CS 540 Introduction to Artificial Intelligence
Logic (continued) & Natural Language Processing

University of Wisconsin–Madison
Fall 2025, Section 3
September 17, 2025

Announcements

- **HW 1 online:**
  - Writing assignment---nothing too stressful
  - Deadline **Friday, 9/19, 11:59PM**
- **HW 2 released Friday 9/19**
  - Probability & Statistics

Class Roadmap

Probability & Statistics
Linear Algebra
Principal Component Analysis (PCA)
Logic
Natural Language Processing (NLP)
Machine Learning: Introduction

Mostly Foundations

Today’s Class

- Review & Finish Propositional Logic
- First-Order Logic

- Introduction to NLP and Language Modeling

Last Class: Propositional Logic

- Defined in terms of **syntax** & **semantics**
- Syntax:
  - Which sentences are valid?
- Semantics:
  - Given a possible world, which sentences are true?

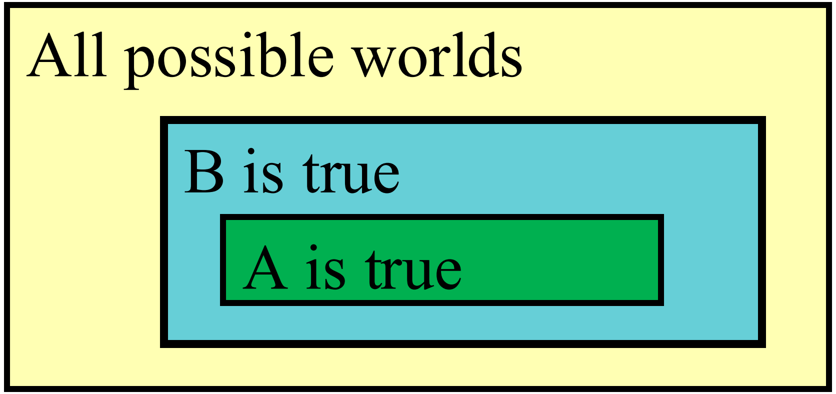
Last Class: Propositional Logic

- Syntax:
  - Atomic symbols: P, Q, R ...
  - Connectives: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
  - Valid sentences built out of these
- Semantics
  - Defined by truth table

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

Last Class: Entailment

- Let A, B be valid sentences
- We write $A \vDash B$ if:
  - in every possible world where A is TRUE…
  - then B is also TRUE



Logical Inference

- Given knowledge base: $\{A_1, A_2, \ldots, A_n\}$
- Common goal: Does KB entail sentence B?
- More generally: produce new sentences

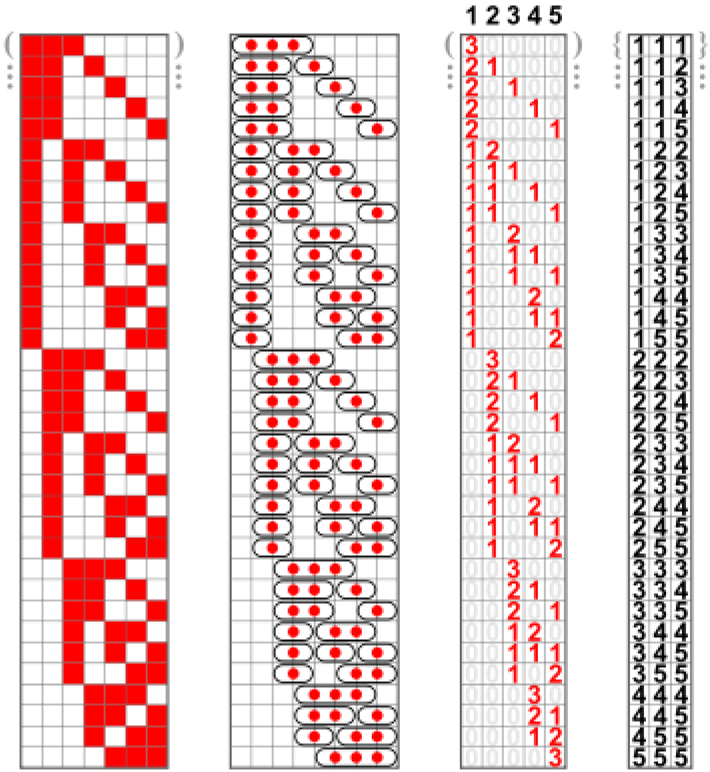
- Challenges:
  - Soundness
  - Completeness
  - Efficiency

