



# CS 540 Introduction to Artificial Intelligence

## **Games I**

University of Wisconsin-Madison  
**Spring 2024**

# Announcements

- **Homework:**
  - HW8 due on **Tuesday 16<sup>th</sup> at 11 AM**

- Class roadmap:

Thursday, Apr. 11 <sup>th</sup>	Games – Part I
Tuesday, Apr. 16 <sup>th</sup>	Games – Part II
Thursday, Apr. 18 <sup>th</sup>	Reinforcement Learning

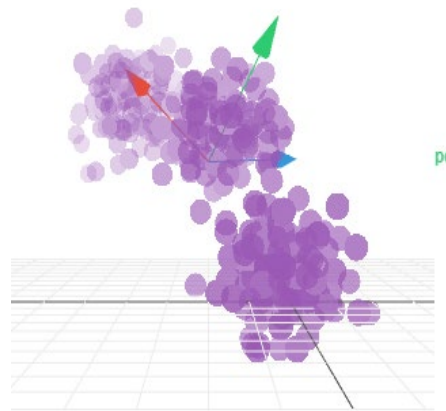
# Outline

- Introduction to game theory
  - Properties of games, mathematical formulation
- Simultaneous-Move Games
  - Normal form, strategies, dominance, Nash equilibrium

# So Far in The Course

We looked at techniques:

- **Unsupervised:** See data, do something with it. Unstructured.
- **Supervised:** Train a model to make predictions. More structure (labels).
- **Planning and Games:** Much more structure.



Victor Powell



indoor

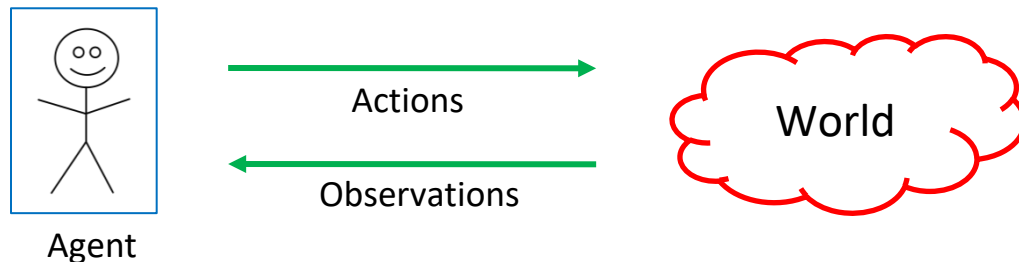


outdoor



# More General Model

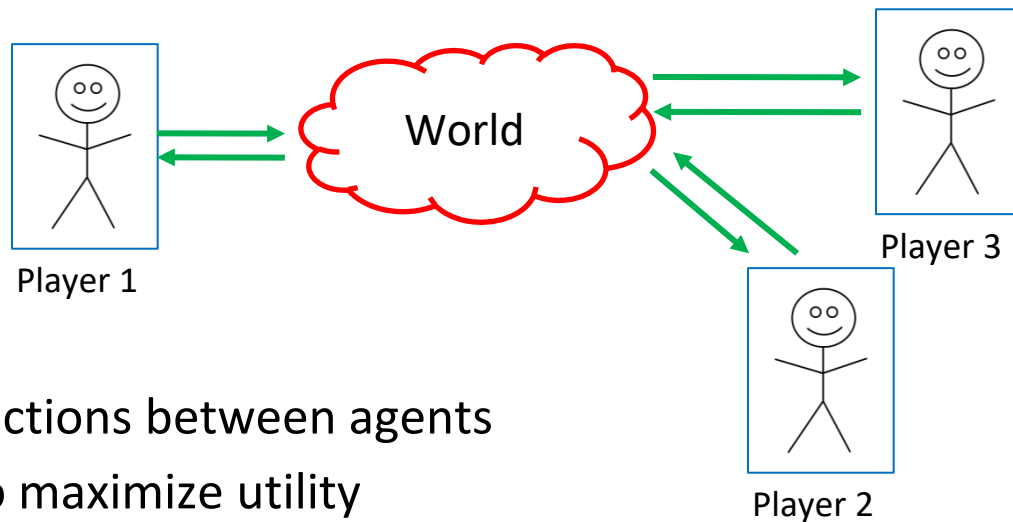
Suppose we have an **agent** **interacting** with the **world**



