CS 540 Introduction to Artificial Intelligence
Logic

University of Wisconsin–Madison
Fall 2025, Section 3
September 15, 2025

Announcements

- **HW 1 online:**
  - Writing assignment---nothing too stressful
  - Deadline **Friday, 9/19, 11:59PM**
- **HW 2 released Friday 9/19**
  - Probability & Statistics

Class Roadmap

Probability & Statistics	Mostly Foundations
Linear Algebra	
Principal Component Analysis (PCA)	
Logic	
Natural Language Processing (NLP)	
Machine Learning: Introduction	

Logic & Artificial Intelligence

Why are we studying logic?

- Traditional approach to AI ('50s-'80s)
  - "Symbolic AI"
  - The Logic Theorist – 1956
    - Proved a bunch of theorems!
- Logic also the language of:
  - Programming languages, databases, etc.

Logic, AI, and the Future of Math

Tools of logic might allow AI to write new, formally verifiable proofs

JUNE 8, 2024 | 12 MIN READ

AI Will Become Mathematicians' 'Co-Pilot'

Fields Medalist Terence Tao explains how proof checkers and AI programs are dramatically changing mathematics

BY CHRISTOPH DRÖSSER EDITED BY GARY STIX

Just_Super/Getty Images

AI-Driven Formal Theorem Proving

Code | Docs | Models | Dataset (Lean 3) | Dataset (Lean 4)

The Grand Challenge

The integration of artificial intelligence with formal mathematics presents a critical research challenge in bridging two fundamentally different computational paradigms. Large Language Models demonstrate remarkable capabilities in mathematical reasoning and proof generation, yet suffer from inconsistencies and hallucinations that compromise logical reliability. Formal proof assistants such as Lean provide absolute verification through mechanized type theory, ensuring every mathematical statement is rigorously validated by a trusted kernel. **Our central ambition** is to combine the power of LLMs with Lean to produce more verifiable mathematics, code, and scientific reasoning for a wide range of downstream applications.

Our Research Program

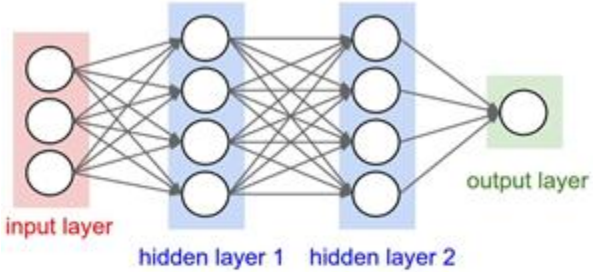
Our laboratory has developed several different research projects that systematically explores different facets of AI-assisted theorem proving. This work is primarily driven by researchers at Caltech, under the leadership of Professor Anima Anandkumar.

https://www.scientificamerican.com/article/ai-will-become-mathematicians-co-pilot/
https://leandojo.org/

