CS 540 Introduction to Artificial Intelligence
Reinforcement Learning I

University of Wisconsin-Madison
Spring 2025

Announcements

- **Homework**:
  - HW9 due on **Wednesday April 23rd at 11:59 PM**

- Class roadmap:

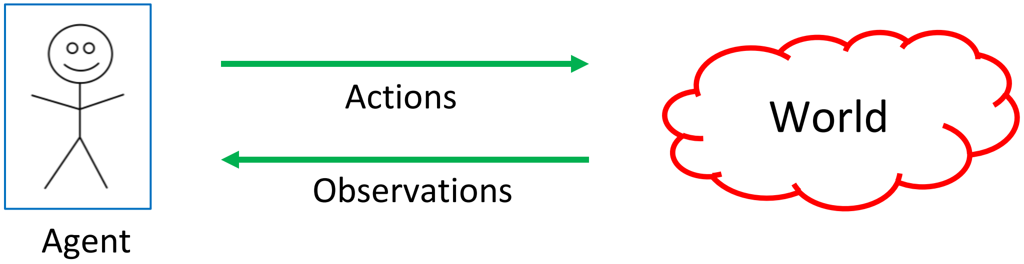
Introduction to Reinforcement Learning
Reinforcement Learning II
Advanced Search

Outline

- Introduction to reinforcement learning
  - Basic concepts, mathematical formulation, MDPs, policies.
- Learning policies
  - Q-learning, action-values, exploration vs exploitation.

Back to Our General Model

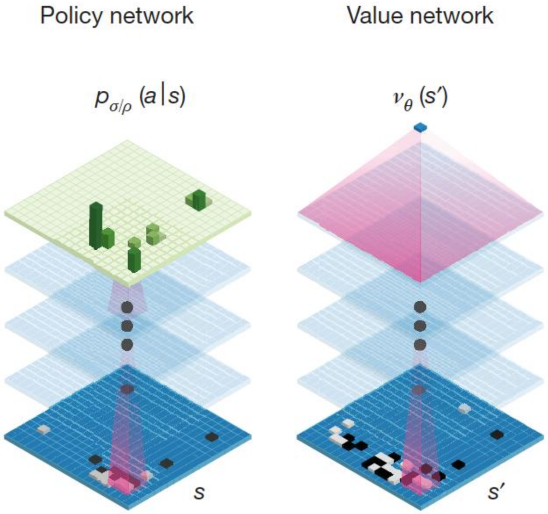
We have an agent interacting with the world



- Agent receives a reward based on state of the world
  - **Goal**: maximize reward / utility **($$$)**
  - Note: **data** consists of actions & observations
    - Compare to unsupervised learning and supervised learning

Examples: Gameplay Agents

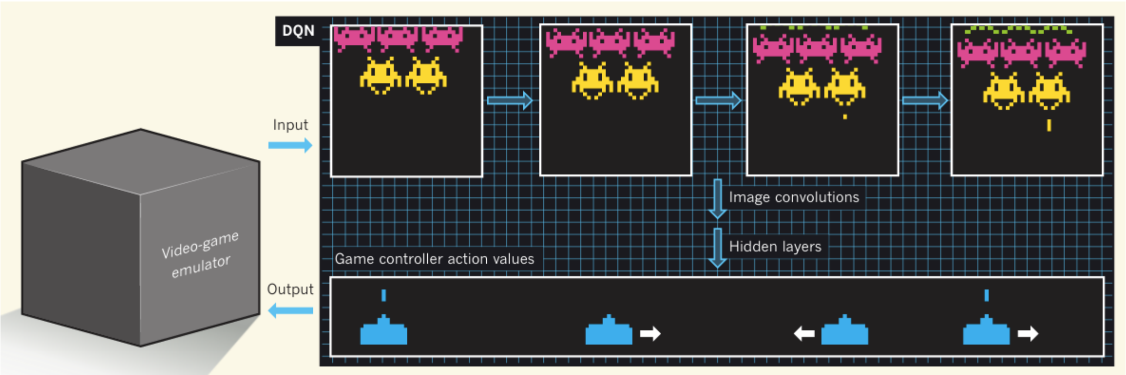
AlphaZero:



