Intro to Category Theory: Limits and Universal Properties

1 Category theory as a useful generalization of mathematics

Category theory can be used as a common language for all of mathematics. However, just because something can generalize everything doesn't necessarily make it very useful. *Natural language* can capture all mathematics, but that doesn't make it a very useful mathematical framework. At a meta-level, what makes a good foundation of mathematics? You need to strike a balance between having a system of axioms that are not restrictive enough as to exclude interesting areas of mathematical study, and having axioms for which you can derive some interesting theorems. These theorems shed light on properties shared by all areas of mathematics. Natural language is too weak, and abelian groups are too restrictive.

Why is category theory a useful generalization? On the outset it doesn't seems how it can be. They're very close to directed graphs with three simple axioms: arrows compose, composition is associative, and objects have identity arrows. It doesn't seem there's enough structure here to derive some interesting theorems. We can formulate classes of objects as categories, such as **Set** and **Grp**, but when we do all it seems we are left with is a spaghetti of indecipherable arrows.

Arguably universal properties is the answer to what category theory is useful for. They are the common way to start and disentangle the arrows of a category. A universal property will define an object in a category that is the "most efficient solution" to a certain problem. At a high-level a universal object is an object that satisfies a property such that are other objects that satisfy the property can be "factorized" through the universal object. In other words, if you want to "understand" how an object satisfies a property you can construct this understanding by going through the universal object.

2 The Equaliser

Before we go on with the category theory, let's begin with an example in set theory.

Definition 2.1. Consider sets X and Y, and two functions $f : X \to Y$ and $g : X \to Y$. The *equaliser* of f and g is the set of elements x of X such that $f(x) = g(x)$.

$$Eq(f, g) := \{x \in X | f(x) = g(x)\}$$

The idea is that there is some subset of elements of X for which f and g match, and the equaliser is the largest such subset.

Example 2.1. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4, 5\}$, with

$$\begin{aligned} f(a) &= 2 & g(a) &= 1 \\ f(b) &= 2 & g(b) &= 2 \\ f(c) &= 4 & g(c) &= 5 \\ f(d) &= 5 & g(d) &= 5 \end{aligned}$$

In this case f and g match on b and d, thus $Eq(f, g) = \{b, d\}$.

2.1 An alternative definition

To start moving in the categorical direction, let's observe that we can characterize the equaliser as satisfying two characteristics. First, the equalizer satisfies the property that it is a subset of X and for all $x \in Eq(f, g)$, $f(x) = g(x)$. However, note that the equaliser is not the only set that satisfies this property. Going to the previous example, $\{b\}$ also has that for all elements $f(x) = g(x)$, same thing for $\{\}$. What's special about the equaliser is that it is the largest subset of X which satisfies this property. Stated another way for all other sets S, such that $\forall s \in S\ f(s) = g(s)$, then $S \subseteq Eq(f, g)$.

We're not quite at the categorical definition yet, but this is one way to start to think of universal objects (or properties). A universal object is the solution to an optimization problem. A universal object satisfies a property, and for all other objects which satisfy that same property the universal object is "best". We will describe what "best" means in a moment.

To get a little more categorical, we will define the equaliser using functions rather than sets. A heads up, we will *not* define a set $Eq(f, g)$, instead we are going to define a *function eq* that essentially performs the same job. The first thing we need is to consider how to define a particular subset using functions. Consider any function $h : A \to X$. At this moment, A can be any set and h can be any function. Denote the range of h as $range(h) = \{h(a) | a \in A\}$. Then $range(h) \subseteq X$. This is true for any h and any A.

To define the right subset for the equaliser we just need to pick the right h and A. Let consider such functions and sets that satisfy the first property of the equaliser. That is such functions $h : A \to X$ with $f(x) = g(x)$ for all $x \in range(h)$. Such functions pick out the elements for which f and g match. These are all the functions for which $range(h) \subseteq Eq(f, g)$. We can also write this fact using function composition. If $range(h) \subseteq Eq(f, g)$, then $f \circ h = g \circ h$. Conversely, if $f \circ h = g \circ h$ then $range(h) \subseteq Eq(f, g)$.

