Quadratic Support Functions and Extended Mathematical Programs

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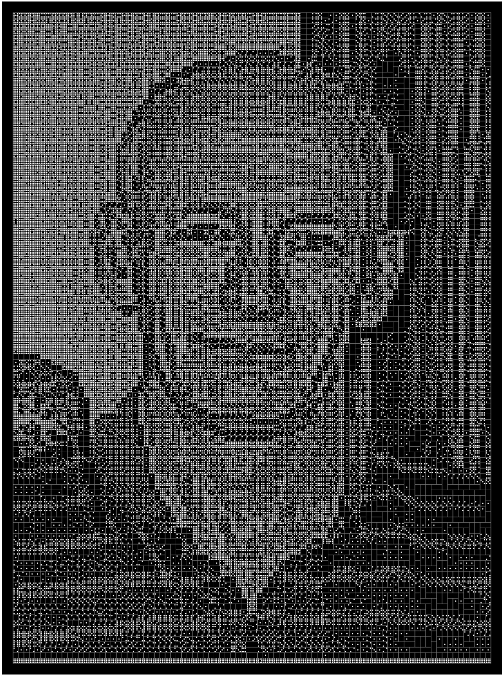
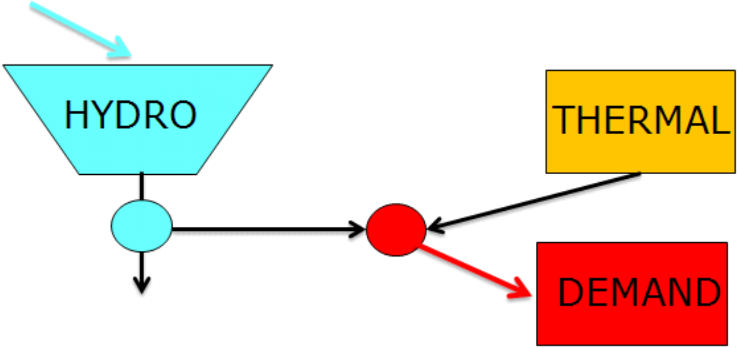
- Happy Birthday Jim
- Mentor, colleague and friend
- Now dominoized via NEOS!



- Even though not a convex composite optimization or exact penalization, this really does use constrained optimization, and a fairly recent image!

Hydro-Thermal System (Philpott/F./Wets)



- Competing agents (consumers, or generators in energy market)
- Each agent minimizes objective independently (cost)
- Market prices are function of all agents activities

Simple electricity "system optimization" problem

$$
\begin{aligned}
\text{SO:} \quad \max_{d_k, u_i, v_j, x_i \geq 0} \quad & \sum_{k \in \mathcal{K}} W_k(d_k) - \sum_{j \in \mathcal{T}} C_j(v_j) + \sum_{i \in \mathcal{H}} V_i(x_i) \\
\text{s.t.} \quad & \sum_{i \in \mathcal{H}} U_i(u_i) + \sum_{j \in \mathcal{T}} v_j \geq \sum_{k \in \mathcal{K}} d_k, \\
& x_i = x_i^0 - u_i + h_i^1, \quad i \in \mathcal{H}
\end{aligned}
$$

- u_i water release of hydro reservoir $i \in \mathcal{H}$
- v_j thermal generation of plant $j \in \mathcal{T}$
- x_i water level in reservoir $i \in \mathcal{H}$
- prod fn U_i (strictly concave) converts water release to energy
- $C_j(v_j)$ denote the cost of generation by thermal plant
- $V_i(x_i)$ future value of terminating with storage x (assumed separable)
- $W_k(d_k)$ utility of consumption d_k