Next slides: three types of algorithms for inference in propositional logic

Methods of Inference: **1. Enumeration**

- Enumerate all interpretations; look at the truth table
  - "Model checking"

- Downside: 2^n interpretations for n symbols



Wiki

Methods of Inference: **2. Using Rules**

- *Modus Ponens*: (A ⇒ B, A) ⊨ B

$$\frac{A \Rightarrow B \qquad A}{B}$$

- And-elimination: (A ∧ B) ⊨ A

$$\frac{A \wedge B}{B}$$

- Other rules on the next page
  - Commutativity, associativity, de Morgan’s laws, distribution for conjunction/disjunction

Logical equivalences

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) && \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) && \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) && \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) && \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) && \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) && \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) && \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) && \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

You can use these equivalences to modify sentences.

Methods of Inference: **3. Resolution**

- Only one rule (the **Resolution Rule**)
- Write every sentence in a special format (Conjunctive Normal Form, **CNF**)

- Foundation of many practical implementations

Resolution and Conjunctive Normal Form

- Everything needs to be in **Conjunctive Normal Form** (CNF)
  - "AND" of clauses; each clause an "OR" of literals

$$\underbrace{(\neg A \lor B \lor C)}_{\text{a clause}} \land (\neg B \lor A) \land (\neg C \lor A)$$

- New sentence may be very long!

The Resolution Rule

- “Resolve” the conflict between two clauses
  - For example:

$$\frac{A \vee B \qquad \neg B \vee C}{A \vee C}$$

- The rule is **sound** (everything we infer is entailed)
- In practice: need to decide where to apply the rule

Logical Inference with Resolution

- Resolution is **complete.**
  - **Theorem:** If a set of clauses is unsatisfiable, then repeatedly applying resolution eventually yields the empty clause.

$$\frac{A \qquad \neg A}{\emptyset}$$

Contradiction!

- To check if $A_1, \ldots, A_n \vDash B$,
  run resolution on $\{A_1, \ldots, A_n, \neg B\}$

First-Order Logic

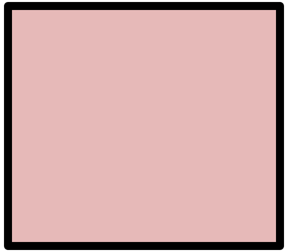
First-Order Logic (FOL)

Propositional logic has some limitations

- Ex: how to say “all squares have four sides”?
- No context, hard to generalize; express facts

FOL is a more expressive logic; works over

- Facts, Objects, Relations, Functions



First-Order Logic Syntax

- **Term**: an object in the world
  - **Constant**: Alice, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**($term_1$, …, $term_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.
  - Constants or functions of constants

FOL Syntax

- **Atom**: smallest T/F expression
  - **Predicate**($term_1$, …, $term_n$)
    - Manager(Alice, Bob), Blue(table)
    - Convention: read “Alice (is) Manager (of) Bob”
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - **$term_1$ = $term_2$**
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object

FOL Syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\wedge$ $\vee$ $\neg$ $\Rightarrow$ $\Leftrightarrow$
    - Less(x,22) $\wedge$ Less(y,33)
  - Complex sentence using quantifiers $\forall$, $\exists$
- Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets define the predicates

FOL Quantifiers

- Universal quantifier: ∀
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is ⇒
  - Forms if-then rules
  - "all humans are mammals"

    ∀x human(x) ⇒ mammal(x)
  - Means if x is a human, then x is a mammal

FOL Quantifiers

- Existential quantifier: $\exists$
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is $\wedge$
  - "some humans are male"

    $$\exists x\ \mathbf{human(x)} \wedge \mathbf{male(x)}$$

  - Means there is an x who is a human and is a male

Break and Quiz

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg P \Rightarrow \neg Q) \wedge R$

(ii) $(\neg P \vee \neg Q) \rightarrow (P \vee \neg R)$

- A. Both
- B. Neither
- C. Just (i)
- D. Just (ii)

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Plug interpretation into each sentence.

For (i): $(\neg p \rightarrow \neg q)$ will be false so $\neg(\neg p \rightarrow \neg q)$ will be true and r is true by assignment.