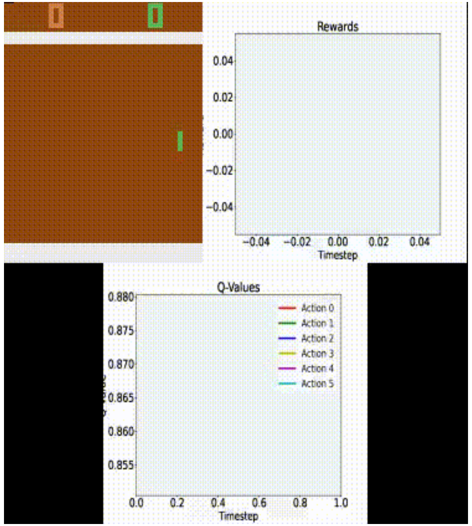
https://deepmind.com/research/alphago/

Examples: Video Game Agents

Pong, Atari



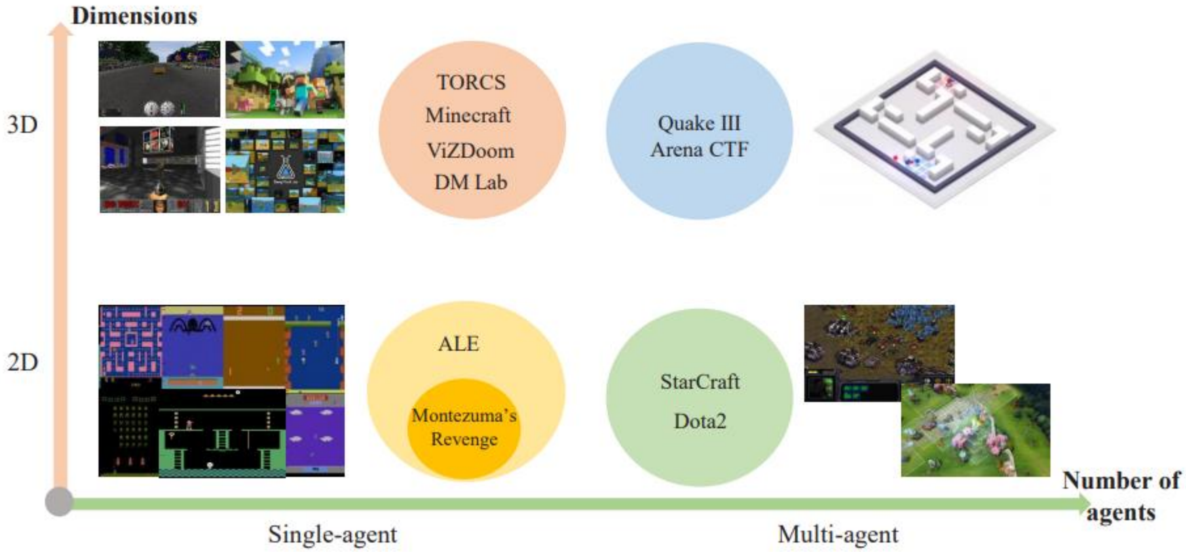
Mnih et al, “Human-level control through deep reinforcement learning”



A. Nielsen

Examples: Video Game Agents

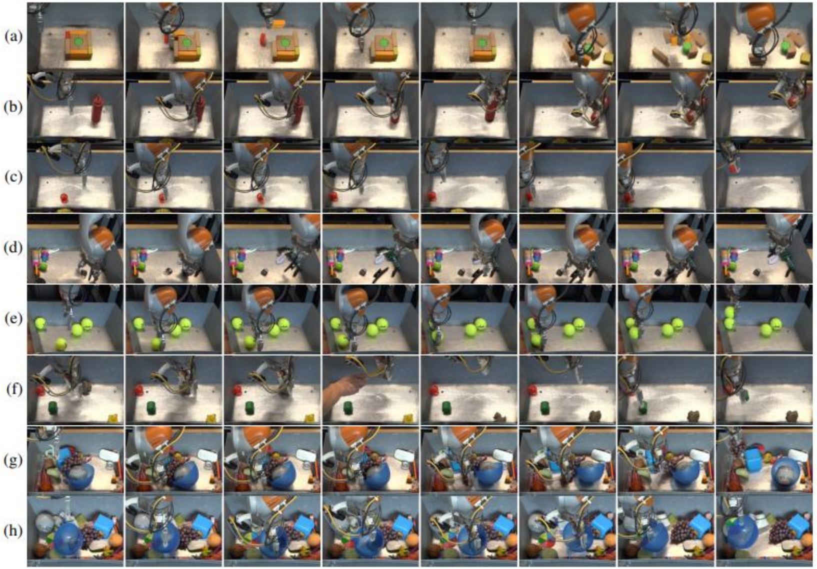
Minecraft, Quake, StarCraft, and more!