SO equivalent to CE (price takers)

$$\text{Consumers } k \in \mathcal{K} \text{ solve CP}(k)\text{:} \quad \max_{d_k \geq 0} \quad W_k(d_k) - p^T d_k$$

$$\text{Thermal plants } j \in \mathcal{T} \text{ solve TP}(j)\text{:} \quad \max_{v_j \geq 0} \quad p^T v_j - C_j(v_j)$$

$$\text{Hydro plants } i \in \mathcal{H} \text{ solve HP}(i)\text{:} \quad \max_{u_i, x_i \geq 0} \quad p^T U_i(u_i) + V_i(x_i)$$

$$\text{s.t.} \quad x_i = x_i^0 - u_i + h_i^1$$

Perfectly competitive (Walrasian) equilibrium is a MOPEC

$$\begin{aligned} \text{CE:} \quad & d_k \in \arg\max \text{CP}(k), & k \in \mathcal{K}, \\ & v_j \in \arg\max \text{TP}(j), & j \in \mathcal{T}, \\ & u_i, x_i \in \arg\max \text{HP}(i), & i \in \mathcal{H}, \\ & 0 \leq p \perp \sum_{i \in \mathcal{H}} U_i(u_i) + \sum_{j \in \mathcal{T}} v_j \geq \sum_{k \in \mathcal{K}} d_k. \end{aligned}$$

MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, p) \text{ s.t. } g_i(x_i, x_{-i}, p) \leq 0, \forall i$$

$$p \text{ solves } \mathrm{VI}(h(x, \cdot), C)$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h p cons
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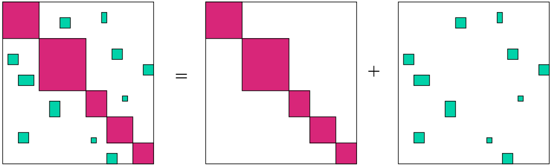
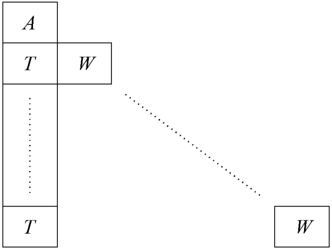
- (Generalized) Nash
- Reformulate optimization problem as first order conditions (complementarity)
- Use nonsmooth Newton methods to solve
- Solve overall problem using "individual optimizations"?



Stochastic: Agents have recourse?

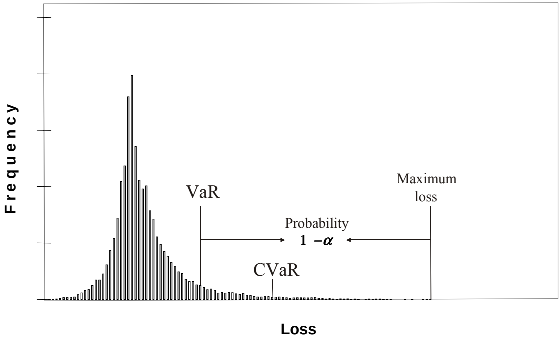
- Agents face uncertainties in reservoir inflows
- Two stage stochastic programming, x^1 is here-and-now decision, recourse decisions x^2 depend on realization of a random variable
- ρ is a risk measure (e.g. expectation, CVaR)

$$\begin{aligned} \text{SP: } \min \quad & c(x^1) + \rho[q^T x^2] \\ \text{s.t.} \quad & Ax^1 = b, \quad x^1 \geq 0, \\ & T(\omega)x^1 + W(\omega)x^2(\omega) \geq d(\omega), \\ & x^2(\omega) \geq 0, \forall \omega \in \Omega. \end{aligned}$$



Risk Measures

- Modern approach to modeling risk aversion uses concept of risk measures
- $\overline{CVaR}_\alpha$: mean of upper tail beyond α-quantile (e.g. $\alpha = 0.95$)



- mean-risk, mean deviations from quantiles, VaR, CVaR
- Much more in mathematical economics and finance literature
- Optimization approaches still valid, different objectives, varying convex/non-convex difficulty

Dual Representation of Risk Measures

- Dual representation (of coherent r.m.) in terms of risk sets

$$\rho(Z) = \sup_{\mu \in \mathcal{D}} \mathbb{E}_{\mu}[Z]$$

- If $\mathcal{D} = \{p\}$ then $\rho(Z) = \mathbb{E}[Z]$
- If $\mathcal{D}_{\alpha,p} = \{\lambda : 0 \leq \lambda_i \leq p_i/(1-\alpha), \sum_i \lambda_i = 1\}$, then

$$\rho(Z) = \overline{CVaR}_{\alpha}(Z)$$

- Special case of a Quadratic Support Function

$$\rho(y) = \sup_{u \in U} \langle u, By + b \rangle - \frac{1}{2} \langle u, Mu \rangle$$