- Agent receives a reward based on state of the world
  - **Goal:** maximize reward / utility (\$\$\$)
  - Note: now **data** consists of actions, observations, and rewards
  - Setup for decision theory, reinforcement learning, planning

# Games: Multiple Agents

Games setup: **multiple** agents



- Now: interactions between agents
- Still want to maximize utility
- Requires **strategic** decision making.

# Modeling Games: Properties

Let's work through **properties** of games

- **Number** of agents/players
- Action space: **finite** or **infinite**
- **Deterministic** or **random**
- **Zero**-sum or **general**-sum
- **Sequential** or **simultaneous moves**

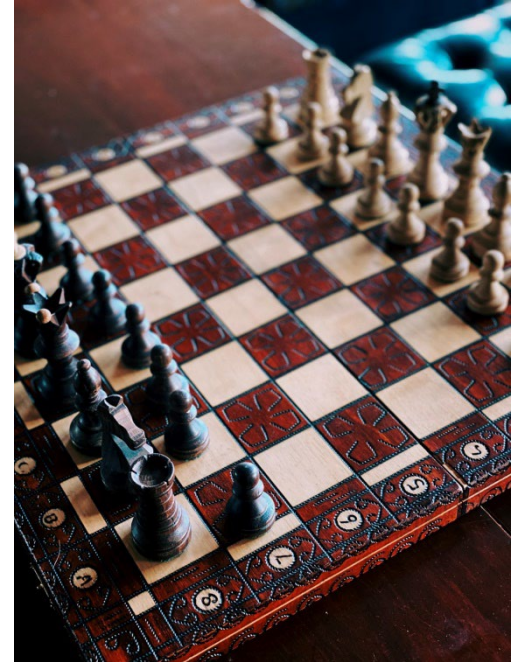


Wiki

# Property 1: **Number** of players

1 or more players

- Usually interested in  $\geq 2$  players
- Typically a finite number of players





## Property 2: Action Space

Action space: set of possible actions an agent can choose from.

Can be finite or infinite.

Examples:

- Rock-paper-scissors
- Tennis

## Property 3: **Deterministic** or **Random**

- Is there **chance** in the game?
  - Poker
  - Chess
  - Scrabble



## Property 4: **Sum of payoffs**

- Two basic types: zero sum vs. general sum.
- Zero sum: one player's win is the other's loss
  - Pure competition.
  - Example: rock-paper-scissors
- General sum
  - Example: driving to work, prisoner's dilemma

## Property 5: **Sequential** or **Simultaneous Moves**

- Simultaneous: all players take action at the same time
- Sequential: take turns
  - But payoff is often only revealed at end of game

# Quiz break:

Give the properties of the game shown on the right:

- Number of players?
- Deterministic or stochastic?
- Sum of pay-offs?
- Finite or infinite action-space?
- Sequential or simultaneous?



# Normal Form Game

Mathematical description of simultaneous games.

- $n$  players  $\{1, 2, \dots, n\}$
- Player  $i$  chooses strategy  $a_i$  from action space  $A_i$ .
- Strategy profile:  $a = (a_1, a_2, \dots, a_n)$
- Player  $i$  gets rewards  $u_i(a)$ 
  - **Note:** reward depends on other players!
- We consider the simple case where all reward functions are common knowledge.

# Example of Normal Form Game

## Ex: Prisoner's Dilemma

Player 1	Player 2	
	<i>Stay silent</i>	<i>Betray</i>
<i>Stay silent</i>	-1, -1	-3, 0
<i>Betray</i>	0, -3	-2, -2

- 2 players, 2 actions: yields 2x2 payoff matrix
- Strategy set: {Stay silent, betray}

# Strictly Dominant Strategies

Let's analyze such games. Some strategies are better than others!

