CS 540 Introduction to Artificial Intelligence
Games I (continued)

University of Wisconsin–Madison

Fall 2025, Section 3

November 12, 2025

Announcements

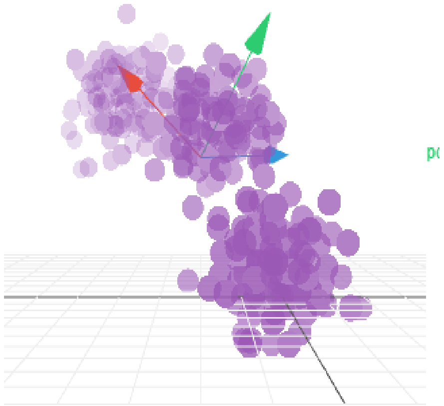
- **Homework**:
  - HW7 due on **Friday, Nov 14th at 11:59 PM**
- Class roadmap:

Search
Games – Part I
Games – Part II
Reinforcement Learning

So Far in The Course

We looked at techniques:

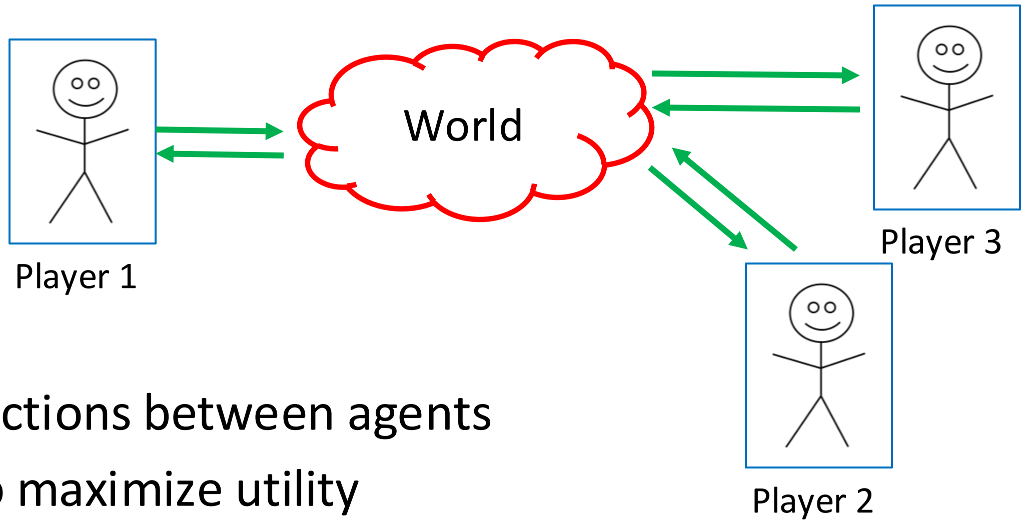
- **Unsupervised:** See data, do something with it. Unstructured.
- **Supervised:** Train a model to make predictions. More structure (labels).
- **Planning and Games**: Much more structure.



Victor Powell

Games: Multiple Agents

Games setup: **multiple** agents



- Now: interactions between agents
- Still want to maximize utility
- Requires **strategic** decision making.

Modeling Games: Properties

Let's work through **properties** of games

- **Number** of agents/players
- Action space: **finite** or **infinite**
- **Deterministic** or **random**
- **Zero**-sum or **general**-sum
- **Sequential** or **simultaneous moves**

Wiki

Normal Form Game

Mathematical description of simultaneous games.

- $\boldsymbol{n}$ players $\{1,2,\ldots,n\}$
- Player i chooses strategy/action $\boldsymbol{a_i}$ from action space $\boldsymbol{A_i}$.
- Strategy profile: $\boldsymbol{a = (a_1, a_2, \ldots, a_n)}$
- Player i gets rewards $\boldsymbol{u_i(a)}$
  - **Note**: reward depends on other players!

- We consider the simple case where all reward functions are common knowledge.

Example of Normal Form Game

Ex: Prisoner’s Dilemma

Player 2 Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

- 2 players, 2 actions: yields 2x2 payoff matrix
- Strategy set: {Stay silent, betray}

Absolute Best Responses

Player i's best response to strategy to a_{-i}

$$BR(a_{-i}) \stackrel{\text{def}}{=} \operatorname*{argmax}_b u_i(b, a_{-i})$$

BR(player2=silent) = betray

BR(player2=betray) = betray

Player 2 Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

a_i^* is the dominant strategy for player *i*, if

$$a_i^* = BR(a_{-i}), \forall\, a_{-i}$$

Nash Equilibrium

$a*$ is a Nash equilibrium if no player has an incentive to **unilaterally deviate**