Going back to the example, all functions $h : A \to X$ with $f(x) = g(x)$ for all $x \in range(h)$, have $range(h) \subseteq \{b, d\}$. Conversely, any function $h : A \to X$ with $range(h) \subseteq \{b, d\}$ has $f(x) = g(x)$ for all $x \in range(h)$. We are still a ways from the definition of the equaliser. Here are some potential functions which satisfy the property.

$$even_b : \mathbb{Z} \to X \qquad even_b(n) \begin{cases} b & \text{if } n \text{ is even} \\ d & \text{otherwise} \end{cases}$$

$$empty : \emptyset \to X \qquad empty = \emptyset$$

$$eq_{f,g} : E = \{\text{☹}, \text{☺}\} \to X \qquad \begin{aligned} eq_{f,g}(\text{☹}) &= b \\ eq_{f,g}(\text{☺}) &= d \end{aligned}$$

$$select_b : \{\bullet\} \to X \qquad select_b(\bullet) = b$$

For each of the above functions we have that their range's are a subset of $Eq(f, g)$. Thus we have

$$\begin{aligned} f \circ even_b &= g \circ even_b \\ f \circ empty &= g \circ empty \\ f \circ eq_{f,g} &= g \circ eq_{f,g} \\ f \circ select_b &= g \circ select_b \end{aligned}$$

Because each of these functions have $f \circ h = g \circ h$ each of them satisfy the first property of an equaliser. However, these functions are not equally good as far as the equaliser is concerned. For starters, we have that $eq_{f,g}$ and $even_b$ are better than the other two because we have $range(eq_{f,g}) = range(even_b) = Eq(f, g)$, but $range(empty) \subsetneq Eq(f, g)$ and $range(select_b) \subsetneq Eq(f, g)$. Perhaps naturally, $even_b$ doesn't seem as good as $eq_{f,g}$ because it is "too big". That is, $even_b$ represents more information than is necessary. In set theory, one way we can make the distinction is by saying that $eq_{f,g}$ is an injection whereas $even_b$ is not. Stated another way $E = \{\text{☹}, \text{☺}\}$ is isomorphic to $Eq(f, g)$, whereas $\mathbb{Z}$ is not.

The categorical definition of an equaliser will show what make $eq_{f,g}$ special. The idea is that E and $eq_{f,g}$ completely embodies what it means to equalize f and g. That is while there are other functions that equalize f and g, the can all be understood through the lens of E and $eq_{f,g}$. E and $eq_{f,g}$ ***uniquely factorizes*** all other pseudo-equalisers. This factorization property is what makes $eq_{f,g}$ "best" and represents the solution to the optimization property of a universal property. For example, while it is the case that $\{\bullet\}$ and $select_b$ have some equalization property this can be understood through E and $eq_{f,g}$. Stated with functions, there exists a function $u_{select} : \{\bullet\} \to E$, such that $select_b = eq_{f,g} \circ u_{select_b}$. That is there's some piece of $eq_{f,g}$ "inside" $select_b$. In this case, $u_{select_b}(\bullet) = \text{☹}$. There is a u for each of the functions $even_b$, $empty$, and $select_b$.

$$u_{even_b} : \mathbb{Z} \to E \qquad u_{even_b}(n) \begin{cases} \frownie & \text{if } n \text{ is even} \\ \smiley & \text{otherwise} \end{cases}$$

$$u_{empty} : \emptyset \to E \qquad u_{empty} = \emptyset$$

$$u_{select_b} : \{\bullet\} \to E \qquad u_{select_b}(\bullet) = \frownie$$

Go ahead and verify that $even_b = eq_{f,g} \circ u_{even_b}$, $empty = eq_{f,g} \circ u_{empty}$, and $select_b = eq_{f,g} \circ u_{select_b}$. In fact, because E and $eq_{f,g}$ are universal, it means for *any* function $m : O \to X$ with $f \circ m = g \circ m$, then there exists a function $u_m : O \to E$, with $m = eq_{f,g} \circ u_m$. However, the existence of a factorizing function u is not totally what defines a universal object. Using the previous example, $\mathbb{Z}$ and $even_b$ also satisfies this property.

$$u_{empty} : \emptyset \to \mathbb{Z} \qquad u_{empty} = \emptyset$$

$$u_{eq_{f,g}} : E \to \mathbb{Z} \qquad u_{eq_{f,g}}(\frownie) = 0, \quad u_{eq_{f,g}}(\smiley) = 1$$