- EMP allows any Quadratic Support Function to be defined and facilitates a model transformation to a tractable form for solution

Addition: compose equilibria with QS functions

- Add soft penalties to objectives and/or within constraints:

$$\min_x \theta(x) + \rho_O(F(x))$$
$$\text{s.t. } \rho_C(g(x)) \leq 0$$

```
QS g rhoC udef B M
...

QSF cvarup F rhoO theta p
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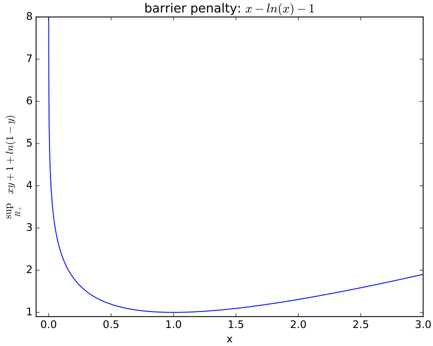
- $batinclude QSprimal modname using emp min obj
- Allow modeler to compose QS functions automatically

- Can solve using MCP or primal reformulations
- More general conjugate functions also possible:



The link to MOPEC

$$\min_{x \in X} \theta(x) + \rho(F(x))$$

$$\rho(y) = \sup_{u \in U} \langle u, y \rangle - \frac{1}{2} \langle u, Mu \rangle$$

$$0 \in \partial\theta(x) + \nabla F(x)^T \partial\rho(F(x)) + N_X(x)$$

$$0 \in \partial\theta(x) + \nabla F(x)^T u + N_X(x)$$

$$0 \in -u + \partial\rho(F(x)) \iff 0 \in -F(x) + Mu + N_U(u)$$

This is a MOPEC, and we have multiple copies of this for each agent

$$\text{CP:} \quad \min_{d^1 \geq 0} \quad p^1 d^1 - W(d^1)$$

$$\text{TP:} \quad \min_{v^1 \geq 0} \quad C(v^1) - p^1 v^1$$

$$\text{HP:} \quad \min_{u^1, x^1 \geq 0} \quad -p^1 U(u^1)$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

Two stage stochastic MOPEC (1,1,1)

$$\text{CP:} \quad \min_{d^1, d^2_\omega \geq 0} \quad p^1 d^1 - W(d^1) + \rho_C \left[p^2_\omega d^2_\omega - W(d^2_\omega) \right]$$

$$\text{TP:} \quad \min_{v^1, v^2_\omega \geq 0} \quad C(v^1) - p^1 v^1 + \rho_T \left[C(v^2_\omega) - p^2_\omega v^2(\omega) \right]$$

$$\text{HP:} \quad \min_{\substack{u^1, x^1 \geq 0 \\ u^2_\omega, x^2_\omega \geq 0}} \quad -p^1 U(u^1) + \rho_H \left[-p^2(\omega) U(u^2_\omega) - V(x^2_\omega) \right]$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

$$x^2_\omega = x^1 - u^2_\omega + h^2_\omega$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

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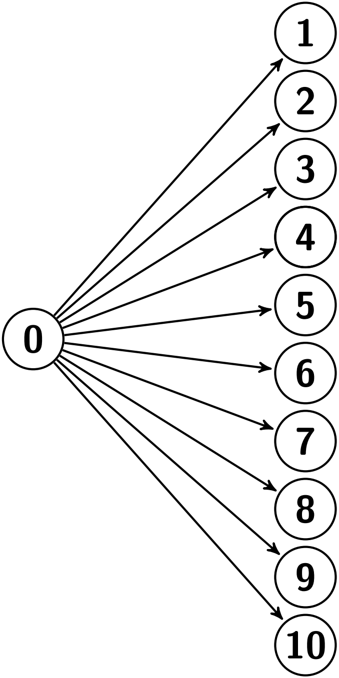
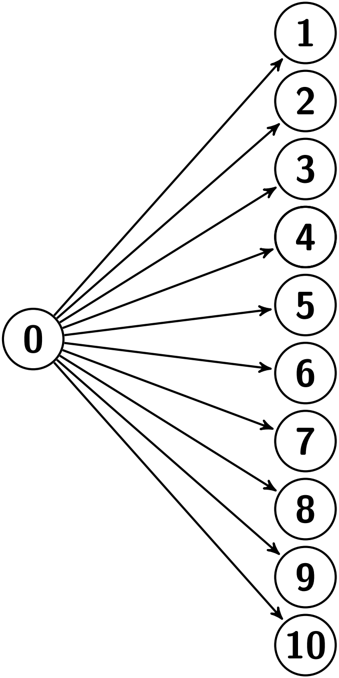
$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

$$0 \leq p^2_\omega \perp U(u^2_\omega) + v^2_\omega \geq d^2_\omega, \forall \omega$$



- Single hydro, thermal and representative consumer
- Initial storage 10, inflow of 4 to 0, equal prob random inflows of i to node i
- Risk neutral: SO equivalent to CE (key point is that each risk set is a singleton, and that is the same as the system risk set)



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- Each agent has its own risk measure, e.g. $0.8EV + 0.2CVaR$
- Is there a system risk measure?
- Is there a system optimization problem?

$$\min \sum_i C(x_i^1) + \rho_i \left(C(x_i^2(\omega)) \right) ????$$

Equilibrium or optimization?

Theorem

If (d, v, u, x) solves (risk averse) SO, then there exists a probability distribution σ_k and prices p so that (d, v, u, x, p) solves (risk neutral) CE(σ)