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i$$

Player 2 / Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Nash Equilibrium: Best Response to Each Other

$a*$ is a Nash equilibrium:

$$\forall i, \forall b \in A_i: u_i(a_i^*, a_{-i}^*) \geq u_i(b, a_{-i}^*)$$

(no player has an incentive to **unilaterally deviate**)

- Equivalently, for each player i:

$$a_i^* \in BR(a_{-i}^*) = argmax_b\, u_i(b, a_{-i}^*)$$

- Compared to DSE (a DSE is a NE, the other direction is generally not true):

$$a_i^* = BR(a_{-i}), \forall\, a_{-i}$$

Nash Equilibrium: Best Response to Each Other

$a*$ is a Nash equilibrium:

$$\forall i, \forall b \in A_i : u_i(a_i^*, a_{-i}^*) \geq u_i(b, a_{-i}^*)$$

(no player has an incentive to **unilaterally deviate**)

- Pure Nash equilibrium:
  - A **pure strategy** is a deterministic choice (no randomness).
  - Later: we will consider **mixed** strategies
  - In pure Nash equilibrium, players can only play pure strategies.

Finding (pure) Nash Equilibria by hand

- As player 1: For each column, find the best response, underscore it.

Player 2 Player 1	*L*	*R*
T	<u>2, 1</u>	0, 0
B	0, 0	<u>1, 2</u>

Finding (pure) Nash Equilibria by hand

- As player 2: For each row, find the best response, upper-score it.

Player 2 / Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Finding (pure) Nash Equilibria by hand

- Entries with both lower and upper bars are pure NEs.

Player 2 / Player 1	*L*	*R*
T	2, 1	0, 0
B	0, 0	1, 2

Pure Nash Equilibrium may not exist

So far, pure strategy: each player picks a deterministic strategy. But:

Player 2 / Player 1	*rock*	paper	*scissors*
rock	0, 0	-1, 1	1, -1
paper	1, -1	0, 0	-1, 1
scissors	-1, 1	1, -1	0, 0

Mixed Strategies

Can also randomize actions: "**mixed**"

- Player *i* assigns probabilities x_i to each action

$$x_i(a_i), \text{ where } \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0$$

- Now consider **expected rewards**

$$u_i(x_i, x_{-i}) = E_{a_i \sim x_i, a_{-i} \sim x_{-i}} u_i(a_i, a_{-i}) = \sum_{a_i} \sum_{a_{-i}} x_i(a_i) x_{-i}(a_{-i}) u_i(a_i, a_{-i})$$

Mixed Strategy Nash Equilibrium

Consider the mixed strategy $x^* = (x_1^*, \ldots, x_n^*)$

- This is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-1}^*) \geq u_i(x_i, x_{-i}^*) \quad \forall x_i \in \Delta_{A_i}, \forall i \in \{1, \ldots, n\}$$

Better than doing anything else, "**best response**"

Space of probability distributions over strategies.

- Intuition: nobody can **increase expected reward** by changing only their own strategy.

Mixed Strategy Nash Equilibrium

Example: $x_1^*(\cdot) = x_2^*(\cdot) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Player 2 Player 1	*rock*	paper	*scissors*
rock	0, 0	-1, 1	1, -1
paper	1, -1	0, 0	-1, 1
scissors	-1, 1	1, -1	0, 0

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Example: Two Finger Morra. Show 1 or 2 fingers. The "even player" wins if the sum is even, and vice versa.

odd / even	*f1*	f2
f1	2, -2	-3, 3
f2	-3, 3	4, -4

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Two Finger Morra. Two-player zero-sum game. No pure NE:

odd / even	*f1*	f2
f1	2, -2	-3, 3
f2	-3, 3	4, -4

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Suppose odd's mixed strategy at NE is (q, 1-q), and even's (p, 1-p)

By definition, p is best response to q: $u_1(p, q) \geq u_1(p', q) \forall p'$.

Note $u_1(p, q) = pu_1(f_1, q) + (1 - p)u_1(f_2, q)$

Average is no greater than components

→ $u_1(p, q) = u_1(f_1, q) = u_1(f_2, q)$

	odd / even	q	1-q
		f1	f2
p	*f1*	2, -2	-3, 3
1-p	f2	-3, 3	4, -4

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

$$u_1(f_1, q) = u_1(f_2, q)$$

$$2q + (-3)(1 - q) = (-3)q + 4(1 - q)$$

$$q = \frac{7}{12}$$

Similarly, $u_2(p, f_1) = u_2(p, f_2)$

$$p = \frac{7}{12}$$

At this NE, even gets -1/12, odd gets 1/12.