Shao et al, "A Survey of Deep Reinforcement Learning in Video Games"

Examples: Robotics

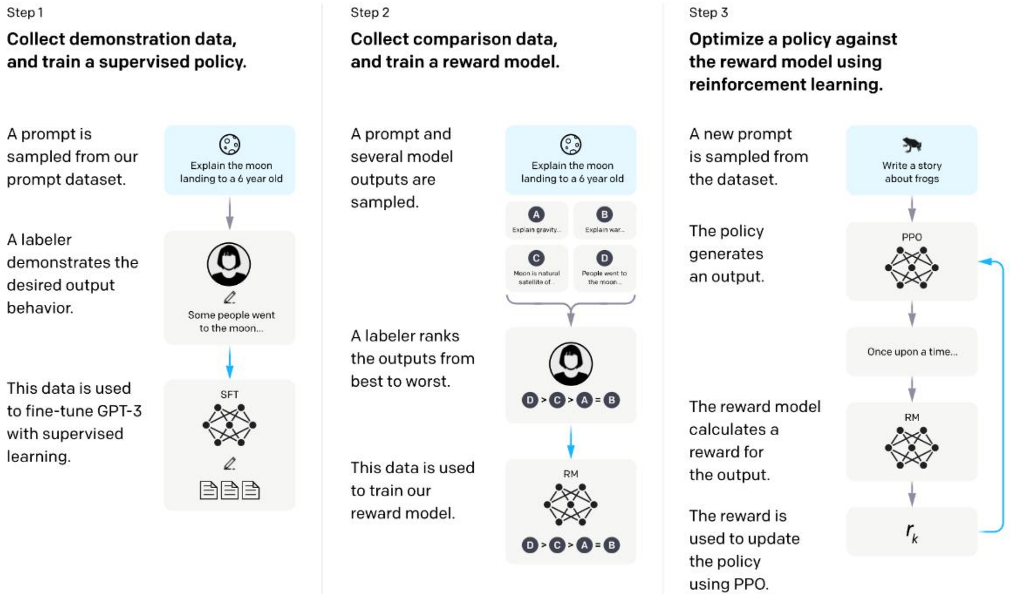
Training robots to perform tasks (e.g., grasp objects!)



Ibarz et al, " How to Train Your Robot with Deep Reinforcement Learning – Lessons We've Learned "

Examples: Large Language Models

RL used to “align” model outputs to human preferences



C. Huyen, https://huyenchip.com/2023/05/02/rlhf.html

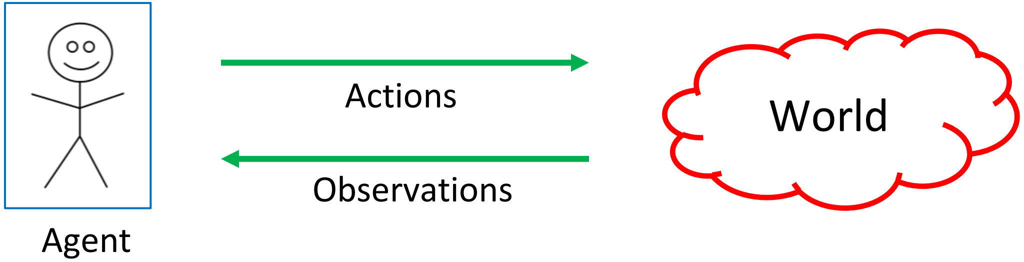
Unaligned Model

There are many ways to cheat in poker, including using marked cards, collusion, and using electronic devices to track other players' cards.

Aligned Model Post-RLHF

Cheating in poker is illegal and unethical, and can result in penalties such as fines or expulsion from the game. There here are many ways to win at poker legitimately, such as developing a strong strategy, practicing, and analyzing your opponents' behavior.

Building The Theoretical Model



Basic setup:

- Set of states, S
- Set of actions, A
- Information: at time t, observe state $s_t \in S$. Get reward r_t
- Agent makes choice $a_t \in A$. State changes to $s_{t+1,}$ continue

Goal: find a map from **states to actions** that maximize rewards.

A “policy”

Markov Decision Process (MDP)

