Risk-averse multi-stage stochastic equilibria: models and algorithms

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The MOPEC problem (GNE)

Assume there are N agents, find $(x_1^*, \dots, x_N^*, \pi^*)$ such that for each agent:

$$\begin{aligned} x_a^* \in \arg\min \quad & f_a(x_a; x_{-a}^*, \pi^*) \\ \text{s.t.} \;\; & x_a \in X_a(x_{-a}^*, \pi^*) \end{aligned}$$

and a market equilibrium constraint:

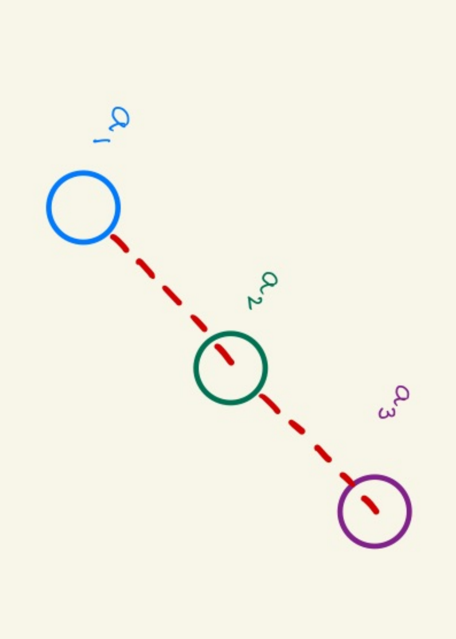
$$0 \in H(\pi^*; x^*) + N_P(\pi^*)$$

Variables:

- x_a: variable controlled by each agent a
- $x_{-a} = (x_1, x_2, \dots, x_{a-1}, x_{a+1}, \dots, x_N)$: action of other agents
- price variable π, set by the market equilibrium constraint

- Optimizations might be large LP or QP models of particular sectors
- Extensive literature, hard problems (non-monotone) even if f_a strongly convex

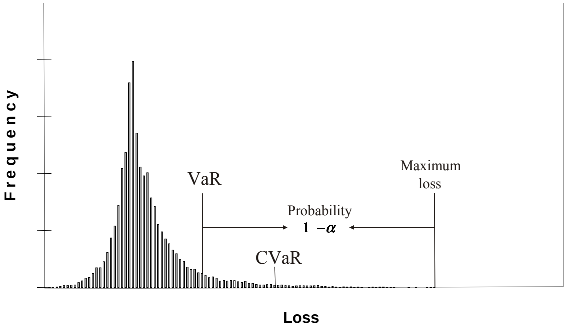
Structure



- Agent optimization problems at nodes
- Complementarity links across agents

Risk modeling

- Modern approach to modeling risk aversion uses concept of risk measures
- Considers not only the expected value of the uncertain quantities, but also more "extreme events"



- $\overline{CVaR}_\alpha$: mean of upper tail beyond α-quantile (e.g. $\alpha = 0.95$)
- Dual representation (of coherent r.m.) in terms of risk sets: $\mathcal{D}$ [4]

$$\rho(Z) = \sup_{\mu \in \mathcal{D}} \mathbb{E}_\mu[Z]$$

- Different agents have different risk profiles

One example: MOPEC equilibrium

Agents (e.g): 'fos', 'ren', 'trns', 'dem':

$$
\begin{array}{llll}
\mathrm{S}(a): & \min \ \rho_a(\psi_a) & \text{s.t.} & (z_a, y_a, q_a, r_a) \in \mathcal{X}_a \\
& \quad \psi_a(\omega) & = & \mathcal{C}_a(z_a) + \mathcal{Z}_a(y_a, q_a, r_a, \omega) \\
& & & +\pi_e(\omega)\left(d_a(\omega) - q_a(\omega) - r_a(\omega)\right) \\
& & & +\pi_c(\omega)\mathcal{E}(y_a, \omega)
\end{array}
$$

