Study Group – MSTs

Name: ______________________ Wisc id: ______________________

Truth or Lie?

Which of the following statements are TRUE and which of them are FALSE?

1. Prim's and Kruskal's will always return the same MST. T — F
2. If the edges in a graph have different weights, then the MST is unique. T — F
3. There is a MST that can be used to find the shortest path between 2 vertices in a graph. T — F
4. Adding a constant to every edge weight does not change the MSTs. T — F
5. A MST does not contain the maximum cost edge of a graph. T — F

Maximizing Gold Dragons

In the midst of the War of the Five Kings, you are a cunning maester serving under the rule of a corrupt and tyrannical king. The Iron Throne's communication network, consisting of a series of raven rookeries, is the key to controlling information and thus the key to power. However, the network is outdated and inefficient, with many rookeries isolated and unable to communicate with each other.

Seeing an opportunity, you propose a plan to the Iron Throne: to modernize and expand the raven network, connecting the rookeries in a way that maximizes the reach and efficiency of the network. Unbeknownst to the Iron Throne, you have also struck a secret deal with the spymasters from the Free Cities across the Narrow Sea. They are interested in the political secrets of Westeros and are willing to pay you handsomely in gold dragons for ensuring that certain key rookery connections are utilized in the network, allowing them to intercept and decipher the messages sent between them.

Your task is to design a new network that not only satisfies the Iron Throne's need for communication but also maximizes the gold dragons you can receive from the spymasters. Given a raven network (weighted graph), G, where each vertex represents a raven rookery and each edge represents a potential connection between two rookeries, your task is to maximize the amount of gold dragons you can receive. The weight of each edge in G corresponds to the number of gold dragons the spymasters are willing to pay you for using that edge in the network.

Data approximation application

Suppose, for a data approximation application, you are given a set of points in the plane and you would like to partition these points into two sets, A and B, such that each point in A is as close or closer to another point in A than it is to any point in B, and vice versa. Describe an efficient algorithm for doing this partitioning.

Number of MSTs in a complete graph

Consider a complete graph where all edges have the same cost. Let T_n be the number of MSTs on such complete graphs of order n. Prove that $T_n = n^{n-2}$.

Kruskal's algorithm

Algorithm: KRUSKAL'S(G)

Input: G, a graph $G = (V, E)$
begin
  Let $T = \emptyset$
  Let $L = E$ and sort L
  for *each edge* $e \in L$ *in ascending order* **do**
    if $H = (V, T \cup \{e\})$ *is an acylic graph* **then**
      $T \leftarrow T \cup \{e\}$
  return $H = (V, T)$

Prim's Algorithm

Algorithm: PRIM'S(G, v)

Input: G, a graph $G = (V, E)$
begin
  Let $U = \{v\}$, where $v \in V$
  Let $T = \emptyset$
  while $U \neq V$ **do**
    Find least-weight edge $e = (u, v) : u \in U \wedge v \in V \setminus U$
    $T \leftarrow T \cup \{e\}$
    $U \leftarrow U \cup \{v\}$
  return $H = (U, T)$