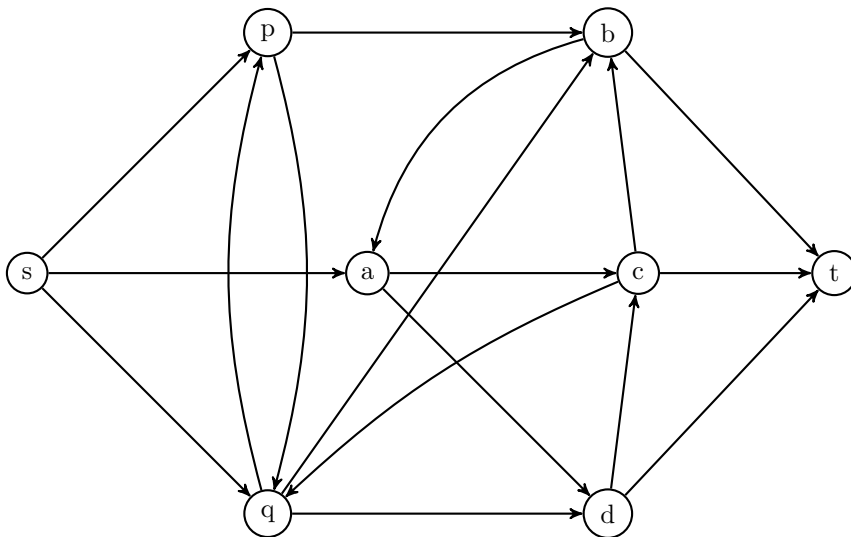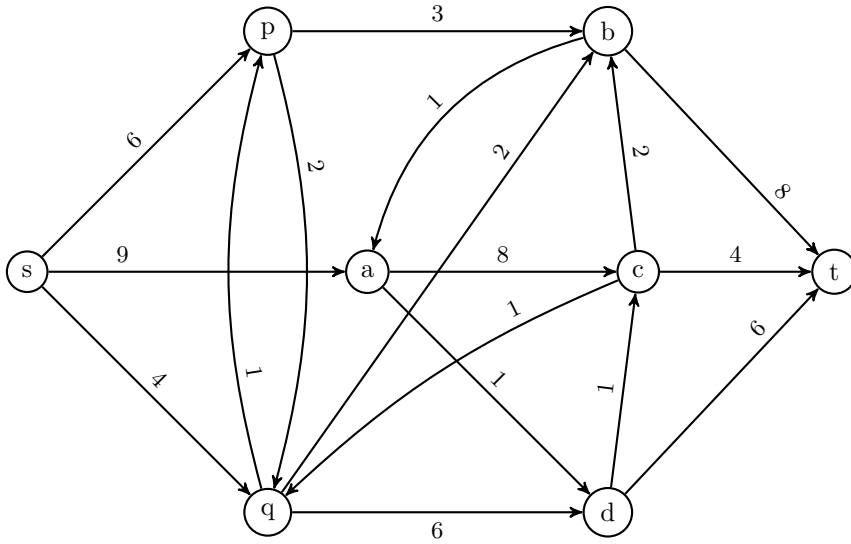# Study Group – Network Flow

Name: _____   Wisc id: _____

Example

Cook, Cunningham, Pulleyblank, Schrijver. Combinatorial Optimization.
Find a maximum flow and a minimum cut for the digraph.

Cook, Cunningham, Pulleyblank, Schrijver. Combinatorial Optimization.
Prove that there is no maximum flow (i.e., you can send an arbitrarily amount of flow from s to t) if and only if there is an (s, t)-path, each of whose edges e has $c_e = \infty$.

Secret Raven Messages

The realm of Westeros is fraught with the threat of espionage. To maintain secure lines of communication, the maesters of the Citadel have established a network of secret raven routes. These routes are known only to a trusted few and are used to send vital messages across the land.

You are tasked with determining the maximum number of raven messages that can be sent in one day from the Citadel to the Eyrie through this network of routes. There are n towns, where town 1 is the Citadel and town n is the Eyrie. Each route between town i and j has a limited carrying capacity of c_{ij}, defined as the maximum number of ravens that can safely travel the route per day without raising suspicion.

The Frey Wedding Alliances

Walder Frey, the Lord of the Crossing, commands a large and ambitious family. With numerous sons at his behest, he aims to extend the influence of House Frey by forging matrimonial alliances with other prestigious houses of Westeros. These marriages are not only a matter of political maneuvering but also of personal preference, as each of Lord Frey's sons has a certain affinity for the daughters of specific noble houses.