- Strictly dominant strategy: if  $a_i$  strictly better than  $b$  *regardless* of what other players do,  $a_i$  is **strictly dominant**
- I.e.,  $u_i(a_i, a_{-i}) > u_i(b, a_{-i}), \forall b \neq a_i, \forall a_{-i}$



All of the other entries  
of  $a$  excluding  $i$

- Sometimes a dominant strategy does not exist!



# Strictly Dominant Strategies Example

## Back to Prisoner's Dilemma

- Examine all the entries: betray strictly dominates
- Check:

Player 1	Player 2	
	<i>Stay silent</i>	<i>Betray</i>
<i>Stay silent</i>	-1, -1	-3, 0
<i>Betray</i>	0, -3	-2, -2

# Dominant Strategy Equilibrium

$a^*$  is a (strictly) dominant strategy equilibrium (DSE), if every player  $i$  has a strictly dominant strategy  $a_i^*$

- Rational players will play at DSE, if one exists.

Player 1	Player 2	
	<i>Stay silent</i>	<i>Betray</i>
<i>Stay silent</i>	-1, -1	-3, 0
<i>Betray</i>	0, -3	-2, -2

# Dominant Strategy: Absolute Best Responses

Player  $i$ 's best response to strategy to  $a_{-i}$   $BR(a_{-i}) = \underset{b}{\operatorname{argmax}} u_i(b, a_{-i})$

$BR(\text{player2=silent}) = \text{betray}$

$BR(\text{player2=betray}) = \text{betray}$

Player 2		
Player 1	<i>Stay silent</i>	<i>Betray</i>
	<i>Stay silent</i>	<i>Betray</i>
<i>Stay silent</i>	-1, -1	-3, 0
<i>Betray</i>	0, -3	-2, -2

$a_i^*$  is the dominant strategy for player  $i$ , if

$$a_i^* = BR(a_{-i}), \forall a_{-i}$$

# Dominant Strategy Equilibrium

Dominant Strategy Equilibrium does not always exist.

		Player 2	
		$L$	$R$
Player 1	$T$	2, 1	0, 0
	$B$	0, 0	1, 2

# Nash Equilibrium

$a^*$  is a Nash equilibrium if no player has an incentive to **unilaterally deviate**

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i$$

Player 2			
		$L$	$R$
Player 1	$T$	2, 1	0, 0
	$B$	0, 0	1, 2

# Nash Equilibrium: Best Response to Each Other

$a^*$  is a Nash equilibrium:

$$\forall i, \forall b \in A_i: u_i(a_i^*, a_{-i}^*) \geq u_i(b, a_{-i}^*)$$

(no player has an incentive to **unilaterally deviate**)

- Equivalently, for each player  $i$ :

$$a_i^* \in BR(a_{-i}^*) = \operatorname{argmax}_b u_i(b, a_{-i}^*)$$

- Compared to DSE (a DSE is a NE, the other direction is generally not true):

$$a_i^* = BR(a_{-i}), \forall a_{-i}$$

# Nash Equilibrium: Best Response to Each Other

$a^*$  is a Nash equilibrium:

$$\forall i, \forall b \in A_i: u_i(a_i^*, a_{-i}^*) \geq u_i(b, a_{-i}^*)$$

(no player has an incentive to **unilaterally deviate**)

- Pure Nash equilibrium:
  - A **pure strategy** is a deterministic choice (no randomness).
  - Later: we will consider **mixed** strategies
  - In pure Nash equilibrium, players can only play pure strategies.

# Finding (pure) Nash Equilibria by hand

- As player 1: For each column, find the best response, underscore it.