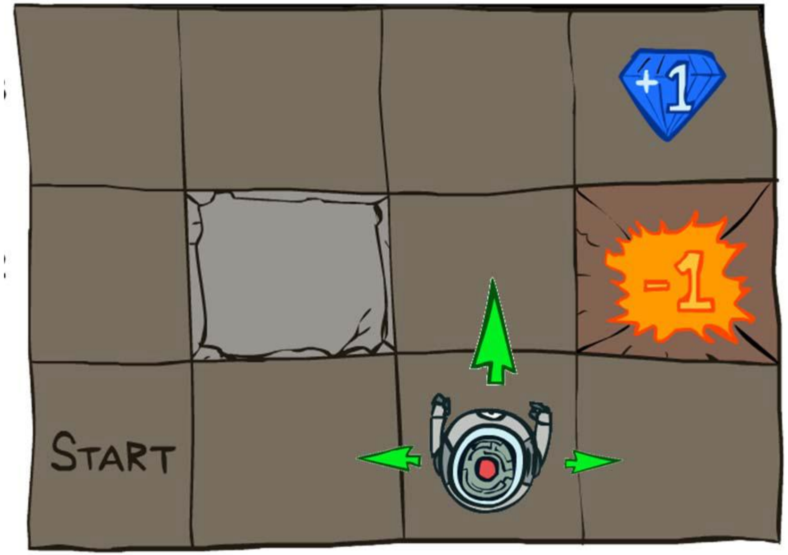
The formal mathematical model:

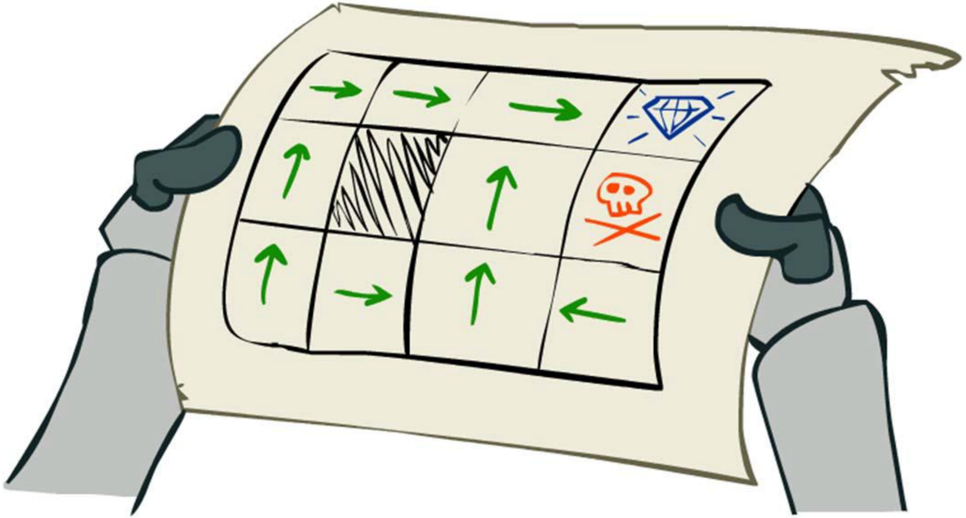
- **State set** S. Initial state $s_{0.}$ **Action set** A
- **Reward function:** $r(s_t)$
- **State transition model:** $P(s_{t+1}|s_t, a_t)$
  - Markov assumption: transition probability only depends on s_t and a_t, and not earlier history (previous actions or states)
- More generally: $r(s_t, a_t)$, potentially random
- **Policy**: $\pi(s) : S \to A$ action to take at a particular state