For (ii): $(\neg p \vee \neg q)$ is true and $(p \vee \neg r)$ is false which makes the implication false.

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- a. $A \lor (\neg A \rightarrow B)$
- b. $A \lor B$
- c. $A \lor (A \rightarrow B)$
- d. $A \rightarrow B$

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Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- **a. $A \lor (\neg A \rightarrow B)$**
- **b. $A \lor B$ (equivalent!)**
- c. $A \lor (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- **a. A V (¬A → B)**
- **b. A V B (equivalent!)**
- c. A V (A → B)
- d. A → B

Answer a. is the exact translation of the English sentence into a logic sentence. You can see that answer b. is also correct by writing out the truth table for all answers and seeing that a and b have the same truth tables.

Or you can use the fact that $\neg A \rightarrow B = A \vee B$ and that $A \vee A \vee B = A \vee B$ to prove equivalence.

Break & Quiz

Q 2.1: Which has more rows: a truth table on *n* symbols, or a joint distribution table on *n* binary random variables?

- A. Truth table
- B. Distribution
- C. Same size
- D. It depends

Break & Quiz

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Q 2.1: How many entries does a truth table have for a FOL sentence with k variables where each variable can take on n values?

- A. Truth tables are not applicable to FOL.
- B. 2^k
- C. n^k
- D. It depends

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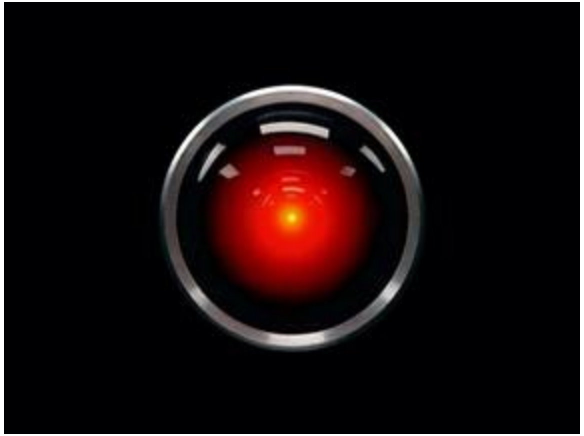
Must have one entry for every possible assignment of values to variables. That number is (C).

Natural Language Processing

What is **NLP**?

Combining computing with human language. Want to:

- Answer questions
- Summarize or extract information
- Translate between languages
- Generate dialogue/language
- Write stories automatically



Why is it **hard**?

Many reasons:

- Ambiguity: "*Mary saw the duck with the telescope in the park*". Several meanings.
- Understanding of the world
  - "Bob and Joe are fathers".
  - "Bob and Joe are brothers".

Approaches to NLP

A brief history

- Symbolic NLP: 50’s to 90’s
- Statistical/Probabilistic: 90’s to present
    - Neural nets: 2010’s to present
    - Large Language Model (LLM): GPT etc.

Lots of progress!

Lots more to work to do

ELIZA program

Language Models

- Basic idea: use probabilistic models to **assign a probability to a sentence W**

$$P(W) = P(w_1, w_2, \ldots, w_n) \text{ or } P(w_{\text{next}} | w_1, w_2 \ldots)$$

- Goes back to Shannon
  - Information theory: letters

Zero-order approximation	XFOML RXKHRJFFJUJ ALPWXFWJXYJ FFJEYVJCQSGHYD QPAAMKBZAACIBZLKJQD
First-order approximation	OCRO HLO RGWR NMIELWIS EU LL NBNESEBYA TH EEI ALHENHTTPA OOBTTVA NAH BRL
Second-order approximation	ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO TIZIN ANDY TOBE SEACE CTISBE
Third-order approximation	IN NO IST LAT WHEY CRATICT FROURE BIRS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA OF CRE
First-order word approximation	REPRESENTING AND SPEEDILY IS AN GOOD APT OR COME CAN DIFFERENT NATURAL HERE HE THE A IN CAME THE TO OF TO EXPERT GRAY COME TO FURNISHES THE LINE MESSAGE HAD BE THESE

Training The Model

Recall the chain rule of probability:

$$P(w_1, w_2, \ldots, w_n) = P(w_1)P(w_2|w_1)\ldots P(w_n|w_{n-1}\ldots w_1)$$

- How do we estimate these probabilities?
  - I.e., “training” in machine learning.
- From data (text corpus)
  - Can’t estimate reliably for long histories.