(Observe that each agent must maximize their own expected profit using probabilities σ_k that are derived from identifying the worst outcomes as measured by SO. These will correspond to the worst outcomes for each agent only under very special circumstances)

- High initial storage level (15 units)
  - Worst case scenario is 1: lowest system cost, smallest profit for hydro
  - SO equivalent to CE
- Low initial storage level (10 units)
  - Different worst case scenarios
  - SO different to CE (for large range of demand elasticities)
- Attempt to construct agreement on what would be the worst-case outcome by trading risk

Contracts in MOPEC (Philpott/F./Wets)

- Can we modify (complete) system to have a social optimum by trading risk?
- How do we design these instruments? How many are needed? What is cost of deficiency?
- Facilitated by allowing contracts bought now, for goods delivered later (e.g. Arrow-Debreu Securities)
- Conceptually allows to transfer goods from one period to another (provides wealth retention or pricing of ancillary services in energy market)
- Can investigate new instruments to mitigate risk, or move to system optimal solutions from equilibrium (or market) solutions

$$\text{CP:} \quad \min_{d^1, d^2_\omega \geq 0} \quad p^1 d^1 - W(d^1) + \rho_C \left[p^2_\omega d^2_\omega - W(d^2_\omega) \right]$$

$$\text{TP:} \quad \min_{v^1, v^2_\omega \geq 0} \quad C(v^1) - p^1 v^1 + \rho_T \left[C(v^2_\omega) - p^2_\omega v^2(\omega) \right]$$

$$\text{HP:} \quad \min_{\substack{u^1, x^1 \geq 0 \\ u^2_\omega, x^2_\omega \geq 0}} \quad - p^1 U(u^1) + \rho_H \left[-p^2(\omega) U(u^2_\omega) - V(x^2_\omega) \right]$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

$$x^2_\omega = x^1 - u^2_\omega + h^2_\omega$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

$$0 \leq p^2_\omega \perp U(u^2_\omega) + v^2_\omega \geq d^2_\omega, \forall \omega$$

Trading risk: pay σ_ω now, deliver 1 later in ω

$$\text{CP:} \quad \min_{d^1, d^2_\omega \geq 0, t^C} \quad \sigma t^C + p^1 d^1 - W(d^1) + \rho_C \left[p^2_\omega d^2_\omega - W(d^2_\omega) - t^C_\omega \right]$$

$$\text{TP:} \quad \min_{v^1, v^2_\omega \geq 0, t^T} \quad \sigma t^T + C(v^1) - p^1 v^1 + \rho_T \left[C(v^2_\omega) - p^2_\omega v^2(\omega) - t^T_\omega \right]$$

$$\text{HP:} \quad \min_{\substack{u^1, x^1 \geq 0 \\ u^2_\omega, x^2_\omega \geq 0, t^H}} \quad \sigma t^H - p^1 U(u^1) + \rho_H \left[-p^2(\omega) U(u^2_\omega) - V(x^2_\omega) - t^H_\omega \right]$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

$$x^2_\omega = x^1 - u^2_\omega + h^2_\omega$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

$$0 \leq p^2_\omega \perp U(u^2_\omega) + v^2_\omega \geq d^2_\omega, \forall \omega$$

$$0 \leq \sigma_\omega \perp t^C_\omega + t^T_\omega + t^H_\omega \geq 0, \forall \omega$$

Main Result

Theorem