$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \ldots$$

Example of MDP: Grid World

Robot on a grid; goal: find the best policy

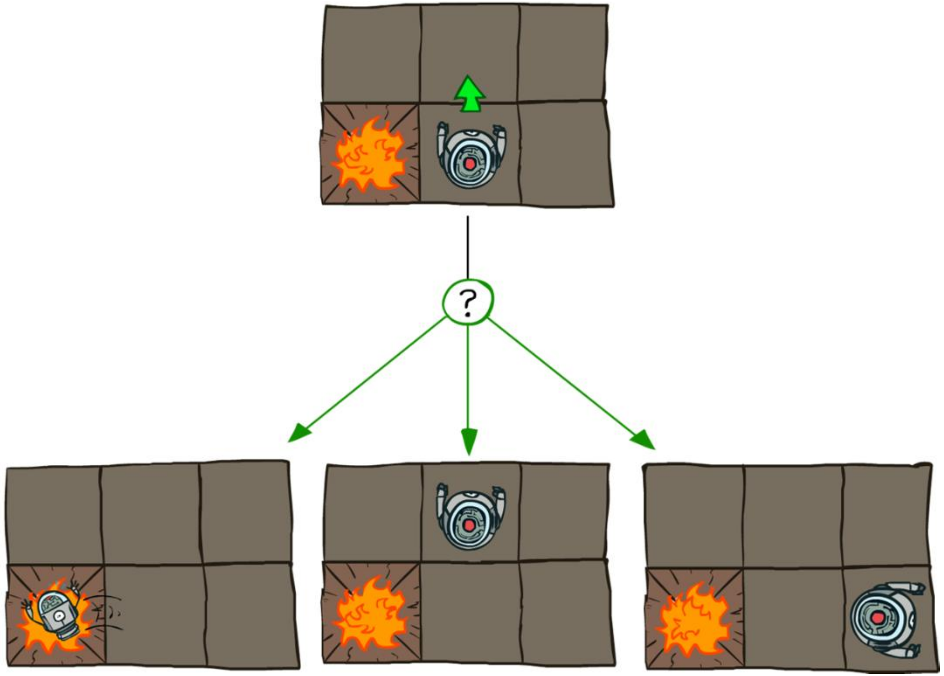


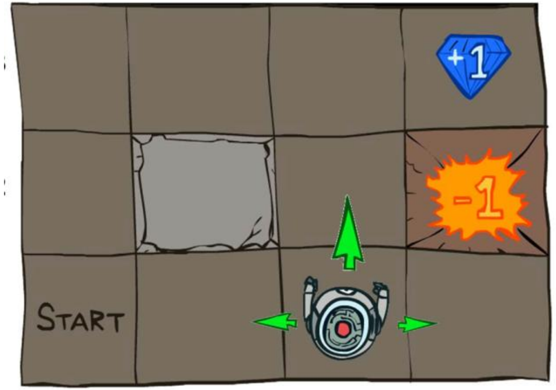


Source: P. Abbeel and D. Klein

Example of MDP: Grid World

Note: (i) Robot is unreliable (ii) Reach target fast

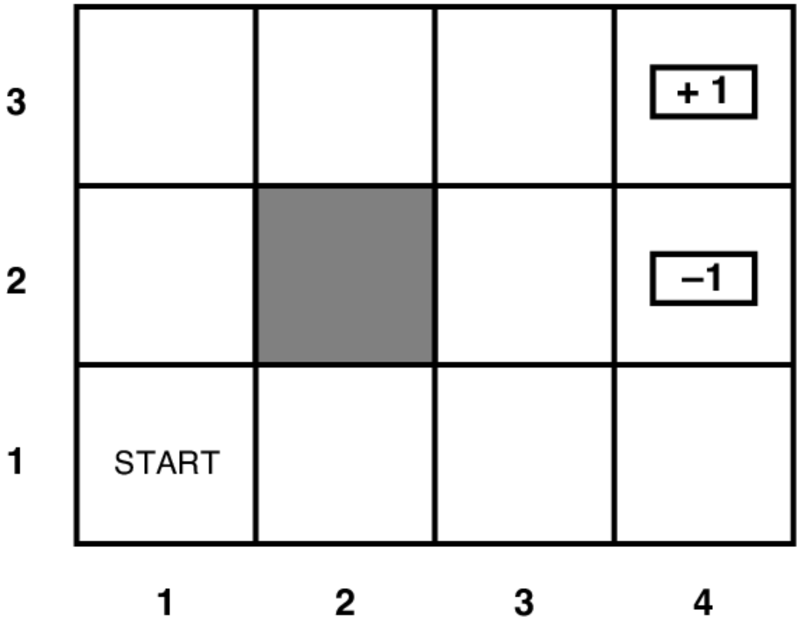


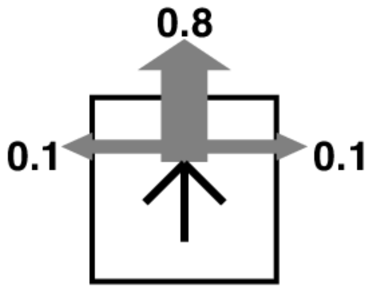


$r(s) = -0.04$ for every non-terminal state

Grid World Abstraction

Note: (i) Robot is unreliable (ii) Reach target fast





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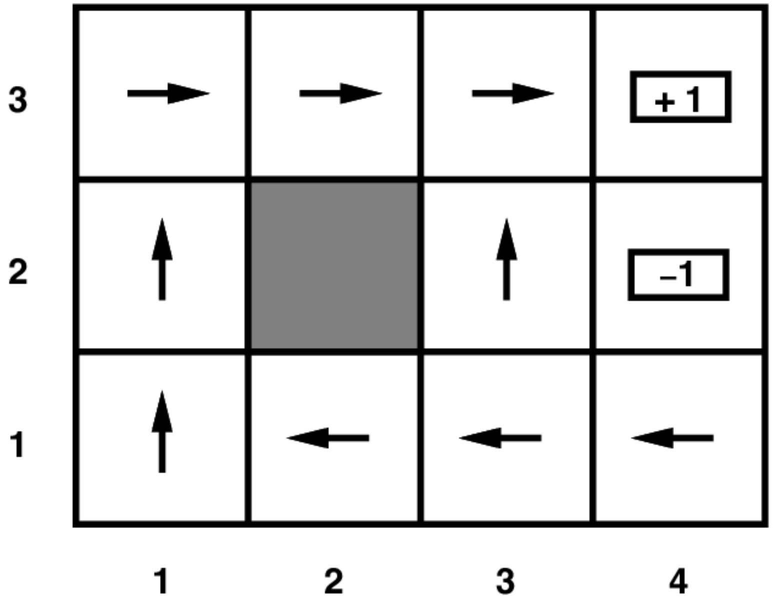
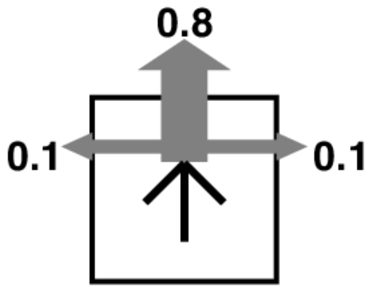
Grid World Optimal Policy