		q	1-q
	odd even	*f1*	f2
p	*f1*	2, -2	-3, 3
1-p	f2	-3, 3	4, -4

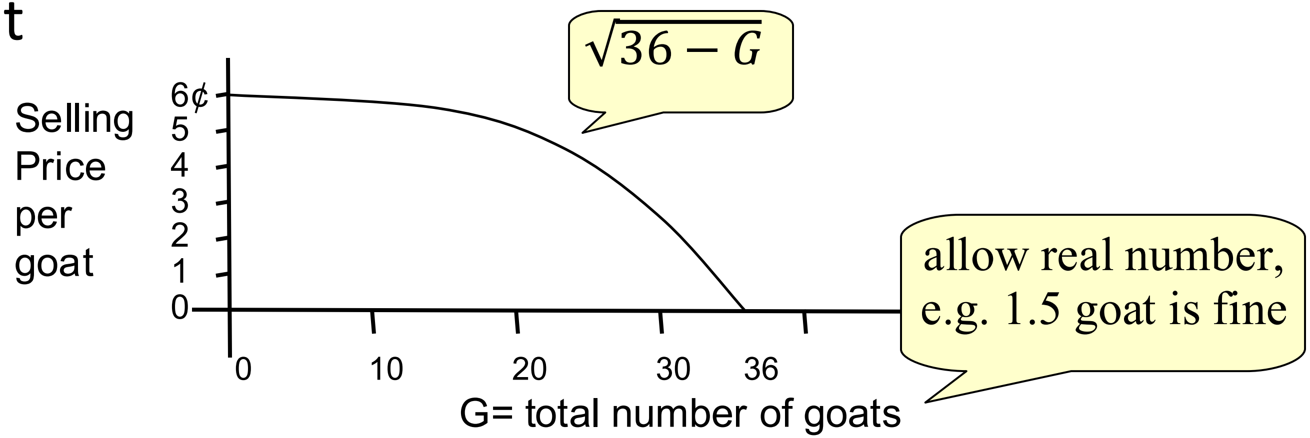
Properties of Nash Equilibrium

Major result: (John Nash ’51)

- Every finite (players, actions) game has at least one Nash equilibrium
  - But not necessarily **pure** (i.e., deterministic strategy)
- Could be more than one
- Searching for Nash equilibria: computationally **hard**.
  - Exception: two-player zero-sum games (can be found with linear programming).

Pure NE in an Infinite game: The tragedy of the Commons

- Price per goat



- How many goats should one (out of *n*) rational farmer graze?
- How much would the farmer earn?

Continuous Action Game

- Each farmer has infinite number of strategies $g_i \in [0,36]$
- The value for farmer *i*, when the *n* farmers play at $(g_1, g_2, \ldots, g_n)$ is

$$u_i(g_1, g_2, \ldots, g_n) = g_i \sqrt{36 - \sum_{j \in [n]} g_j}$$

- Assume a pure Nash equilibrium exists.
- Assume (by apparent symmetry) the NE is $(g^*, g^*, \ldots, g^*)$.

Finding g*

- $u_i(g_1, g_2, \ldots, g_n) = g_i\sqrt{36 - \sum_j g_j}$
- g* is the best response to others (g*,…, g*)

$$g^* = argmax_{h\in[0,36]} u_i(g^*, \ldots, h, \ldots, g^*)$$

i-th argument

$$= argmax_h h\sqrt{36 - (n-1)g^* - h}$$

Finding g*

$$g^* = argmax_h h\sqrt{36 - (n-1)g^* - h}$$

- Taking derivative w.r.t. *h* of the RHS, setting to 0:

$$g^* = \frac{72 - 2(n-1)g^*}{3}$$

$$g^* = \frac{72}{2n+1}$$

So what?