Player 2		
Player 1	<i>L</i>	<i>R</i>
<i>T</i>	<u>2, 1</u>	0, 0
<i>B</i>	0, 0	<u>1, 2</u>



# Finding (pure) Nash Equilibria by hand

- As player 2: For each row, find the best response, upper-score it.

Player 1	Player 2	
	<i>L</i>	<i>R</i>
<i>T</i>	<u>2, 1</u>	0, 0
<i>B</i>	0, 0	<u>1, 2</u>

# Finding (pure) Nash Equilibria by hand

- Entries with both lower and upper bars are pure NEs.

Player 1	Player 2	
	<i>L</i>	<i>R</i>
<i>T</i>	<u>2, 1</u>	0, 0
<i>B</i>	0, 0	<u>1, 2</u>

# Pure Nash Equilibrium may not exist

So far, pure strategy: each player picks a deterministic strategy. But:

Player 2		<i>rock</i>	<i>paper</i>	<i>scissors</i>
Player 1				
<i>rock</i>		0, 0	<u>-1, 1</u>	<u>1, -1</u>
<i>paper</i>		<u>1, -1</u>	0, 0	-1, 1
<i>scissors</i>		<u>-1, 1</u>	<u>1, -1</u>	0, 0

# Mixed Strategies

Can also randomize actions: “**mixed**”

- Player  $i$  assigns probabilities  $x_i$  to each action

$$x_i(a_i), \text{ where } \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0$$

- Now consider **expected rewards**

$$u_i(x_i, x_{-i}) = E_{a_i \sim x_i, a_{-i} \sim x_{-i}} u_i(a_i, a_{-i}) = \sum_{a_i} \sum_{a_{-i}} x_i(a_i) x_{-i}(a_{-i}) u_i(a_i, a_{-i})$$

# Mixed Strategy Nash Equilibrium

Consider the mixed strategy  $x^* = (x_1^*, \dots, x_n^*)$

- This is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*) \quad \forall x_i \in \Delta_{A_i}, \forall i \in \{1, \dots, n\}$$



Better than doing  
anything else,  
“**best response**”



Space of probability  
distributions over  
strategies.

- Intuition: nobody can **increase expected reward** by changing only their own strategy.

# Mixed Strategy Nash Equilibrium

Example:  $x_1^*(\cdot) = x_2^*(\cdot) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Player 1 \ Player 2			
	<i>rock</i>	<i>paper</i>	<i>scissors</i>
<i>rock</i>	0, 0	-1, 1	1, -1
<i>paper</i>	1, -1	0, 0	-1, 1
<i>scissors</i>	-1, 1	1, -1	0, 0

# Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Example: Two Finger Morra. Show 1 or 2 fingers. The “even player” wins if the sum is even, and vice versa.

	odd		
		$f1$	$f2$
even			
	$f1$	2, -2	-3, 3
	$f2$	-3, 3	4, -4

# Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Two Finger Morra. Two-player zero-sum game. No pure NE:

	odd	$f1$	$f2$
		$f1$	$f2$
even	$f1$	<u>2, -2</u>	-3, 3
	$f2$	-3, 3	<u>4, -4</u>



# Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Suppose odd's mixed strategy at NE is  $(q, 1-q)$ , and even's  $(p, 1-p)$

By definition,  $p$  is best response to  $q$ :  $u_1(p, q) \geq u_1(p', q) \forall p'$ .

Note  $u_1(p, q) = pu_1(f_1, q) + (1 - p)u_1(f_2, q)$

Average is no greater than components

$\rightarrow u_1(p, q) = u_1(f_1, q) = u_1(f_2, q)$

		q	1-q
even	odd	f1	f2
	even		
p	f1	<u>2, -2</u>	-3, 3
1-p	f2	<u>-3, 3</u>	<u>4, -4</u>