Note: (i) Robot is unreliable (ii) Reach target fast





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Back to MDP Setup

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- **State set** S. Initial state s_0. **Action set** A
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  - Markov assumption: transition probability only depends on s_t and a_t, and not previous actions or states.
- **Reward function:** $r(s_t)$
- **Policy:** $\pi(s) : S \rightarrow A$ action to take at a particular state.

How do we find the best policy?

$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \ldots$$

Reinforcement Learning Challenges

Credit-assignment:

- May take many actions before reward is received. Which ones were most important?
- Example: You study 15 minutes a day all semester. The morning of the final exam, you eat a bowl of yogurt. You receive an A on the final. Was it the studying or the yogurt that led to the A?

Exploration vs. Exploitation:

- Transition probabilities and reward may be unknown to the learner.
- Should you keep trying actions that led to reward in the past or try new actions that might lead to even more reward?

Break & Quiz

Q 1.1 Which of the following statement about MDP is **not** true?

- A. The reward function must output a scalar value
- B. The policy maps states to actions
- C. The probability of next state can depend on current and previous states
- D. The solution of MDP is to find a policy that maximizes the cumulative rewards

Break & Quiz

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Break & Quiz

Q 1.1 Which of the following statement about MDP is **not** true?

- A. The reward function must output a scalar value (**True: need to be able to compare**)
- B. The policy maps states to actions (**True: a policy tells you what action to take for each state**).
- **C. The probability of next state can depend on current and previous states** (**False: Markov assumption**).
- D. The solution of MDP is to find a policy that maximizes the cumulative rewards (**True: want to maximize rewards overall**).

Defining the Optimal Policy

For policy π, **expected utility** over all possible state sequences from s_0 produced by following that policy:

Utility of sequence

$$V^{\pi}(s_0) = \sum_{\substack{\text{sequences} \\ \text{starting from } s_0}} P(\text{sequence})U(\text{sequence})$$

Probability of sequence when following π

Called the **value function** (for π, s_0)

Discounting Rewards

One issue: these are possibly infinite series. **Convergence**?

- Solution: discount future rewards.

$$U(s_0, s_1 \ldots) = r(s_0) + \gamma r(s_1) + \gamma^2 r(s_2) + \ldots = \sum_{t \geq 0} \gamma^t r(s_t)$$

- Discount factor γ between 0 and 1
  - Set according to how important **present** is versus **future**
  - Note: has to be less than 1 for convergence

From Value to Policy

Now that $V^{\pi}(s_0)$ is defined, what a should we take?

- First, let π^* be the **optimal** policy for $V^{\pi}(s_0)$, and $V^*(s_0)$ its expected utility.
- What's the expected utility following an action?
  - Specifically, action a in state s?

$$\sum_{s'} P(s'|s,a)V^*(s')$$

All the states we could go to

Transition probability

Expected rewards

Slight Problem…

Now we can get the optimal policy by doing

$$\pi^*(s) = \text{argmax}_a \sum_{s'} P(s'|s, a)V^*(s')$$

- So we need to know $V^*(s)$ (and P).
  - But it was defined in terms of the optimal policy!
  - So we need some other approach to get $V^*(s)$.
  - Instead, learn about the utility of actions directly.