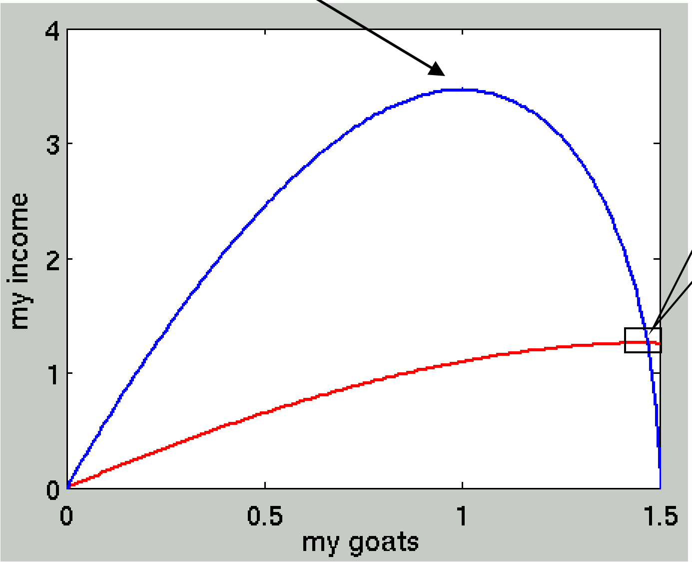
The tragedy of the Commons

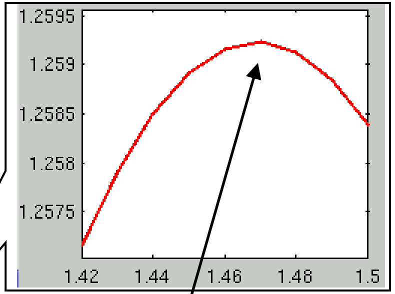
- Say there are n=24 farmers.
  Each would rationally graze $g_i^* = 72/(2*24+1) = 1.47$ goats
- Each would get 1.25¢

- But if they cooperate and each grazes only 1 goat
- Each would get 3.46¢

The tragedy of the Commons

If all 24 farmers agree on the same number of goats to raise, 1 goat per farmer would be optimal



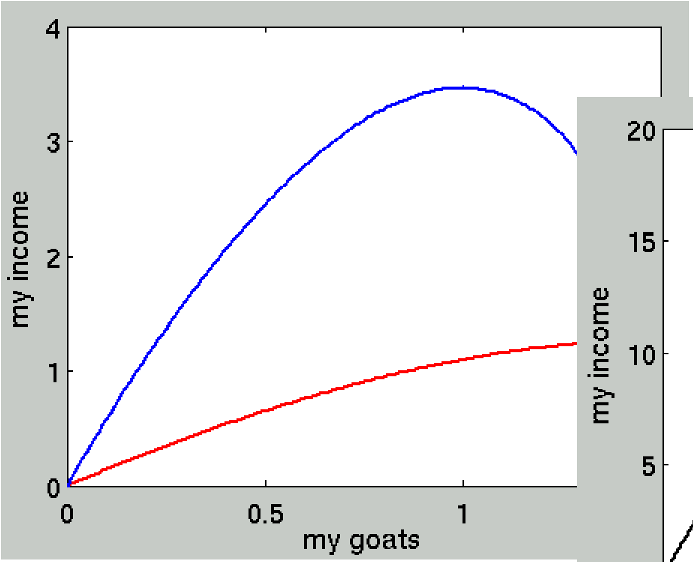


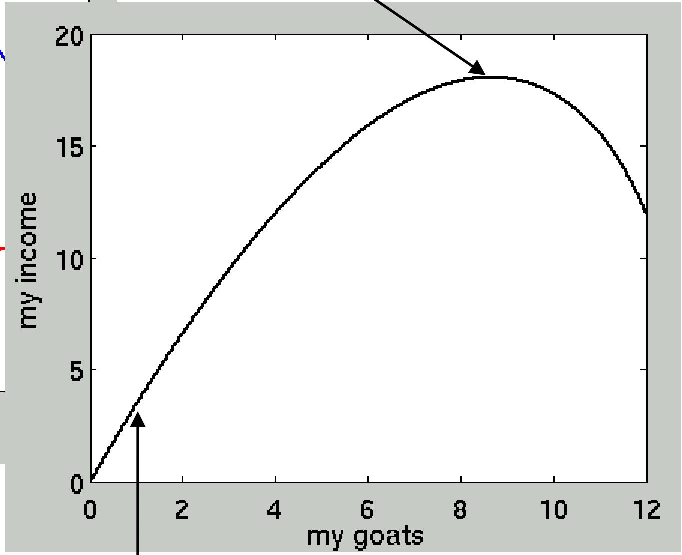
If the other 23 farmers play the N.E. of 1.47 goats each, 1.47 goats would be optimal

The tragedy of the Commons

If all 24 farmers agree on the same number of goats to raise, 1 goat per farmer would be optimal

But this is not a N.E.! A farmer can benefit from cheating (other 23 play at 1):





'by rule'

The tragedy

- Rational behaviors lead to sub-optimal solutions!
- Maximizing individual welfare not necessarily maximizes social welfare
- What went wrong?

Shouldn't have allowed free grazing?

Mechanism design: designing the rules of a game

Summary

- Intro to game theory
  - Characterize games by various properties
- Mathematical formulation for simultaneous games
  - Normal form, dominance, Nash equilibria, mixed vs pure