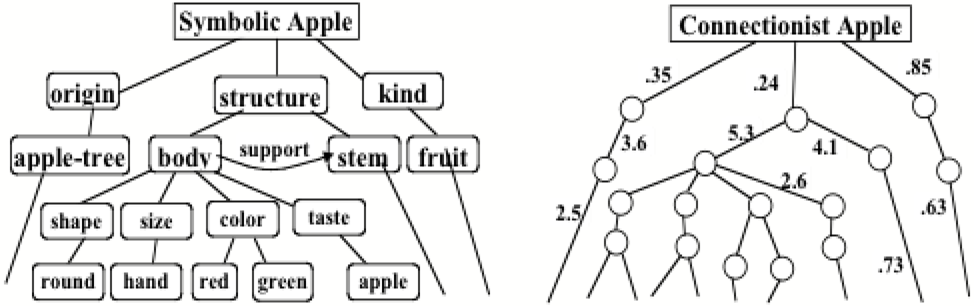
Symbolic vs Connectionist

Rival approach: **connectionist**

- Probabilistic models
- Neural networks
- **Extremely popular** last 20 years



Stanford CS231n

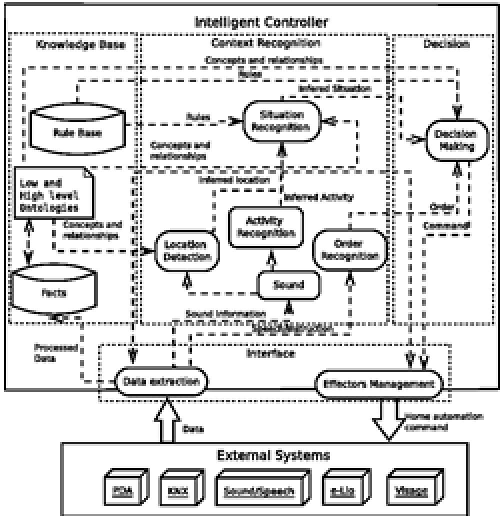


M. Minsky

Symbolic vs Connectionist

Best of both worlds?

- “Neurosymbolic AI”
- **Example**: Markov Logic Networks



Outline

- Introduction to Logic
  - Arguments, validity, soundness
- Propositional Logic
  - Sentences, semantics, inference
- First-Order Logic (FOL)
  - Predicates, objects, formulas, quantifiers

Basics of Logic

- Syntax

Well-formed	Not well-formed
“$x + y = 4$”	“$x4y +=$”

- Semantics

“$x + y = 4$” is true when x is 2 and y is 2

- Possible Worlds

All possible values for x and y

A possible world is called a **model** $\boldsymbol{m}$

- Satisfaction

If m has $x = 2$ and $y = 2$,
then m **satisfies** “$x + y = 4$”

Basics of Logic

- Entailment
- Inference

"$x = 0$" entails "$xy = 0$"

In any possible world where this is true… …this is also true.

How do we:

- Check if sentence P entails sentence Q?
- Make new conclusions?
- Prove theorems from assumptions?

We'll see examples soon.

Propositional Logic: A Very Simple Logic

Logic Vocabulary:

- Sentences, symbols, connectives, parentheses
  - Symbols: P, Q, R, … (**atomic** sentences)
  - Connectives:

∧	and	[conjunction]
∨	or	[disjunction]
⇒	implies	[implication]
⇔	is equivalent	[biconditional]
¬	not	[negation]

  - Literal: P or negation ¬P

Propositional Logic Basics

Examples:

- $(P \lor Q) \Rightarrow S$
  - "If it is cold or it is raining, then I need a jacket"
- $Q \Rightarrow P$
  - "If it is raining, then it is cold"
- $\neg R$
  - "It is not hot"

Propositional Logic Basics

Several rules in place

- Precedence: ¬, ∧, ∨, ⇒, ⇔
- Use parentheses when needed
- Syntax: **well-formed** or not well-formed:
  - P ⇒ Q ⇒ S **not well-formed (not associative!)**

Sentences & Semantics

- Sentences: built up from symbols with connectives

  - **Interpretation**: assigning True / False to symbols (a row in truth table)
  - **Semantics**: interpretations for which sentence evaluates to True
  - **Model**: (of a set of sentences) interpretation for which all sentences are True

Another kind of model :)

Evaluating a Sentence

- Example:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

- Note:
  - If P is false, P $\Rightarrow$ Q is true regardless of Q ("5 is even implies 6 is odd" is True!)
  - Causality not needed: "5 is odd implies the Sun is a star" is True!)