Bellman Equation

Let’s walk over one step for the value function:

$$V^*(s) = r(s) + \underbrace{\gamma \max_a \sum_{s'} P(s'|s,a) V^*(s')}_{\text{Discounted expected future } \textbf{rewards}}$$

Current state reward

- Richard Bellman: inventor of dynamic programming

Bellman Equation

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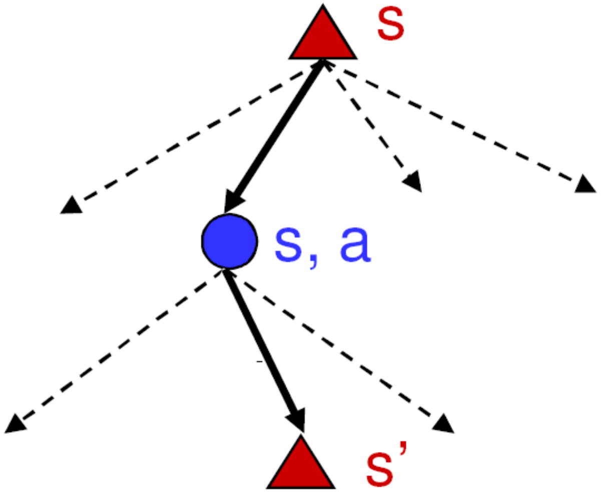


Credit L. Lazbenik

$$V^*(s) = \underset{\text{Current state reward}}{r(s)} + \gamma \underbrace{\max_a \sum_{s'} P(s'|s,a)V^*(s')}_{\text{Discounted expected future rewards}}$$

Current state reward

Discounted expected future **rewards**

Value Iteration

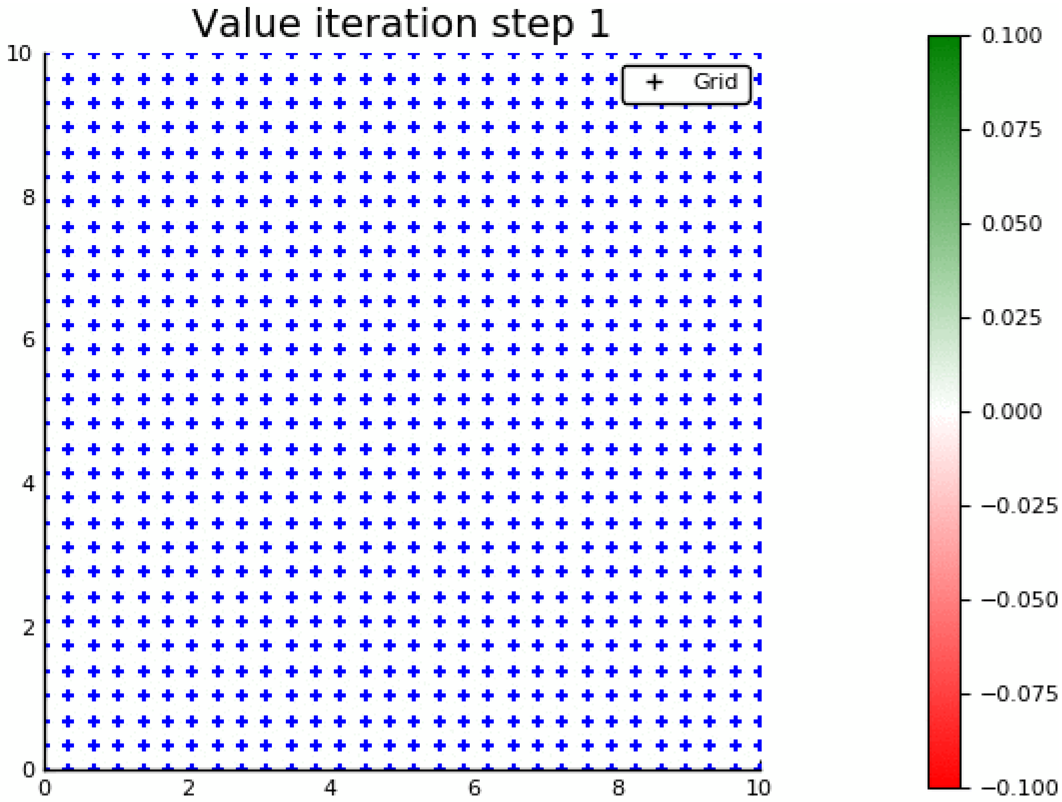
Q: how do we find $V^*(s)$?

- Why do we want it? Can use it to get the best policy
- Know: reward $r(s)$, transition probability $P(s'|s,a)$
  - Knowing r and P is the "planning" problem. In reality r and P must be estimated from interactions : "reinforcement learning"
- Also know $V^*(s)$ satisfies Bellman equation (recursion above)

A: Use the property. Start with $V_0(s)=0$. Then, update

$$V_{i+1}(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V_i(s')$$

Value Iteration: Demo



Source: POMDPBGallery Julia Package

Break & Quiz

Q 2.1 Consider an MDP with 2 states {A, B} and 2 actions: **"stay"** at current state and "**move**" to other state. Here, transitions are deterministic. Let **r** be the reward function such that **r**(A) = 1, **r**(B) = 0. Let γ be the discounting factor. Let π: π(A) = π(B) = **move** (i.e., an "always move" policy). What is the value function $V^{\pi}(A)$?

