Nonlinear Complementarity Problems and Extensions

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Abstract

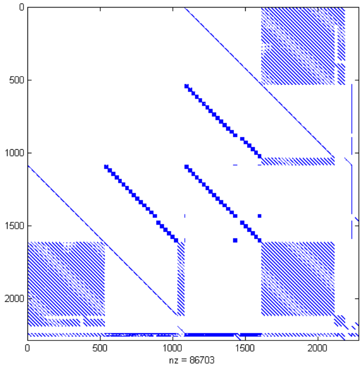
While optimizers are familiar with complementary slackness within the optimality conditions of linear and nonlinear programming, there are numerous complementarity problems arising naturally in many practical applications from engineering and economics. These include applied general equilibrium modeling, traffic network design, structural engineering and finance. Several examples will be outlined, together with an overview of modern, practical modeling and solution techniques within this field. Since complementarity allows for competition among players, optimization problems that involve complementarity constraints, and models with embedded complementarities are becoming increasingly important within applications. We outline these new ideas, highlight several computational schemes and explain their utility by application.

Modeling languages: state-of-the-art

- Optimization models improve understanding of underlying systems and facilitate operational/strategic improvements
- Key link to applications, prototyping of optimization capability
- Widely used in:
  - engineering - operation/design
  - economics - policy/energy modeling
  - military - operations/planning
  - finance, medical treatment, supply chain management, etc.
- Interface to solutions: facilitates automatic differentiation, separation of data, model and solver
- Modeling languages no longer novel: typically represent another tool for use within a solution process.

World Bank Project (Uruguay Round)

- 24 regions, 22 commodities
  - Nonlinear complementarity problem
  - Size: 2200 x 2200
- Short term gains $53 billion p.a.
  - Much smaller than previous literature
- Long term gains $188 billion p.a.
  - Number of less developed countries loose in short term
- Unpopular conclusions - forced concessions by World Bank



Modeling languages: an example

$$\min c^T x \text{ s.t. } a_i^T x \leq b_i, i = 1, 2, \ldots, m$$

```
set i, j; parameter b(i), c(j), A(i,j);

variables obj, x(j);
equations defobj, cons(i);

defobj.. obj =e= sum(j, c(j)*x(j));

cons(i).. sum(j, A(i,j)*x(j)) =l= b(i);

model lpmod /defobj, cons/;
solve lpmod using lp min obj;
```

Modeling language limitations

- Data (collection) remains bottleneck in many applications
  - Tools interface to databases, spreadsheets, Matlab
- Problem format is old/traditional

$$\min_x f(x) \text{ s.t. } g(x) \leq 0, h(x) = 0$$

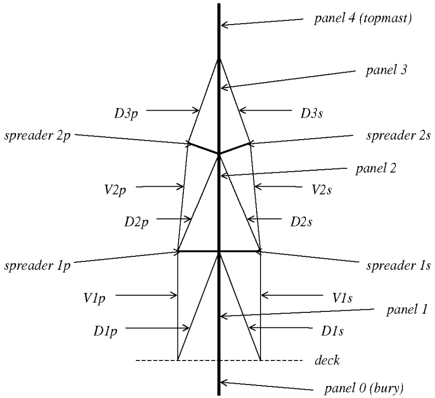
  - Support for integer, sos, semicontinuous variables
  - Limited support for logical constructs
  - Support for complementarity constraints
- Optimization assumes you control the complete system

Optimal Yacht Rig Design

- Current mast design trends use a large primary spar that is supported laterally by a set of tension and compression members, generally termed the rig
- Reduction in either the weight of the rig or the height of the VCG will improve performance
- Design must work well under a variety of weather conditions

Complementarity feature

- Stays are tension only members (in practice) - Hookes Law
- When tensile load becomes zero, the stay goes slack (low material stiffness)
- $0 \geq s \perp s - k\delta \leq 0$
  - s axial load
  - k member spring constant
  - δ member extension
- Either $s_i = 0$ or $s_i = k\delta_i$



MPEC: complementarity constraints

$$\begin{aligned} \min_{x,s} \quad & f(x,s) \\ \text{s.t.} \quad & g(x,s) \leq 0, \\ & 0 \geq s \perp h(x,s) \leq 0 \end{aligned}$$

- g, h model "engineering" expertise: finite elements, etc
- $\perp$ models "turning on/off" of Hooke's Law
- Complementarity "$\perp$" constraints available in AMPL and GAMS
- NLPEC: use the convert tool to automatically reformulate as a parameteric sequence of NLP's
- Solution by repeated use of standard NLP software
- Southern Spars Company (NZ): improved from 5-0 to 5-2 in America's Cup!

