The Case for Competition: Efficient Computation of Equilibrium

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PATH for Nonlinear Complementarity Problems

- Given $F : \Re^n \to \Re^n$
- Find $x \in \Re^n$ such that

$$0 \leq F(x) \qquad x \geq 0$$

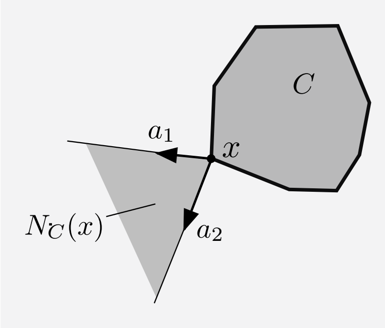
$$x^T F(x) = 0$$

- Compactly written

$$0 \leq F(x) \quad \perp \quad x \geq 0$$

- Preprocessing to simplify without changing underlying problem
- Crashing method to quickly identify basis
- Nonmonotone pathsearch with watchdog
- Perturbation scheme for rank deficiency
- Restart strategy
- Projected gradient searches

The Normal Cone



- $\mathcal{C} = \{x : a_i^T x \le b_i, i = 1, \ldots, m\}$ polyhedral
- $N_{\mathcal{C}}(x) = \left\{ \sum_{i=1}^{m} \lambda_i a_i : 0 \le b_i - a_i^T x \perp \lambda_i \ge 0 \right\}$
- $\perp$ identifies active set, i.e.

$$(b_i - a_i^T x) > 0 \implies \lambda_i = 0$$

- The normal cone captures complementarity relationships
- $-F(x) \in N_{\Re^n_+}(x)$ if and only if

$$0 \le F(x) \quad \perp \quad x \ge 0$$

The good news!

- PATH solves rectangular VI

$$-F(x) \in N_{\mathcal{I}_1 \times \ldots \times \mathcal{I}_m}(x)$$

(feasible set is a Cartesian product of possibly unbounded intervals)

- PATHVI solves VI

$$-F(x) \in N_{\mathcal{C}}(x)$$

by identifying

$$\mathcal{C} = \{x \in P : g(x) \in K\}$$

and reformulating as

$$\begin{aligned} x^* \text{ solves } \mathrm{VI}(F, \mathcal{C}) &\iff 0 \in F(x^*) + N_{\mathcal{C}}(x^*) \\ &\iff 0 \in \begin{bmatrix} F(x^*) + \nabla g(x^*)\lambda \\ -g(x^*) \end{bmatrix} + N_{P \times K^\circ}(x^*, \lambda) \end{aligned}$$

- Use Newton method, each step solves an affine variational inequality

Experimental results: AVI vs MCP

- Run PATHVI over AVI formulation.
- Run PATH over rectangular form (poorer theory as $\text{rec}(\mathcal{C})$ larger).
- Structure knowledge leads to improved reliability

Name	(#cons,#vars)	Number of iterations (time/secs)	
		PATHVI	PATH
CVXQP1_M	(500, 1000)	3119 (0.459)	fail
CVXQP2_M	(250, 1000)	33835 (2.927)	fail
CVXQP3_M	(750, 1000)	360 (0.105)	3603 (1.992)
CONT-050	(2401, 2597)	11 (2.753)	382 (272.429)
CONT-100	(9801,10197)	3 (174.267)	fail

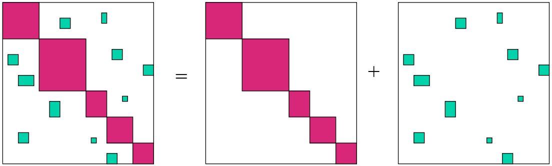
MOPEC

$$\min_{x_i} f_i(x_i, x_{-i}, \pi) \text{ s.t. } g_i(x_i, x_{-i}, \pi) \leq 0, \forall i$$

$$\pi \text{ solves VI}(h(x, \cdot), \mathcal{C})$$

```
equilibrium
min f(1) x(1) g(1)
...
min f(m) x(m) g(m)
vi h pi cons
```

- (Generalized) Nash
- (x, π) solves all problems simultaneously
- Reformulate optimization problem as first order conditions (complementarity)
- Use nonsmooth Newton methods to solve



PATHVI on Nash Equilibria

Name	Elapsed time (secs)		
	PATHVI	PATH	PATHVI/ UMFPACK
vimod1	0.372	4.129	0.437
vimod2	1.098	24.134	0.645
vimod3	3.208	60.553	1.639
vimod4	127.194	66.427	18.319
vimod5	327.970	325.558	40.285
vimod6	2341.193	1841.642	109.960

Shared Constraints: river basin example

What if agents have shared knowledge? Three agents near a river, maximizing profit by producing some commodities. Each agent can throw pollutant in the river, but limited by two shared constraints θ

$$x_i^* \in \arg\max_{x_i} p(\sum_j x_j)^T x_i - c_i(x_i) \text{ s.t. } x_i \geq 0, x_{-i} = x_{-i}^*, \theta(x) \leq 0$$

What are the multipliers on the blue shared constraint?