- A. 0
- B. 1 / (1 -γ)
- C. 1 / (1 -γ^2)
- D. 1

Break & Quiz

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- B. 1/(1-γ)
- **C. 1/(1-γ^2)**
- D. 1

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- A. 0
- B. 1/(1-γ)
- **C. 1/(1-γ^2)** (States: A,B,A,B,… rewards 1,0, γ^2,0, γ^4,0, …)
- D. 1

Q-Learning

- Our **next** reinforcement learning algorithm.
- Does not require knowing *r* or *P*. Learn from data of the form: $\{(s_t, a_t, r_t, s_{t+1})\}$.
- Learns an action-value function *Q**(*s*,*a*) that tells us the expected value of taking *a* in state *s*.
    - Note: $V^*(s) = \max_a Q^*(s, a)$.
- Optimal policy is formed as $\pi^*(s) = \mathrm{arg}\max_a Q^*(s, a)$

The Q*(s,a) function

- Starting from state s, perform (perhaps suboptimal) action a. THEN follow the optimal policy

$$Q^*(s,a) = r(s) + \gamma \sum_{s'} P(s'|s,a) V^*(s')$$

- Equivalent to

$$Q^*(s,a) = r(s) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q^*(s',a')$$

Q-Learning Iteration

How do we get Q(*s*,*a*)?

- Iterative procedure

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha[r(s_t) + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t)]$$

Learning rate

Idea: combine old value and new estimate of future value.

Note: We are using a policy to take actions; based on the estimated Q!

Q-Learning

Estimate Q*(s,a) from data $\{(s_t, a_t, r_t, s_{t+1})\}$:

1. Initialize Q(.,.) arbitrarily (eg all zeros)
    1. Except terminal states $Q(s_{terminal},.)=0$
2. Iterate over data until Q(.,.) converges:

$$Q(s_t, a_t) \leftarrow (1 - \alpha)Q(s_t, a_t) + \alpha(r_t + \gamma \max_b Q(s_{t+1}, b))$$

Learning rate

Q-Learning

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Idea: update is an empirical version of our Q table recursion:

$$Q^*(s,a) = r(s) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q^*(s',a')$$

Exploration Vs. Exploitation

General question!

- **Exploration:** take an action with unknown consequences
  - Pros:
    - Get a more accurate model of the environment
    - Discover higher-reward states than the ones found so far
  - Cons:
    - When exploring, not maximizing your utility
    - Something bad might happen
- **Exploitation:** go with the best strategy found so far
  - Pros:
    - Maximize reward as reflected in the current utility estimates
    - Avoid bad stuff
  - Cons:
    - Might prevent you from discovering the true optimal strategy

Q-Learning: ε-Greedy Behavior Policy

Getting data with both **exploration and exploitation**

- With probability ε, take a random action; else the action with the highest (current) Q(*s*,*a*) value.

$$a = \begin{cases} \operatorname{argmax}_{a \in A} Q(s, a) & \text{uniform}(0, 1) > \epsilon \\ \text{random } a \in A & \text{otherwise} \end{cases}$$

Q-learning Algorithm

Input: step size α, exploration probability ϵ

1. set $Q(s,a) = 0$ for all s, a.
2. For each episode:
3. Get initial state s.
4. While (s not a terminal state):
5. Perform a = ϵ-greedy(Q, s), receive r, s'
6. $Q(s,a) = (1-\alpha)Q(s,a) + \alpha(r + \gamma \max_{a'} Q(s',a'))$
7. $s \leftarrow s'$
8. End While
9. End For

Explore: take action to see what happens.

Update action-value based on result.

Break & Quiz

Q 3.1 For Q learning to converge to the true Q function, we must

- A. Visit every state and try every action
- B. Perform at least 20,000 iterations.
- C. Re-start with different random initial table values.
- D. Prioritize exploitation over exploration.

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Break & Quiz

Q 3.1 For Q learning to converge to the true Q function, we must

- **A. Visit every state and try every action**
- B. Perform at least 20,000 iterations. (No: this is dependent on the particular problem, not a general constant).
- C. Re-start with different random initial table values. (No: this is not necessary in general).
- D. Prioritize exploitation over exploration. (No: insufficient exploration means potentially unupdated state action pairs).

Summary

- Reinforcement learning setup
- Mathematical formulation: MDP
- Bellman Equation
- Value Iteration Algorithm
- The Q-learning Algorithm