Evaluating a Sentence: Truth Table

- **Ex**:

P	Q	R	¬P	Q∧R	¬P∨Q∧R	¬P∨Q∧R⇒Q
0	0	0	1	0	1	0
0	0	1	1	0	1	0
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	0	0	1
1	1	1	0	1	1	1

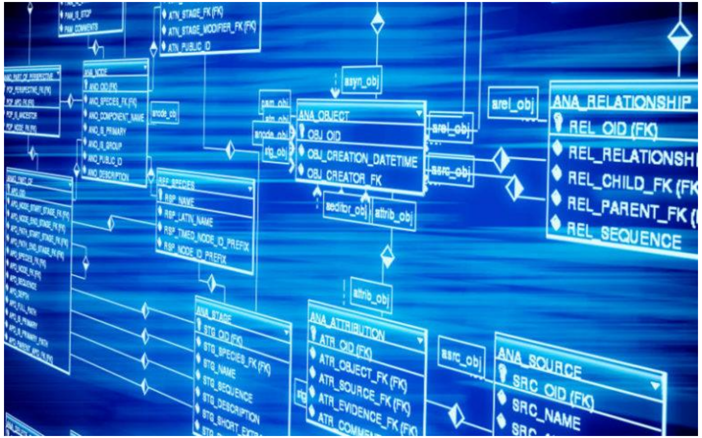
- **Satisfiable**
  - There exists some interpretation where the sentence is true.

Knowledge Bases

- **Knowledge Base (KB)**: A set of sentences $\{A_1, \ldots, A_n\}$
  - Like a long sentence, connect with conjunction
  - KB: $A_1 \land A_2 \land \ldots \land A_n$

Model of a KB: interpretations where all sentences are True

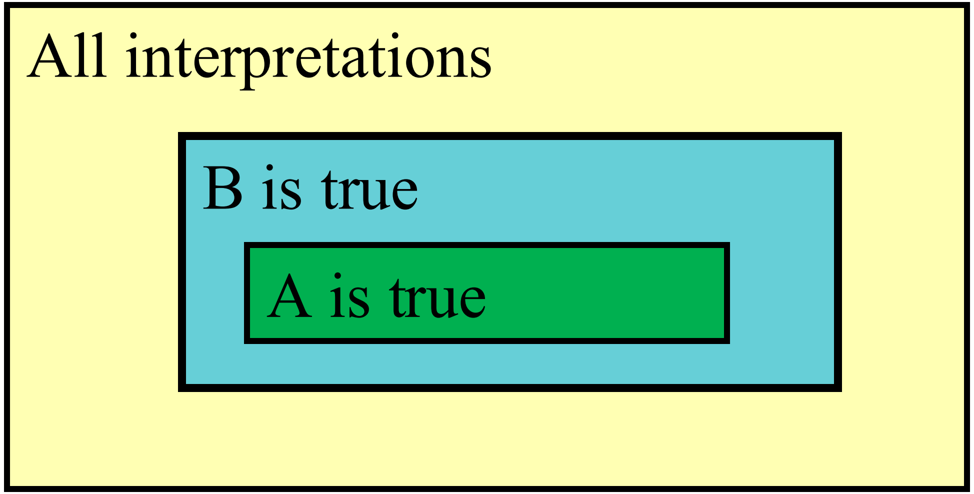
Goal: inference to discover new sentences



Entailment

Entailment: a sentence B logically follows from A

- Write A ⊨ B
- A ⊨ B if in every interpretation where A is true, B is also true



Goals of Logical Inference

- Given knowledge base: $\{A_1, A_2, \ldots, A_n\}$
- Common goal: Does KB entail sentence B?
- More generally: produce new sentences