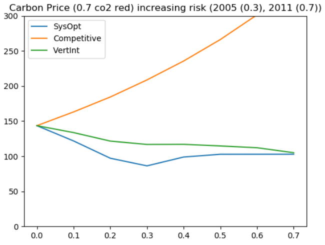
and the prices, production and purchases satisfy the market clearing conditions

$$
\begin{aligned}
0 &\le \sum_a \left(q_a(\omega) + r_a(\omega) - d_a(\omega)\right) \perp \pi_e(\omega) \ge 0, \\
0 &\le E - \sum_a \mathcal{E}(y_a, \omega) \perp \pi_c(\omega) \ge 0.
\end{aligned}
$$

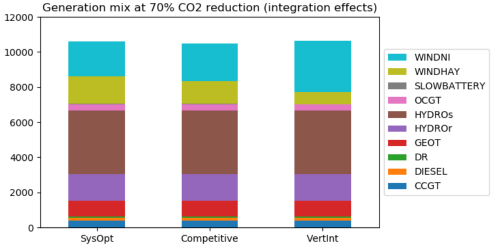
[2] provides theory to show when system optimization is equivalent

Increasing risk aversion: carbon price and investment

- $\rho(Z) = (1-\lambda)\mathbb{E}[Z] + \lambda \text{AVaR}_{0.90}(Z)$
- Same price risk neutral
- Competitive equilibrium: increased price
- VertInt: co-ownership of wind/thermal results in more wind closer to existing thermal



(a) Carbon prices with increasing λ



(b) Ownership at $\lambda = 0.3$

These problems are computationally challenging

Standard methods to solve the MOPEC problem

- Convert the MOPEC problem to mixed complementarity problem (EMP does this) and solve it using PATH solver
- Or traditional decomposition method: splitting, prox-gradient
- EMP/PATH fails to solve large-scale MOPEC problems
- Decompositions usually fail to solve the problem without helpful problem properties, and slow convergence

Solution method: Primal penalty and dual method

- Agent based decomposition (prox gradient)
- Penalty (Augmented Lagrangian) of the constraint $H(x, \pi) \geq 0$ in the primal agents' problems and updating dual in the major iterations.
- Able to solve the problem in situation without having an implicit function $\pi = h(x)$ from the constraint $0 \leq H(x, \pi) \perp \pi \geq 0$.
- Performance mainly depends on the choice of γ. γ small enough enables algorithm to converge to the true solution, but too small γ may cause slow convergence.

Algorithm 1 Gauss-Seidel Primal penalty and dual method

set $k = 0$, define initial point π^0.
while stopping criterion not met **do**
  for $a = 1, 2, \ldots, N$ **do**
    get solution (x_a^{k+1}, y_a^{k+1}) from solving

$$\begin{aligned} \min \quad & f_a(x_a, \bar{x}_{-a}^{k+1}, \pi^k) + y_a^T \pi^k + 0.5\gamma \cdot (y_a)^2 \\ \text{s.t.} \quad & x_a \in X_a(\bar{x}_{-a}^{k+1}, \pi^k) \\ & y_a \geq -H(x_a, \bar{x}_{-a}^{k+1}, \pi^k) \\ & y_a \geq 0 \end{aligned}$$

    here $\bar{x}_{-a}^{k+1} = (x_1^{k+1}, \ldots, x_{a-1}^{k+1}, x_{a+1}^k, \ldots, x_N^k)$.
  end for
  $\pi^{k+1} = \max\{0, \pi^k - \gamma \cdot H(x^{k+1}, \pi^k)\}$
  $k = k + 1$
end while

Comparison between PATH and Primal-Dual method

risk neutral

size	PATH	Primal-Dual		
	time(secs)	γ	# Iter	time(secs)
62K $\times$ 22K	1795.79	0.005	75	333.21

risk averse

size	residual	PATH	Primal-Dual		
		time(secs)	γ	# Iter	time(secs)
114 $\times$ 62	1e-6	-	0.05	264	35.87
114 $\times$ 62	1e-6	-	0.1	162	20.97
114 $\times$ 62	1e-6	-	0.5	334	45.45
21K $\times$ 8.5K	< 1	-	0.005	32	165.76

- The stopping criterion is **1e-6**
- In risk-averse setting, PATH fails to find a solution without good initial point even in small cases