Use of complementarity

- Pricing electricity markets and options
- Video games: model contact problems (with friction)
- Structure design - how springy is concrete?
- Computer networks - the price of anarchy

What can we model via CP?

$$\min\,(G(x), H(x)) \leq y$$
$$\min\,\left(F^1(x), F^2(x), \ldots, F^m(x)\right) = 0$$
$$\text{kth-largest}\,\left(F^1(x), F^2(x), \ldots, F^m(x)\right) = 0$$
$$\text{Switch on/off: } g(x)h(x) \leq 0, h(x) \geq 0$$

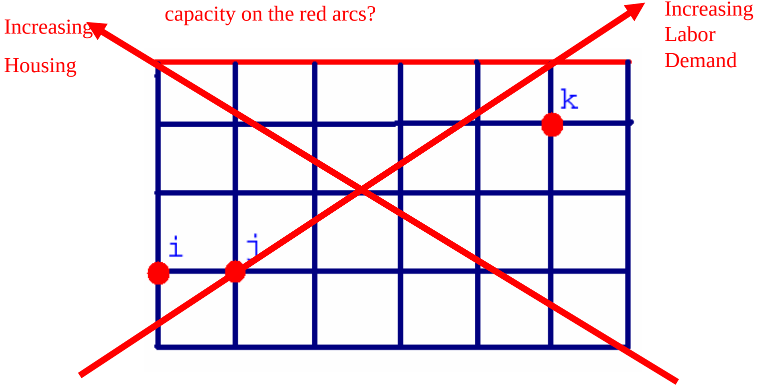
So what's my point?

- Can solve practical models due to availability of good software
- Design (optimization) under our control
- Weather conditions treated via "scenarios"
- Hookes Law turned on/off beyond control of designer

- Complementarity facilitates modeling of competition, nonsmoothness and "switching"
- Large scale models involving complementarity now solvable
- Do you (or should you) care?

Walras meets Wardrop

What is the effect on housing prices of increasing capacity on the red arcs?



Features

- We buy a house to "optimize" some measure
  - Price driven by market
  - We compete against each other
- Driver's choose routes to "optimize" travel time
  - Choices affect congestion
  - Your choice affects me!
- Production processes are "optimized"
- But the road designer does not control any of these!

Simple AGE model

$$(P) : \min_{y \geq 0} \quad c^T y \qquad \text{s.t. } Ay \geq d \quad (\perp p \geq 0)$$

$$(C) : \max_{d \geq 0} \quad u(d) \qquad \text{s.t. } p^T d \leq I$$

- In equilibrium, the optimal demand d from (C) will be the demand in (P), and the sales price p in (C) will be the marginal price on production from (P)
- Complementarity conditions of (P) and (C) have both primal and dual variables
- Optimization models linked by variables and multipliers
- Equilibrium problem solvable as a complementarity problem
- Can add "other features" such as taxation, transportation, tolls.

EMP: Embedded models

- Model has the format:

$$\begin{array}{lll} \min_{x} & f(x,y) & \\ \text{s.t.} & g(x,y) \leq 0 & (\perp \lambda \geq 0) \\ & & \\ H(x,y,\lambda) = 0 & & (\perp y \text{ free}) \end{array}$$

- Difficult to implement correctly, particularly when multiple optimization models present
- Can do automatically - simply annotate equations
- EMP tool automatically creates an MCP

$$\nabla_x f(x,y) + \lambda^T \nabla g(x,y) = 0$$
$$0 \leq -g(x,y) \perp \lambda \geq 0$$
$$H(x,y,\lambda) = 0$$

The Hollywood perspective: beautiful mind

- Nash Games: x^* is a Nash Equilibrium if

$$x_i^* \in \arg\min_{x_i \in X_i} \ell_i(x_i, x_{-i}^*, q), \forall i \in \mathcal{I}$$

  x_{-i} are the decisions of other players.