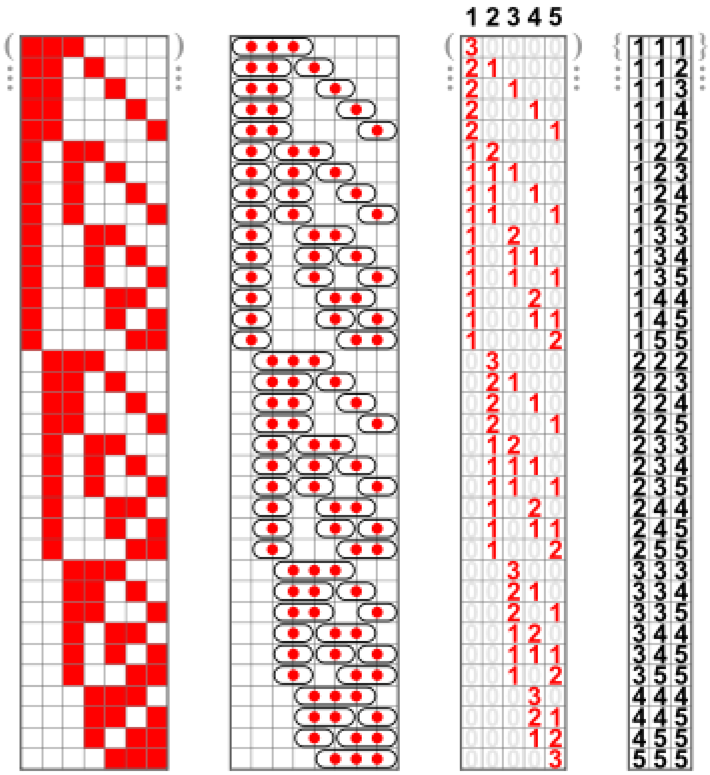
- Challenges:
  - Soundness
  - Completeness
  - Efficiency

Next slides: three types of algorithms for inference in propositional logic

Methods of Inference: **1. Enumeration**

- Enumerate all interpretations; look at the truth table
  - "Model checking"

- Downside: 2^n interpretations for n symbols



Wiki

Methods of Inference: **2. Using Rules**

- *Modus Ponens*: (A ⇒ B, A) ⊨ B

$$\frac{A \Rightarrow B \qquad A}{B}$$

- And-elimination: (A ∧ B) ⊨ A

$$\frac{A \land B}{B}$$

- Other rules on the next page
  - Commutativity, associativity,
    de Morgan's laws, distribution for
    conjunction/disjunction

Logical equivalences

$(\alpha \land \beta)$	$\equiv$	$(\beta \land \alpha)$	commutativity of $\land$
$(\alpha \lor \beta)$	$\equiv$	$(\beta \lor \alpha)$	commutativity of $\lor$
$((\alpha \land \beta) \land \gamma)$	$\equiv$	$(\alpha \land (\beta \land \gamma))$	associativity of $\land$
$((\alpha \lor \beta) \lor \gamma)$	$\equiv$	$(\alpha \lor (\beta \lor \gamma))$	associativity of $\lor$
$\neg(\neg\alpha)$	$\equiv$	α	double-negation elimination
$(\alpha \Rightarrow \beta)$	$\equiv$	$(\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta)$	$\equiv$	$(\neg\alpha \lor \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta)$	$\equiv$	$((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \land \beta)$	$\equiv$	$(\neg\alpha \lor \neg\beta)$	de Morgan
$\neg(\alpha \lor \beta)$	$\equiv$	$(\neg\alpha \land \neg\beta)$	de Morgan
$(\alpha \land (\beta \lor \gamma))$	$\equiv$	$((\alpha \land \beta) \lor (\alpha \land \gamma))$	distributivity of $\land$ over $\lor$
$(\alpha \lor (\beta \land \gamma))$	$\equiv$	$((\alpha \lor \beta) \land (\alpha \lor \gamma))$	distributivity of $\lor$ over $\land$

You can use these equivalences to modify sentences.

Methods of Inference: **3. Resolution**

- Only one rule (the **Resolution Rule**)
- Write every sentence in a special format (Conjunctive Normal Form, **CNF**)

- Foundation of many practical implementations

Resolution and Conjunctive Normal Form

- Everything needs to be in **Conjunctive Normal Form** (CNF)
  - "AND" of clauses; each clause an "OR" of literals

$$(\neg A \lor B \lor C) \land (\neg B \lor A) \land (\neg C \lor A)$$

a clause

- New sentence may be very long!

The Resolution Rule

- "Resolve" the conflict between two clauses
  - For example:

$$\frac{A \vee B \qquad \neg B \vee C}{A \vee C}$$

- The rule is **sound** (everything we infer is entailed)
- In practice: need to decide where to apply the rule

Logical Inference with Resolution

- Resolution is **complete.**
  - **Theorem:** If a set of clauses is unsatisfiable, then repeatedly applying resolution eventually yields the empty clause.

$$\frac{A \qquad \neg A}{\emptyset}$$

Contradiction!

