GAMS (also MILES, NLPEC)
- Matlab, Callable library, NEOS

- MCPLIB
- GAMSWORLD

Historical events (omitted)

- KKT conditions / Complementary slackness
- Lemke's method (bimatrix / Nash games, selfish routing, electricity pricing)
- Triangulation/homotopy
- Complexity: Lemke exponential, NP-Hard, specializations polynomial - interior points
  - Murty; Cottle, Pang and Stone;
  - Facchinei and Pang