# Finding Mixed NE in 2-Player 2-action Zero-Sum Game

$$u_1(f_1, q) = u_1(f_2, q)$$

$$2q + (-3)(1 - q) = (-3)q + 4(1 - q)$$

$$q = \frac{7}{12}$$

$$\text{Similarly, } u_2(p, f_1) = u_2(p, f_2)$$

$$p = \frac{7}{12}$$

At this NE, even gets  $-1/12$ , odd gets  $1/12$ .

		q	1-q
		f1	f2
p	even	<u>2, -2</u>	<u>-3, 3</u>
	odd	<u>-3, 3</u>	<u>4, -4</u>

# Properties of Nash Equilibrium

Major result: (John Nash '51)

- Every **finite** (players, actions) game has at least one Nash equilibrium
  - But not necessarily **pure** (i.e., deterministic strategy)
- Could be more than one
- Searching for Nash equilibria: computationally **hard**.
  - Exception: two-player zero-sum games (can be found with linear programming).

# Break & Quiz

**Q 2.1:** Which of the following is **false**?

- (i) Rock/paper/scissors has a dominant pure strategy
- (ii) There is a Nash equilibrium for rock/paper/scissors

- A. Neither
- B. (i) but not (ii)
- C. (ii) but not (i)
- D. Both

# Break & Quiz

**Q 2.1:** Which of the following is **false**?

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- A. Neither
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# Break & Quiz

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- (i) Rock/paper/scissors has a dominant pure strategy
- (ii) There is a Nash equilibrium for rock/paper/scissors

- A. Neither
- B. (i) but not (ii) (i) is indeed false: easy to check that there's no deterministic dominant strategy; (ii) is true: there is a mixed strategy Nash equilibrium
- C. (ii) but not (i)
- D. Both

# Break & Quiz

**Q 2.2:** Which of the following is **true**?

- (i) Nash equilibria require each player to know other players' possible strategies
  - (ii) Nash equilibria require rational play
- 
- A. Neither
  - B. (i) but not (ii)
  - C. (ii) but not (i)
  - D. Both

# Break & Quiz

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# Break & Quiz

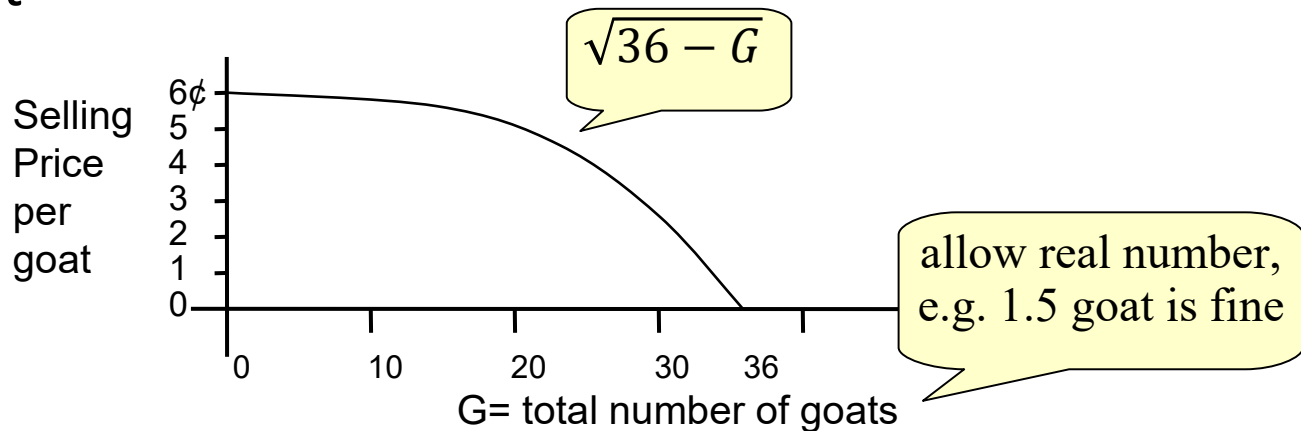
**Q 2.2:** Which of the following is **true**?