Can replicate constraint one for each agent (Generalized Nash)

$$\min_{x_i \in X_i} f_i(x_i, x_{-i})$$
$$\text{s.t. } \theta(x_i, x_{-i}) \leq 0$$

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$$\min_{x_i \in X_i} f_i(x_i, x_{-i})$$
$$\text{s.t. } \theta(x_i, x_{-i}) \leq 0$$

Can force all multipliers to be equal - a MOPEC (variational equilibrium)

$$\min_{x_i \in X_i} f_i(x_i, x_{-i}) + \lambda^T \theta(x_i, x_{-i})$$

$$0 \leq -\theta(x) \perp \lambda \geq 0$$

Different solutions; economists prefer the first one!

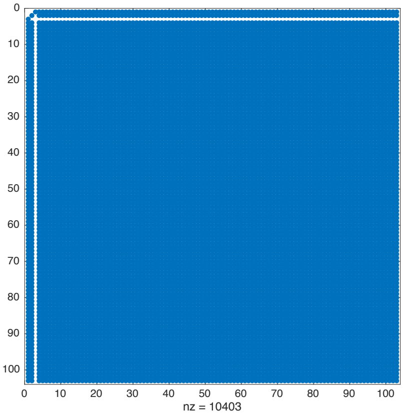
Bad news! Cournot Model (inverse demand function)

$$\max_{x_i} p(\sum_j x_j)^T x_i - c_i(x_i)$$

$$\text{s.t. } B_i x_i = b_i, x_i \geq 0$$

- Cournot model: $|\mathcal{A}| = 5$
- Size $n = |\mathcal{A}| * N_a$

Size (n)	Time (secs)
1,000	35.4
2,500	294.8
5,000	1024.6

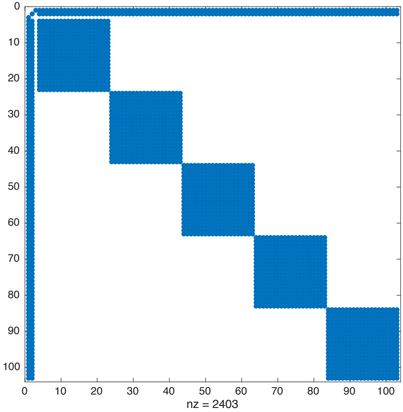


Jacobian nonzero pattern
$n = 100,\ N_a = 20$

Computation: implicit functions

- Use implicit fn: $y(x) = \sum_j x_j$
- Generalization to $h(y, x) = 0$ (via adjoints)
- empinfo: implicit y h

Size (n)	Time (secs)
1,000	2.0
2,500	8.7
5,000	38.8
10,000	> 1080

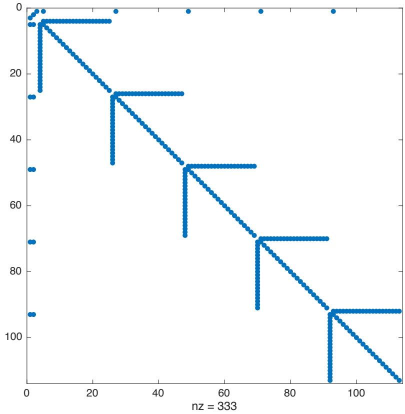


Jacobian nonzero pattern
$n = 100,\ N_a = 20$

Computation: implicit functions and local variables

- Use implicit fn: $y(x) = \sum_j x_j$ (and local aggregation)
- Generalization to $h(y, x) = 0$ (via adjoints)
- empinfo: implicit y h

Size (n)	Time (secs)
1,000	0.5
2,500	0.8
5,000	1.6
10,000	3.9
25,000	17.7
50,000	52.3