Agents a have polyhedral node-dependent risk sets $\mathcal{D}_a(n)$, $n \in \mathcal{N} \setminus \mathcal{L}$ *with nonempty intersection. Now let* $\{u_a^s(n) : n \in \mathcal{N}, a \in \mathcal{A}\}$ *be a solution to SO with risk sets* $D_s(n) = \cap_{a \in \mathcal{A}} \mathcal{D}_a(n)$. *Suppose this gives rise to* μ *(hence* σ*) and prices* $\{p(n) : n \in \mathcal{N}\}$ *where* $p(n)\sigma(n)$ *are Lagrange multipliers. These prices and quantities form a multistage risk-trading equilibrium in which agent a solves OPT(a) with a policy defined by* $u_a(\cdot)$ *together with a policy of trading Arrow-Debreu securities defined by* $\{t_a(n), n \in \mathcal{N} \setminus \{0\}\}$.

- Low storage setting
- If thermal is risk neutral (even with trading) SO equivalent to CE
- If thermal is identically risk averse, there is a CE, but different to original SO
- Trade risk to give optimal solutions for the sum of their positions
- Under a complete market for risk assumption, we may construct a competitive equilibrium with risk trading from a social planning solution

Theory and Observations

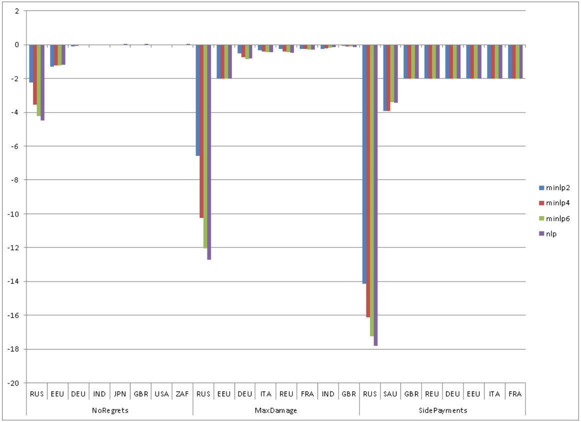
- agent problems are multistage stochastic optimization models
- perfectly competitive partial equilibrium still corresponds to a social optimum when all agents are risk neutral and share common knowledge of the probability distribution governing future inflows
- situation complicated when agents are risk averse
  - utilize stochastic process over scenario tree
  - under mild conditions a social optimum corresponds to a competitive market equilibrium if agents have time-consistent dynamic coherent risk measures and there are enough traded market instruments (over tree) to hedge inflow uncertainty
- Otherwise, must solve the stochastic equilibrium problem
- Research challenge: develop reliable algorithms for large scale decomposition approaches to MOPEC

Optimal Sanctions (Boehringer/F./Rutherford)

- Sanctions can be modeled using similar formulations used for tariff calculations
- Model as a Nash equilibrium with players being countries (or a coalition of countries)
- Demonstrate the actual effects of different policy changes and the power of different economic instruments
- GTAP global production/trade database: 113 countries, 57 goods, 5 factors
- Coalition members strategically choose trade taxes to
  1. optimize their welfare (trade war) or
  2. *minimize* Russian welfare
- Russia chooses trade taxes to *maximize* Russian welfare in response
- Impose (QS) constraints that limit the number of instruments used for each country

Optimal Sanctions: Results

- Resulting Nash equilibrium with trade war, maximize damage, side payments - all have big impact on Russia
- Restricting instruments can change effects (these are the different colored bars)
- Collective (coalition) action significantly better



Same model can used to determine effects of Russian trade sanctions on Turkey

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- dualvar (use multipliers from one agent as variables for another)
- QS functions (both in objectives and constraints)

- Currently available within GAMS
- Some solution algorithms implemented in modeling system - limitations on size, decomposition and advanced algorithms
- QS extensions to Moreau-Yoshida regularization, compositions, composite optimization