- (i) Nash equilibria require each player to know other players' possible strategies
- (ii) Nash equilibria require rational play

- A. Neither (See below)
- B. (i) but not (ii) (Rational play required: i.e., what if prisoners desire longer jail sentences?)
- C. (ii) but not (i) (The basic assumption of Nash equilibria is knowing all of the strategies involved)
- D. **Both**

Pure NE in an Infinite game:  
The tragedy of the Commons

- Price per goat



- How many goats should one (out of  $n$ ) rational farmer graze?
- How much would the farmer earn?

# Continuous Action Game

- Each farmer has infinite number of strategies  $g_i \in [0, 36]$
- The value for farmer  $i$ , when the  $n$  farmers play at  $(g_1, g_2, \dots, g_n)$  is

$$u_i(g_1, g_2, \dots, g_n) = g_i \sqrt{36 - \sum_{j \in [n]} g_j}$$

- **Assume** a pure Nash equilibrium exists.
- **Assume** (by apparent symmetry) the NE is  $(g^*, g^*, \dots, g^*)$ .

# Finding $g^*$

- $u_i(g_1, g_2, \dots, g_n) = g_i \sqrt{36 - \sum_j g_j}$
- $g^*$  is the best response to others ( $g^*, \dots, g^*$ )

$$g^* = \operatorname{argmax}_{h \in [0, 36]} u_i(g^*, \dots, h, \dots, g^*)$$

$$= \operatorname{argmax}_h h \sqrt{36 - (n-1)g^* - h}$$

i-th argument



# Finding $g^*$

$$g^* = \operatorname{argmax}_h h \sqrt{36 - (n-1)g^* - h}$$

- Taking derivative w.r.t.  $h$  of the RHS, setting to 0:

$$g^* = \frac{72 - 2(n-1)g^*}{3}$$

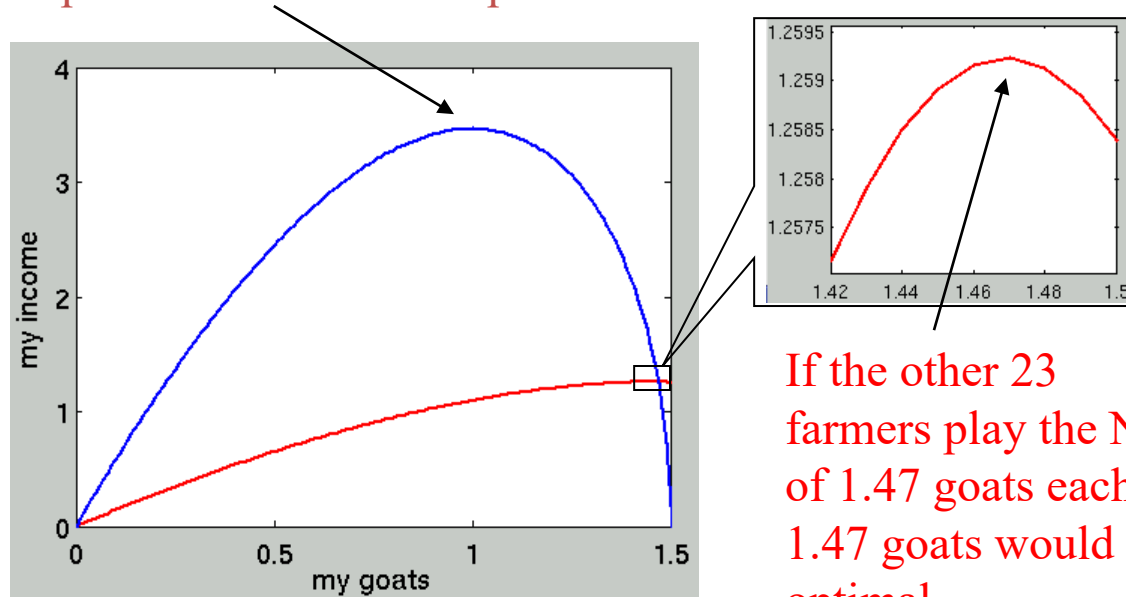
$$g^* = \frac{72}{2n+1} \quad \text{So what?}$$