Risk Measures

Problem type

Objective function or Constraint

$$\min_{x \in X} \theta(x) + \rho(F(x)) \qquad\qquad \min_{x \in X} \theta(x) \text{ s.t. } \rho(F(x)) \leq \alpha$$

- If $\mathcal{D} = \{p\}$ then $\rho(Z) = \mathbb{E}[Z]$
- If $\mathcal{D}_{\alpha,p} = \{\lambda \in [0, p/(1-\alpha)] : \langle \mathbf{1}, \lambda \rangle = 1\}$, then $\rho(Z) = \overline{CVaR}_\alpha(Z)$
- Popular examples include: expectation, Conditional Value at Risk, also known as expected shortfall, Average Value at Risk (AVaR), and expected tail loss (ETL), and mean-upper-absolute semideviation.

Using the algebra of support function, we can create new risk measures from existing ones: for instance

$$(1-\lambda)\mathbb{E} + \lambda \overline{CVaR}_\alpha$$

captures more realistic risk-averse behavior. For $\lambda < 1$, it is strictly monotone (desirable for time-consistency)

The transformation to complementarity

$$\min_{x \in X} \theta(x) + \rho(F(x))$$

$$\rho(y) = \sup_{u \in U} \left\{ \langle u, y \rangle - \frac{1}{2} \langle u, Mu \rangle \right\}$$

conjugate composite function:

$$0 \in \partial\theta(x) + \nabla F(x)^T \partial\rho(F(x)) + N_X(x)$$

calculus:

$$0 \in \partial\theta(x) + \nabla F(x)^T u + N_X(x)$$

$$0 \in -u + \partial\rho(F(x)) \iff 0 \in -F(x) + Mu + N_U(u)$$

- This is a complementarity problem (solvable by PATH)
- Equilibrium formulation
- (Fenchel) duality formulation
- Extreme point formulation

Conjugate composite function (CCF)

$$\rho(y) := \sup_{u \in U} \langle G(y), u \rangle - k(u) \tag{1}$$

$G(y) := By + b$, k is convex, U polyhedral [1]

Conjugate function $\quad G \equiv Id$

ρ is the conjugate function of $\delta_U + k$

Support function $\quad G \equiv Id,\ k \equiv 0$

ρ is the support function of U.

Conversion of constraint to objective

Can extend the conjugacy result to a nested version. Suppose that each component of F has the form $F_i = f_i + \hat{\rho}_i \circ \hat{F}_i$ and consider the CCF composition $\rho \circ F$.
Then, for any $\bar{x} \in \mathrm{dom}(\rho \circ F)$ we have

$$\partial(\rho \circ F)(\bar{x}) = \{\partial\langle v, F\rangle(\bar{x}) \mid v \in \partial\rho(F(\bar{x}))\}.$$

and

$$\partial\langle v, F\rangle(\bar{x}) = \{\langle v, \nabla f\rangle(\bar{x}) + \langle v, w\rangle \quad \text{where } v \in \partial\rho(F(\bar{x})) \\ \text{and } w_i \in \{\partial\langle \hat{v}_i, \hat{F}_i\rangle(\bar{x}) \mid \hat{v}_i \in \partial\hat{\rho}_i(\hat{F}_i(\bar{x}))\} \quad \text{for } i \in \{1, \ldots, q\}\},$$

where f collects all f_i.
So

$$\min_{x \in X} \theta(x) + \delta_{\mathbb{R}_-}(\rho(F(x)) - \alpha) = \min_{x \in X} \theta(x) + \sigma_{\mathbb{R}_+}(\rho(F(x)) - \alpha)$$

and we can apply the nested conjugacy result.

When is $\rho \circ F$ convex?

Uses the concept of K-convexity.

Lemma

Let $F : \mathbb{R}^p \to \mathbb{R}^l_\bullet$ with $F_i : \mathbb{R}^p \to \overline{\mathbb{R}}$ lsc, proper, convex for all $i \in \{1, \ldots, l\}$. Then, for any coherent risk measure ρ, the composition $\rho \circ F$ is lsc, proper, convex and $\mathrm{dom}(F) \subseteq \mathrm{dom}(\rho \circ F)$.