- Quantities q given exogenously, or via complementarity:

$$0 \leq H(x, q) \perp q \geq 0$$

- EMP reformulates automatically for appropriate solvers, e.g. forms KKT conditions

Models are correct and solvable

S	Var	rows	non-zero	dense(%)	Steps	RT (m:s)
20	2400	2568	31536	0.48	5	0 : 03
50	15000	15408	195816	0.08	5	0 : 19
100	60000	60808	781616	0.02	5	1 : 16
200	240000	241608	3123216	0.01	5	5 : 12

Convergence for $S = 200$

Iteration	Residual
0	$1.56(+4)$
1	$1.06(+1)$
2	1.34
3	$2.04(-2)$
4	$1.74(-5)$
5	$2.97(-11)$

Many applications

- Discrete-Time Finite-State Stochastic Games
- Ericson & Pakes (1995) model of dynamic competition in an oligopolistic industry
  - Advertising (Doraszelski & Markovich 2007)
  - Capacity accumulation (Besanko & Doraszelski 2004,...)
  - Collusion (Fershtman & Pakes 2000, 2005, de Roos 2004)
  - Consumer learning (Ching 2002)
  - Firm size distribution (Laincz & Rodrigues 2004)
  - Learning by doing (Benkard 2004,...)
  - Mergers (Berry & Pakes 1993, Gowrisankaran 1999)
  - Network externalities (Jenkins et al 2004,...)
  - Productivity growth (Laincz 2005)
  - Technology adoption (Schivardi & Schneider 2005)
  - International trade (Erdem & Tybout 2003)
  - Finance (Goettler, Parlour & Rajan 2004,...).

EMP: Other new types of constraints

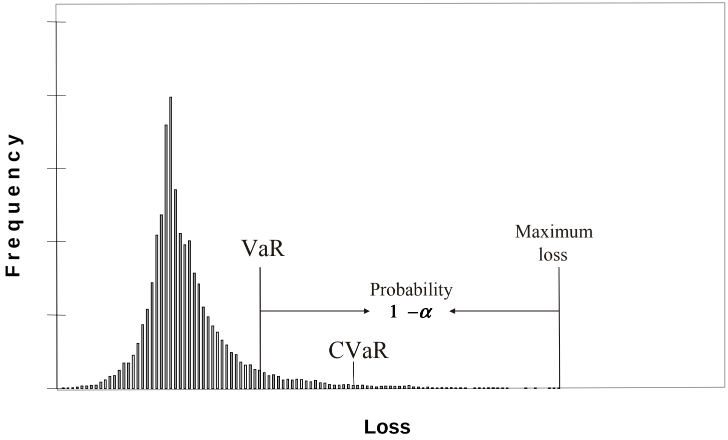
- range constraints $L \leq Ax - b \leq U$
- robust programming (probability constraints, stochastics)

$$f(x, \xi) \leq 0, \forall \xi \in \mathcal{U}$$

- conic programming $a_i^T x - b_i \in K_i$
- soft constraints
- rewards and penalties

Some constraints can be reformulated easily, others not!

CVaR constraints: mean excess dose (radiotherapy)

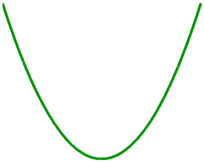


Move mean of tail to the left!

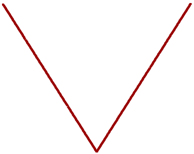
EMP: Extended nonlinear programs

$$\min_{x \in X} f_0(x) + \theta(f_1(x), \ldots, f_m(x))$$

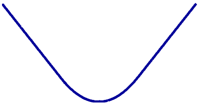
Examples of different θ



least squares,



absolute value,



Huber function

Solution reformulations are very different

Huber function used in robust statistics.

More general θ functions

In general any piecewise linear penalty function can be used: (different upside/downside costs).

General form:

$$\theta(u) = \sup_{y \in Y} \{ y^T u - k(y) \}$$

First order conditions for optimality are an MCP!

EMP: Heirarchical models

- Bilevel programs:

$$\begin{aligned} \min_{x,y} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \le 0, \\ & y \text{ solves } \min_{s} v(x,s) \text{ s.t. } h(x,s) \le 0 \end{aligned}$$

- Model as:
  model bilev /deff,defg,defv,defh/;
  plus empinfo: bilevel y min v defh
- EMP tool automatically creates the MPEC

Conclusions

- Large scale complementarity problems reliably solvable
- Complementarity constraints within optimization problems
- Extended Mathematical Programming available within a modeling system
- System can easily formulate and solve second order cone programs, robust optimization, soft constraints via piecewise linear penalization (with strong supporting theory)
- Embedded optimization models automatically reformulated for appropriate solution engine
- Exploit structure in solvers
- Extend application usage of complementarity solvers