# The tragedy of the Commons

- Say there are  $n=24$  farmers.  
Each would **rationally** graze  $g_i^* = 72/(2*24+1) = 1.47$  goats
- Each would get **1.25¢**
- But if they cooperate and each grazes only 1 goat
- Each would get **3.46¢**

# The tragedy of the Commons

If all 24 farmers agree on the same number of goats to raise, 1 goat per farmer would be optimal



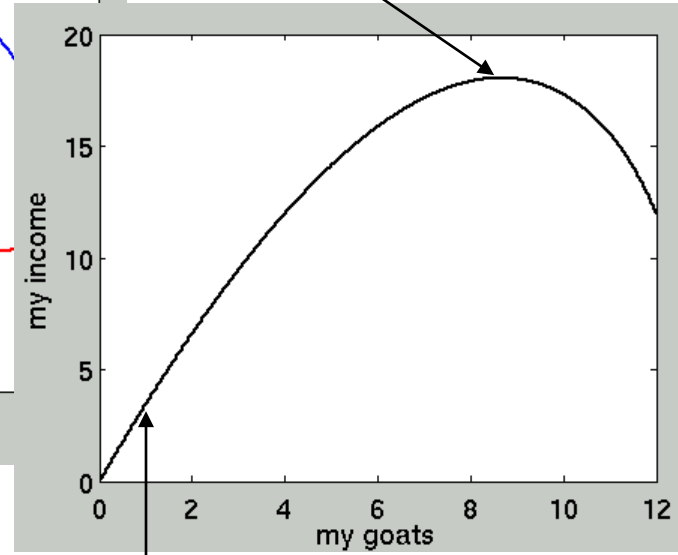
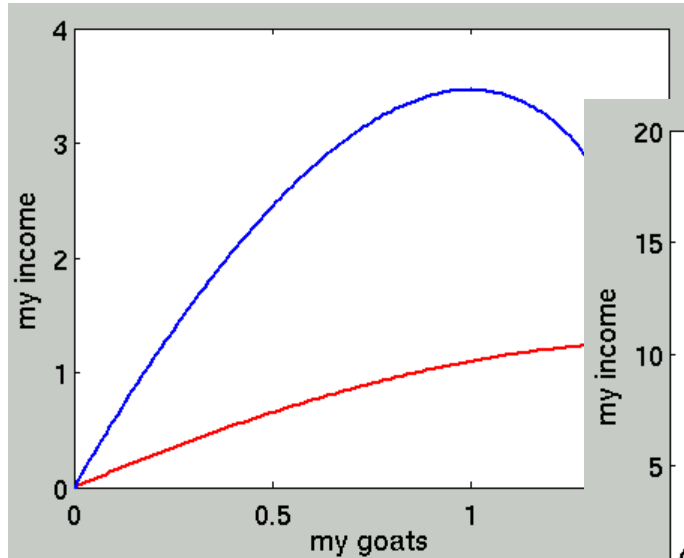
If the other 23 farmers play the N.E. of 1.47 goats each, 1.47 goats would be optimal

# The tragedy of the Commons

If all 24 farmers agree on the same number of goats to raise, 1 goat per farmer would be optimal



But this is not a N.E.! A farmer can benefit from cheating (other 23 play at 1):




'by rule'



## The tragedy

- Rational behaviors lead to sub-optimal solutions!
- Maximizing individual welfare not necessarily maximizes social welfare
- What went wrong?

Shouldn't have allowed **free** grazing?

It's not just the  : the use of the atmosphere and the oceans for dumping of pollutants.

**Mechanism design**: designing the rules of a game

# Summary

- Intro to game theory
  - Characterize games by various properties
- Mathematical formulation for simultaneous games
  - Normal form, dominance, Nash equilibria, mixed vs pure