Training: Make Assumptions

- Markov assumption with shorter history:

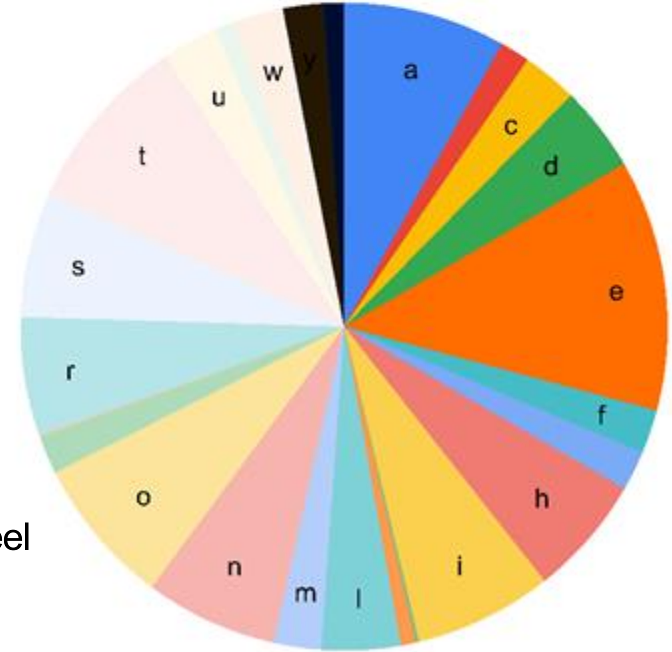
$$P(w_i|w_{i-1}w_{i-2}\dots w_1) = P(w_i|w_{i-1}w_{i-2}\dots w_{i-k})$$

- Present doesn't depend on whole past
  - Just recent past, i.e., *context*.
  - What's ***k=0?***

k=0: **Uni**gram Model

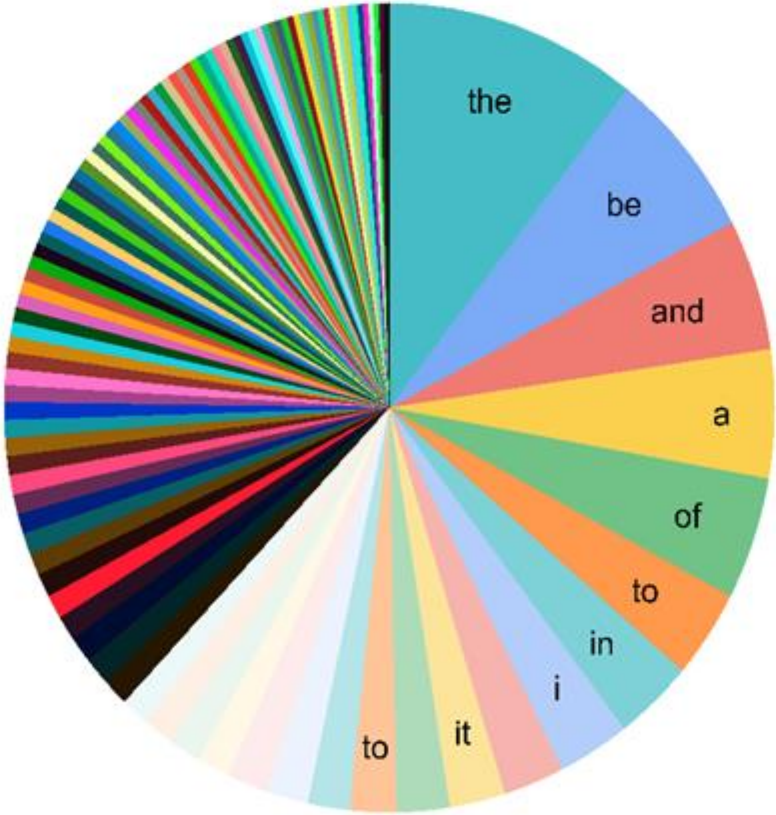
- Full independence assumption:
  - (Present doesn't depend on the past)

$$P(w_1, w_2, \ldots, w_n) = P(w_1)P(w_2)\ldots P(w_n)$$



The English letter frequency wheel

Unigram word model



Example (from Dan Jurafsky's notes)

```
fifth, an, of, futures, the, an, incorporated, a,
a, the, inflation, most, dollars, quarter, in, is,
mass thrift, did, eighty, said, hard, 'm, july,
bullish that, or, limited, the
```