Reformulation via duality

Dualization [3]

$$\max_u \langle u, G(F(x))\rangle - \langle u, Mu\rangle$$
$$Au - b \in K_c$$
$$u \in K_u$$

$$\min_{z,w} \langle b, z\rangle + \tfrac{1}{2}\langle w, Jw\rangle$$
$$G(F(x)) - A^T z - Dw \in K_u^\circ$$
$$z \in K_c^\circ \quad w \text{ free}$$

K_u and K_c convex cones with polar K_u° and K_c°

Improvement to dual QP reformulation

- "The larger K_u, the smaller K_u° is"
- If u is free, then K_u is the whole space and $K_u^\circ = \{0\}$
- Try to use simple bounds to reduce K_u
- Look for $\tilde{u}$ such that $u - \tilde{u} \in \mathbb{R}^n_+$
- $G(F(x)) - A^T z - Dw - M^T \tilde{u} \in \mathbb{R}^n_-$: F convex $\Rightarrow$ convex constraints

Reformulation via conjugacy

ρ as a conjugate function

- ρ is the (Fenchel) conjugate of $k + \delta_U$:

$$\rho(u) = \inf_{u=u_1+u_2} k^*(u_1) + \sigma_U(u_2)$$

- $k(u) = u^T M u = \|L^T u\|_2$ (M psd)

$$\rho(F(x)) = \inf_s \ \tfrac{1}{2}\|s\|_2^2 + \sigma_U(G(F(x)) - Ls) \tag{2}$$

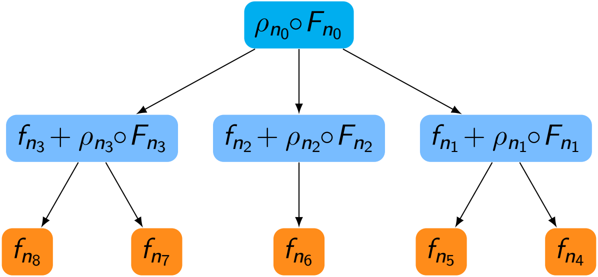
⊕ Problem (2) may be convex if all F_i are convex ($U \subset \mathbb{R}_m^+$)

⊕ Equivalent minimization problem (can use broad range of solvers)

⊖ Need closed-loop expression for σ_U

- Replace σ_U by t and compute vertices V of U and add constraints $\langle v, G(F(x)) - Ls \rangle \leq t \quad \forall v \in V$
- If U is a convex cone, replace σ_U by δ_{U°

Scenario tree with nodes $\mathcal{N} = \{0, 1, \ldots, 8\}$, and $T = 3$



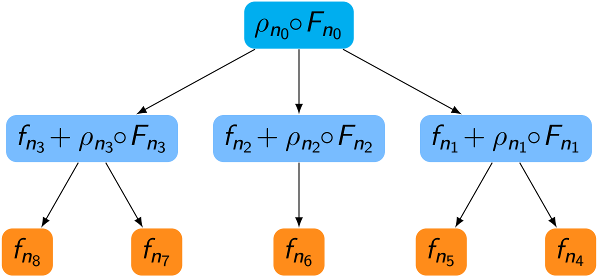
At leaf nodes:

$$\min_{x_{a\ell}} f_{a\ell}(x_{a\ell}; x_{-a\ell}, x_{\cdot\ell\_}, \pi_\ell) \quad \forall a \in \mathcal{A},$$

$$0 \in H_\ell(\pi_\ell; x_{\cdot\ell}) + N_{P_\ell}(\pi_\ell)$$

";" separates variables from parameters in function definition

Stochastic equilibrium (nested definition)

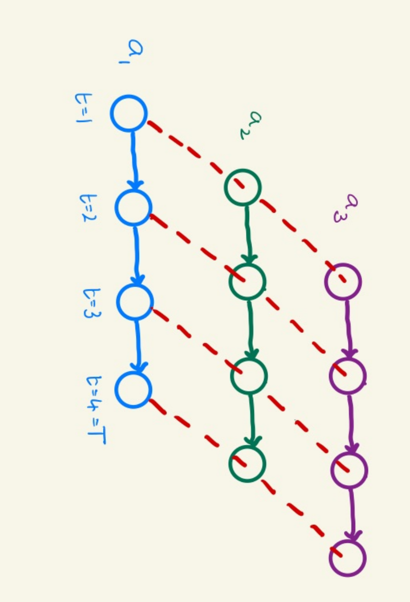


Recursing back to the root node:

$$\begin{aligned}
\min_{x_{a\mathcal{S}(n_0)}} \; & f_{an_0}(x_{an_0}; x_{-an_0}, x_{\cdot n_{0-}}, \pi_{n_0}) \\
& + \mathcal{R}_{an_0}([f_{aj}(x_{aj}; x_{-aj}, x_{\cdot n_0}, \pi_j) \\
& \quad + \mathcal{R}_{aj}([f_{a\ell}(x_{a\ell}; x_{-a\ell}, x_{\cdot \ell_-}, \pi_\ell)]_{\ell \in j_+})]_{j \in n_{0+}}) \qquad \forall a \in \mathcal{A}, \\
0 \in & H_j(\pi_j; x_{\cdot j}) + N_{P_j}(\pi_j), \qquad \forall j \in \mathcal{S}(n_0).
\end{aligned}$$

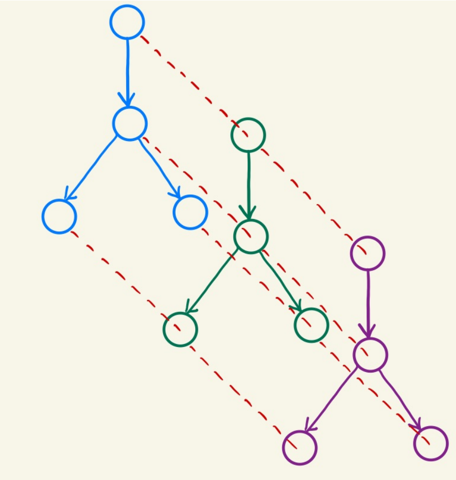
$\mathcal{S}(n)$ is the set of successor nodes of n, including n

Simple dynamics (discrete time, finite horizon)



- Complementarity links nodes across agents
- Dynamics link over time

Scenario trees linked across agents



- Complementarity links nodes of scenario tree across agents
- Dynamics link over time

Example: risk-averse stochastic equilibria

- market equilibrium: price defined by equilibrium constraints
- producers have a random upper bound on their production capacities and their ability to store goods from one stage to the other induces a coupling across stages
- objective function: revenue minus cost of production
- A, the scenario tree has 3 stages with 13 nodes, and there are 5 players in the market with 2 goods.
- B, the scenario tree has 4 stages with 30 nodes, and we have 2 players with 1 good.
- C has 5 stages, 121 nodes, 2 players and 1 good.

	Equilibrium			Duality			Conjugate		
	T (s)	vars	nnz	T (s)	vars	nnz	T (s)	vars	nnz
A	1.6	584	2775	5.2	644	2990	3.8	584	3530
B	9.0	455	2382	3.0	533	2774	Fail	455	2498
C	2.2	1400	8700	Fail	1640	10280	Fail	1400	7736

Different reformulations via option file

Multistage deterministic equivalent

$P(y)$ $\quad \min_{x^t_{an} \in X_{at}} \quad f_{a1}(x^1_{a1}, x^1_{-a1}, \pi^1_1) + \sum_{n \in 1+} y^2_{an} \cdot \left[f_{a2}(x^2_{an}, x^2_{-an}, \pi^2_n, \xi^2_n) + \sum_{m \in n+} y^3_{am} [\ldots] \right]$

$$\text{s.t} \quad h_{a1}(x_{a0}, x^1_{a1}) = 0, \quad g_{a1}(x^1_{a1}, x^1_{-a1}, \pi^1_1) \leq 0,$$

$$h_{at}(x^{t-1}_{an-}, x^t_{an}, \xi^t_n) = 0, \quad g_{at}(x^t_{an}, x^t_{-an}, \pi^t_n, \xi^t_n) \leq 0, \quad \forall t = 2, \ldots, T, \quad \forall n \in \mathcal{N}(t)$$

with the VI constraints

$$0 \leq H_1(\boldsymbol{x^1_1}, \pi^1_1) \quad \perp \pi^1_1 \geq 0$$

$$0 \leq H_t(\boldsymbol{x^t_n}, \pi^t_n, \xi^t_n) \quad \perp \pi^t_n \geq 0, \quad \forall t = 2, \ldots, T, \quad \forall n \in \mathcal{N}(t)$$