- To check if $A_1, \ldots, A_n \vDash B$,
  run resolution on $\{A_1, \ldots, A_n, \neg B\}$

Break and Quiz

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg P \Rightarrow \neg Q) \land R$

(ii) $(\neg P \lor \neg Q) \rightarrow (P \lor \neg R)$

- A. Both
- B. Neither
- C. Just (i)
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

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- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg P \Rightarrow \neg Q) \wedge R$

(ii) $(\neg P \vee \neg Q) \rightarrow (P \vee \neg R)$

- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Plug interpretation into each sentence.

For (i): $(\neg p \rightarrow \neg q)$ will be false so $\neg(\neg p \rightarrow \neg q)$ will be true and r is true by assignment.

For (ii): $(\neg p \vee \neg q)$ is true and $(p \vee \neg r)$ is false which makes the implication false.

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- a. $A \lor (\neg A \rightarrow B)$
- b. $A \lor B$
- c. $A \lor (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- **a. $A \lor (\neg A \rightarrow B)$**
- **b. $A \lor B$ (equivalent!)**
- c. $A \lor (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- **a. A V (¬A → B)**
- **b. A V B (equivalent!)**
- c. A V (A → B)
- d. A → B

Answer a. is the exact translation of the English sentence into a logic sentence. You can see that answer b. is also correct by writing out the truth table for all answers and seeing that a and b have the same truth tables.

Or you can use the fact that $\neg A \rightarrow B = A \vee B$ and that $A \vee A \vee B = A \vee B$ to prove equivalence.

Break & Quiz

Q 2.1: Which has more rows: a truth table on *n* symbols, or a joint distribution table on *n* binary random variables?

- A. Truth table
- B. Distribution
- C. Same size
- D. It depends

Break & Quiz

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Break & Quiz

Q 2.1: How many entries does a truth table have for a FOL sentence with k variables where each variable can take on n values?

- A. Truth tables are not applicable to FOL.
- B. 2^k
- C. n^k
- D. It depends

Break & Quiz

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Break & Quiz

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- C. $\boldsymbol{n^k}$
- D. It depends

Must have one entry for every possible assignment of values to variables. That number is (C).

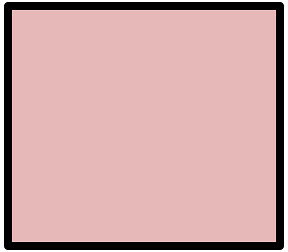
First-Order Logic (FOL)

Propositional logic has some limitations

- Ex: how to say “all squares have four sides”?
- No context, hard to generalize; express facts

FOL is a more expressive logic; works over

- Facts, Objects, Relations, Functions



First-Order Logic Syntax

- **Term**: an object in the world
  - **Constant**: Alice, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**(term$_1$, …, term$_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.
  - Constants or functions of constants

FOL Syntax

- **Atom**: smallest T/F expression
  - **Predicate**($term_1$, …, $term_n$)
    - Manager(Alice, Bob), Blue(table)
    - Convention: read “Alice (is) Manager (of) Bob”
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - **$term_1$ = $term_2$**
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object

FOL Syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\land \lor \neg \Rightarrow \Leftrightarrow$
    - Less(x,22) $\land$ Less(y,33)
  - Complex sentence using quantifiers $\forall$, $\exists$
- Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets define the predicates

FOL Quantifiers

- Universal quantifier: ∀
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is ⇒
  - Forms if-then rules
  - “all humans are mammals”

$$\forall x\ \mathbf{human(x)} \Rightarrow \mathbf{mammal(x)}$$

  - Means if x is a human, then x is a mammal

FOL Quantifiers

- Existential quantifier: ∃
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is ∧
  - "some humans are male"

    ∃x human(x) ∧ male(x)

  - Means there is an x who is a human and is a male