Jacobian nonzero pattern

$n = 100,\ N_a = 20$

Economic Application

- Model is a partial equilibrium, geographic exchange model.
- Goods are distinguished by region of origin.
- There is one unit of region r goods.
- These goods may be consumed in region r or they may be exported.
- Each region solves:

$$\min_{y,x_r} f_r(y,x) \text{ s.t. } h(y,x) = 0,\ x_j = \bar{x}_j, j \neq r$$

  where $f_r(y,x)$ is a quadratic form and $h(y,x)$ defines y uniquely as a function of x.
- $h(y,x)$ defines an equilibrium; here it is simply a set of equations, not a complementarity problem
- Applications: Brexit, modified NAFTA, Russian Sanctions

MCP size of equilibrium problems containing shared variables by formulation strategy

Strategy	Size of the MCP
replication	$(n+2mN)$
switching	$(n+mN+m)$
substitution (explicit)	$(n+m)$
substitution (implicit)	$(n+nm+m)$

Replication:

$$F_i(z) = \begin{bmatrix} \nabla_{x_i} f_i(x, y_i) - (\nabla_{x_i} h(y_i, x))\mu_i \\ \nabla_{y_i} f_i(x, y_i) - (\nabla_{y_i} h(y_i, x))\mu_i \\ h(y_i, x) \end{bmatrix}, \quad z_i = \begin{bmatrix} x_i \\ y_i \\ \mu_i \end{bmatrix}.$$

MCP size of equilibrium problems containing shared variables by formulation strategy

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replication	$(n + 2mN)$
switching	$(n + mN + m)$
substitution (explicit)	$(n + m)$
substitution (implicit)	$(n + nm + m)$

Switching:

$$F_i(z) = \begin{bmatrix} \nabla_{x_i} f_i(x, y) - (\nabla_{x_i} h(y, x))\mu_i \\ \nabla_y f_i(x, y) - (\nabla_y h(y, x))\mu_i \\ h(y, x) \end{bmatrix}, \quad z_i = \begin{bmatrix} x_i \\ \mu_i \\ y \end{bmatrix}.$$

Substitution eliminates μ_i, $\forall i$

Model statistics and performance comparison of the EPEC

MCP statistics according to the shared variable formulation		
Replication	Switching	Substitution
12,144 rows/cols 544,019 non-zeros 0.37% dense	6,578 rows/cols 444,243 non-zeros 1.03% dense	129,030 rows/cols 3,561,521 non-zeros 0.02% dense

PATH			Shared variable formulation (major, time)		
crash	spacer	prox	Replication	Switching	Substitution
✓		✓	7 iters 8 secs	20 iters 22 secs	20 iters 406 secs
		✓	24 iters 376 secs	22 iters 19 secs	21 iters 395 secs
	✓		8 iters 28 secs	8 iters 18 secs	8 iters 219 secs

Decomposition Results

Gauss-Seidel residuals

Iteration	Residual
1	1.526385e+04
2	1.367865e+02
3	2.216626e+00
4	2.192500e-02
5	3.195836e-04
6	8.596711e-06
7	6.048344e-07

Tariff revenue

region	SysOpt	MOPEC
1	0.117	0.012
2	0.517	0.407
3	0.496	0.214
4	0.517	0.407
5	0.117	0.012

- Note that competitive solution produces much less revenue than system optimal solution
- Model has non-convex objective, but each subproblem is solved globally (lindoglobal)
- Timing: 17.2 secs

Conclusion: who knows (and controls) what?

$$\min_{x_i} f_i(x_i, x_{-i}, y(x_i, x_{-i}), \pi) \text{ s.t. } g_i(x_i, x_{-i}, y, \pi) \leq 0, \forall i, \theta(x, y, \pi) = 0$$

$$\pi \text{ solves } \mathrm{VI}(h(x, \cdot), \mathcal{C})$$

- NE/GNE: Generalized Nash Equilibrium (feasible sets of each players problem depends on other players variables)
- Shared constraints: θ is known to all (many) players
- Force all shared constraints to have same dual variable (VI solution)
- Implicit variables: $y(x_i, x_{-i})$ shared
- Can use EMP to write all these problems, and convert to MCP form
- New decomposition algorithms available to modeler (Gauss Seidel, Randomized Sweeps, Gauss Southwell, Grouping of subproblems)
- Enables modelers to convey simple structures to algorithms and allows algorithms to exploit this
- Can evaluate effects of regulations and their implementation in a competitive environment

Spacer steps

- Given (x, y, μ) during iterations
- Compute a unique feasible pair $(\tilde{y}, \tilde{\mu})$
- Evaluate the residual at $(x, \tilde{y}, \tilde{\mu})$
- Choose the point if it has less residual than the one of (x, y, μ)