For any $t = 1, \ldots, T-2, n \in \mathcal{N}(t)$ the dual maximization problem

$D^t_{an}(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{y}_{n++})$: $\quad \max_{\{y^{t+1}_{am}\}_{m \in n+}} \quad \sum_{m \in n+} y^{t+1}_{am} \cdot \left[f_{at+1}(x^{t+1}_{am}, x^{t+1}_{-am}, \pi^{t+1}_m, \xi^{t+1}_m) + \sum_{r \in m+} y^{t+2}_{ar} [\ldots] \right]$

$$\text{s.t} \quad y^{t+1}_a \in \mathcal{D}^{t+1}_a$$

For any $t = T-1, n \in \mathcal{N}(t)$ the dual maximization problem

$D^t_{an}(\boldsymbol{x}, \boldsymbol{\pi})$: $\quad \max_{\{y^{t+1}_{am}\}_{m \in n+}} \quad \sum_{m \in n+} y^{t+1}_{am} \cdot \left[f_{at+1}(x^{t+1}_{am}, x^{t+1}_{-am}, \pi^{t+1}_m, \xi^{t+1}_m) \right]$

$$\text{s.t} \quad y^{t+1}_a \in \mathcal{D}^{t+1}_a$$

Forward backward algorithm

Define $y \in SOL(D(x, \pi)) \iff$

$$\{y_{am}^{t+1}\}_{m \in n+} \in D_{an}^{t}(\boldsymbol{x^k}, \boldsymbol{\pi^k}), \forall t = T-1, n \in \mathcal{N}(t)$$
$$\{y_{am}^{t+1}\}_{m \in n+} \in D_{an}^{t}(\boldsymbol{x^k}, \boldsymbol{\pi^k}, \boldsymbol{y_{n++}}), \forall t = 1, \ldots, T-2, n \in \mathcal{N}(t)$$

Finding a solution of the stochastic MOPEC with risk-averse agents is equivalent to find the solution (x^*, π^*, y^*) of the system

$$\begin{aligned}(x^*, \pi^*) &\in SOL(P(y^*)) \\ y^* &\in SOL(D(x^*, \pi^*))\end{aligned}$$

Detail of Forward backward algorithm

Algorithm 2 Forward-backward algorithm

set $k = 1$, set starting y^0 equal to the probability of risk-neutral case.
while stopping criterion not met **do**
  Solve the MOPEC with fixed risk probabilities $P(y^{k-1})$ to get $(x^k, \pi^k) \in SOL(P(y^{k-1}))$
  for $t = T-1, \ldots, 1$ **do**
    for $n \in \mathcal{N}(t)$ **do**
      if $t = T-1$ **then**
        $\{y_{am}^{k,t+1}\}_{m \in n+} \in D_{an}^t(\boldsymbol{x^k}, \boldsymbol{\pi^k})$
      else
        $\{y_{am}^{k,t+1}\}_{m \in n+} \in D_{an}^t(\boldsymbol{x^k}, \boldsymbol{\pi^k}, \boldsymbol{y_{n++}^k})$
      end if
    end for
  end for
  $k = k + 1$
end while

Numerical experiments

Test problem cases:

- MOPEC properites:
  - Type I: $f_a(x_a, x_{-a}, \pi) = \frac{1}{2}\epsilon||x_a||^2 + c^T x_a - \pi^T x_a + d$, $H(\boldsymbol{x}, \pi) = A\boldsymbol{x} - b$
  - Type II: $f_a(x_a, x_{-a}, \pi) = \frac{1}{2}\epsilon||x_a||^2 + c^T x_a - \pi^T x_a + d$, $H(\boldsymbol{x}, \pi) = A\boldsymbol{x} + B\pi - b$
  - Type III: $f_a(x_a, x_{-a}, \pi) = \frac{1}{2}\epsilon||x_a||^2 + c^T x_a - (B^{-1}(b - A\boldsymbol{x}))^T x_a + d$, no VI constraint and market price variable π
- Coherent risk measure:
  - $\rho(v) = (1-\lambda)\mathbb{E}[v] + \lambda CVaR_{1-\alpha}(v)$, where $CVaR_{1-\alpha}(\cdot)$ is the upper tail risk measure.
- Initial point strategy for PATH solver:
  - Strategy 1: Initial point (x, π, y) is uniformly randomly picked in the feasible region
  - Strategy 2: (x, π) of the initial point is the solution of risk-neutral problem and y is generated so initial basis matrix of PATH is nonsingular.
  - Strategy 3: Run several sweep forward-backward algorithms and use the point achieved as the initial point

Numerical results: performance of different strategies in choosing initial point

MOPEC Type	Ini Stra	total #	success #	success ratio
I	1	1000	375	37.5%
I	2	1000	555	55.5%
I	3(2)	1000	865	86.5%
II	1	1000	539	53.9%
II	2	1000	711	71.1%
II	3(2)	1000	870	87%
III	1	1000	813	81.3%
III	2	1000	892	89.2%
III	3(2)	1000	921	92.1%

- test problem size:
  - agent #: 2
  - scenario tree node size: 39
  - time stage size: 4
  - Corresponding MCP size: 455

Numeral results: changing ϵ and λ with fixed $\alpha = 0.75$

$\lvert\mathcal{N}\rvert$	T	MOPEC type	ϵ	λ	Ini Stra	total #	succ #	succ_r	FB_s #	FB_s_r
39	4	I	0	0.1	3(5)	16	16	100.00%	0	0.00%
39	4	I	0	0.3	3(5)	16	16	100.00%	0	0.00%
39	4	I	0	0.5	3(5)	16	8	50.00%	0	0.00%
39	4	I	0	0.7	3(5)	16	2	12.50%	0	0.00%
39	4	I	0	0.9	3(5)	16	0	0.00%	0	0.00%
39	4	I	1e-2	0.1	3(5)	16	16	100.00%	7	43.75%
39	4	I	1e-2	0.3	3(5)	16	16	100.00%	1	6.25%
39	4	I	1e-2	0.5	3(5)	16	16	100.00%	0	0.00%
39	4	I	1e-2	0.7	3(5)	16	8	50.00%	0	0.00%
39	4	I	1e-2	0.9	3(5)	16	4	25.00%	0	0.00%
39	4	I	1e-1	0.1	3(5)	16	16	100.00%	12	75.00%
39	4	I	1e-1	0.3	3(5)	16	16	100.00%	11	68.75%
39	4	I	1e-1	0.5	3(5)	16	16	100.00%	7	43.75%
39	4	I	1e-1	0.7	3(5)	16	16	100.00%	5	31.25%
39	4	I	1e-1	0.9	3(5)	16	16	100.00%	7	43.75%
39	4	I	1	0.1	3(5)	16	16	100.00%	16	100.00%
39	4	I	1	0.3	3(5)	16	16	100.00%	16	100.00%
39	4	I	1	0.5	3(5)	16	16	100.00%	16	100.00%
39	4	I	1	0.7	3(5)	16	16	100.00%	15	93.75%
39	4	I	1	0.9	3(5)	16	16	100.00%	16	100.00%
39	4	I	10	0.1	3(5)	16	16	100.00%	16	100.00%
39	4	I	10	0.3	3(5)	16	16	100.00%	16	100.00%
39	4	I	10	0.5	3(5)	16	16	100.00%	15	93.75%
39	4	I	10	0.7	3(5)	16	16	100.00%	16	100.00%
39	4	I	10	0.9	3(5)	16	16	100.00%	15	93.75%

Conclusions

- Markets naturally modeled via complementarity
- Solvers exist for medium to large scale problems
- Frameworks (EMP) exist to streamline model transformations
- empinfo: dualvar, bilevel, equilibrium, vi, CCF
- Very large scale models (many agents with many instruments acting strategically) with risk are hard
- Decomposition/diagonalization methods are effective when sensitivity information is exploited
- New algorithms enable solution of more detailed, authentic problems and address underlying policy questions

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