In your esteemed role as the chief advisor to Lord Frey, you are entrusted with the delicate task of orchestrating marriages for his n sons. The landscape of Westeros is complex, with m noble house presenting a unique opportunity for alliance. While every son, denoted by i, has a fondness for the daughter of a particular noble house, denoted by j, it is your duty to navigate through these affinities (given by function $\ell(i, j) = 1$ if i and j like each other, and 0 otherwise) and arrange marriages that maximize the number of alliances with these noble houses.
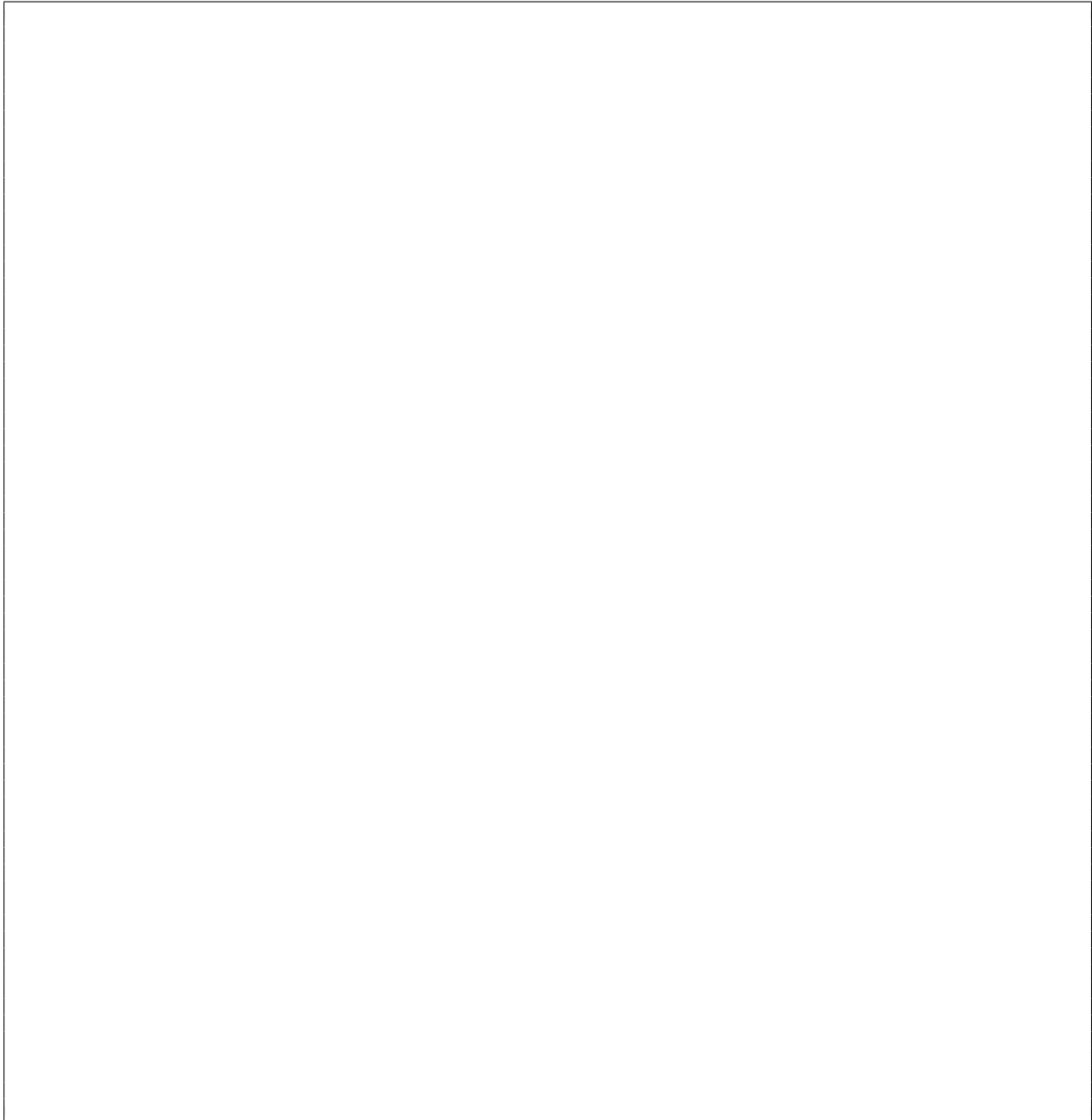
Network flow recap

Flow network

- A directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$.

    - Each edge $e \in E$ has a *capacity* $c_e \geq 0$.
    - A *source* node $s \in V$, and a *sink* node $t \in V$.
    - All other nodes $(V \setminus \{s, t\})$ are internal nodes.
    - Let $C = \sum_{e \text{ out of } s} c_e$.

- *Flow function*: $f : E \to R^+$, where $f(e)$ is the flow across edge e.

    - Valid flow function conditions:

        i *Capacity*: For each $e \in E, 0 \leq f(e) \leq c_e$.
        ii *Conservation*: For each $v \in V \setminus \{s, t\}$, $\sum_{e \text{ into } v} f(e) = f^{\text{in}}(v) = f^{\text{out}}(v) = \sum_{\text{ out of } v} f(e)$.

- The flow starts at s and exits at t.

- Flow value $v(f) = f^{\text{out}}(s) = f^{\text{in}}(t)$.

An $s - t$ cut

- A Cut: Partition of V into sets (A, B) with $s \in A$ and $t \in B$.

- Cut capacity: $c(A, B) = \sum_{e \text{ out of } A} c_e$.

Residual Graph: Given a flow network G and a flow f on G, we define the residual graph G_f:

- Same nodes as G.

- For edge (u, v) in E:

    - Add edge (u, v) with capacity $c_e - f(e)$.
    - Add edge (v, u) with capacity $f(e)$.

Ford-Fulkerson Algorithm: $O(m\,C)$

- Initialize $f(e) = 0$ for all edges.

- While G_f contains an augmenting path P:

    - Update flow f by BOTTLENECK(P, G_f) along P.

Other algorithms:

- Scaled Ford-Fulkerson: $O(m^2 \log C)$.

- Fewest Edges Augmenting Path [BFS] (Edmonds-Karp): $O(m^2 n)$.

- Dinitz 1970: $O\left(\min\left\{ n^{\frac{2}{3}}, m^{\frac{1}{2}} \right\} m \right)$.

- Preflow-Push 1974/1986: $O(n^3)$.

- Best: Orlin 2013: $O(mn)$