k=1: **Bi**gram Model

- Markov Assumption:
  - (Present depends on immediate past)

$$P(w_1, w_2, \ldots, w_n) = P(w_1)P(w_2|w_1)P(w_3|w_2)\ldots P(w_n|w_{n-1})$$

$p(.|q)$: the "after q" wheel

$p(.|j)$: the "after j" wheel

```
texaco, rose, one, in, this, issue,
is, pursuing, growth, in, a, boiler,
house, said, mr., gurria, mexico, 's,
motion, control, proposal, without,
permission, from, five, hundred,
fifty, five, yen outside, new, car,
parking, lot, of, the, agreement,
reached this, would, be, a, record,
november
```

k=n-1: **n-**gram Model

Can do trigrams, 4-grams, and so on

- More expressive as *n* goes up
- Harder to estimate

Training: just count? I.e, for bigram:

$$P(w_i|w_{i-1}) = \frac{\mathrm{count}(w_{i-1}, w_i)}{\mathrm{count}(w_{i-1})}$$

Simple “generative AI” from letter bigram (Markov Chain)

Writing = sampling

- Say we start with q
- Sample from $P(\cdot \mid q)$: spin the “after q” wheel  , we get u
- Sample from $P(\cdot \mid u)$: spin the “after u” wheel, say we get e
- Sample from $P(\cdot \mid e)$: spin the “after e” wheel, say we get r
- ...

Sampling Shakespeare unigram LM

- To him swallowed confess hear both. Which. Of save on trail for are ay device and rote life have
- Every enter now severally so, let
- Hill he late speaks; or! a more to leg less first you enter
- Will rash been and by I the me loves gentle me not slavish page, the and hour; ill let
- Are where exeunt and sighs have rise excellency took of .. sleep knave we. near; vile like

Jurafsky & Martin, Speech and language processing, Prentice Hall, 2000.

Sampling Shakespeare bigram LM

- What means, sir. I confess she? then all sorts, he is trim, captain.
- Why dost stand forth thy canopy, forsooth; he is this palpable hit the King Henry. Live king. Follow.
- What we, hath got so she that I rest and sent to scold and nature bankrupt, nor the first gentleman?
- Enter Menenius, if it so many good direction found'st thou art a strong upon command of fear not a liberal largess given away, Falstaff! Exeunt

Jurafsky & Martin, Speech and language processing, Prentice Hall, 2000.

Sampling Shakespeare trigram LM

- Sweet prince, Falstaff shall die. Harry of Monmouth's grave.
- This shall forbid it should be branded, if renown made it empty.
- What ist that cried?
- Indeed the duke; and had a very good friend.

Jurafsky & Martin, Speech and language processing, Prentice Hall, 2000.

n-gram Training

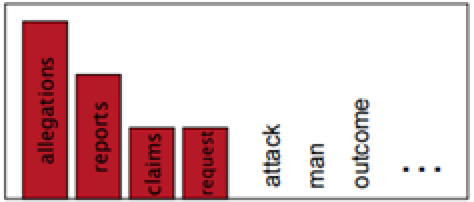
$$P(w_i|w_{i-1}) = \frac{\text{count}(w_{i-1}, w_i)}{\text{count}(w_{i-1})}$$

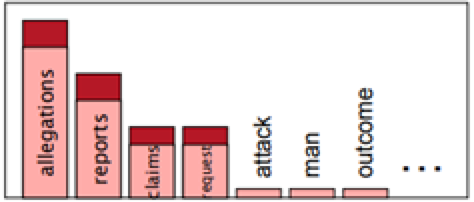
Issues:

- **1**. Multiply tiny numbers?
  - **Solution**: use logs; add instead of multiply
- **2.** n-grams with zero probability?
  - **Solution**: smoothing

$$P(w_i|w_{i-1}) = \frac{\text{count}(w_{i-1}, w_i) + 1}{\text{count}(w_{i-1}) + V}$$

$P(w|\text{denied the})$





Dan Klein

Evaluating Language Models

How do we know we've done a good job?

- Observation
- Train/test on separate data & measure metrics
- **Metrics**:
  - 1. Extrinsic evaluation
  - 2. Perplexity