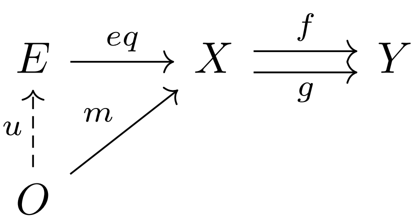
$$u_{select_b} : \{\bullet\} \to \mathbb{Z} \qquad u_{select_b}(\bullet) = 0$$

What excludes $\mathbb{Z}$ and $even_b$ from being the equaliser is that the choice of u in this situation is not unique. Take the factorization for $eq_{f,g}$ for example. There are an infinite amount of choices for the factorizer $u_{eq_{f,g}}$. That is solution is $\frownie$ is mapped to and even number and $\smiley$ is mapped to an odd number, then we will have $eq_{f,g} = even_b \circ u_{eq_{f,g}}$. By requiring that u is unique (at least in **Set**), ensures that E and eq is no larger than required. It is exactly the right size.

Here is the formal alternative definition we've been building up to.

Definition 2.2. Consider sets X and Y, and two functions $f : X \to Y$ and $g : X \to Y$. The *equaliser* of f and g is a set E and a function $eq : E \to X$ with $f \circ eq = g \circ eq$, such that for any other set O and function $m : O \to X$ with $f \circ m = g \circ m$, then there exists a unique function $u : O \to E$ with $m = eq \circ u$.

This definition has the following associated diagram, which we will describe more later:



This pattern of definition will be the pattern for any universal property, so it is important to really sit and dissect the logic of the definition. First there is a set E and a function eq with respect to a *particular* f and g. You give me an f and g and I give you an E and eq with $f \circ eq = g \circ eq$. Furthermore, if you give me any other set O and function m with the property that $f \circ m = g \circ m$ then I can produce a *unique* $u : O \to E$ such that $m = eq \circ u$. The definition is not saying that eq factorizes any $m : O \to X$. Only such m's with $f \circ m = g \circ m$.

Another way to think about it is that you have a set A and a function h that satisfies some property. (In this case the equaliser property $f \circ h = g \circ h$). A and h are universal if for any other set O and function m that satisfies the same property, then A and h uniquely factorizes m.

To relate the alternative definition back to the first one, suppose we have the equaliser E and eq for an f and g. Then $range(eq) = Eq(f, g)$, where $Eq(f, g)$ is the set defined in the first definition. Note that, definition 2.2 does not define a unique set. There are multiple sets and functions that can be equalisers for the same functions, but all such examples are isomorphic.

2.2 A discussion on uniqueness (at least for the equaliser in Set)

I think at this point there still some dissatisfaction with the requirement of a *unique* factorization. All I've said is that uniqueness ensures E and eq aren't "too big", but it's not clear why uniqueness ensures E is just the right size. In this section, I'm going to try and clear this up at least for the equaliser in set. Some of these arguments will generalize to other categories and some will apply to all morphisms in set. I will try and be clear how general each statement is.

First, let's be clear what I mean when I say E and eq aren't too big. In the case of the equaliser, it's saying that E has the same cardinality as $Eq(f, g)$. Equivalently $eq : E \to X$ is an injection. Here's another way to think about the equaliser.

Theorem 1. Suppose we have sets X and Y and two functions $f : X \to Y$ and $g : X \to Y$. Consider then a set E and a function $eq : E \to X$.

$range(eq) = Eq(f, g) \wedge eq$ is an injection $\iff$ E and eq is an equaliser of f and g according to 2.2.

This is one way to see why $\mathbb{Z}$ and $even_b$ is not an equaliser. $range(even_b) = Eq(f, g)$, but $even_b$ is not an injection. Equivalently, $|\mathbb{Z}| \neq |Eq(f, g)|$.

I'm now going to prove the ($\Longleftarrow$) direction. A good exercise is to show the ($\Longrightarrow$) direction. (Hint: $f \circ eq = g \circ eq$ comes from $range(eq) = Eq(f, g)$. Then you must be able to construct a factorizer u for each m. Think about eq^{-1} for the construction of u.)

To show ($\Longleftarrow$) we need the definition of a monomorphism. This is a general category theoretic definition.

Definition 2.3. A monomorphism f is a morphism $f : X \to Y$ such that for all objects Z and all morphisms $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

Lemma 2. In the category **Set**, monomorphisms are injections.

Proof. Suppose a function $f : X \to Y$ is a monomorphism, but not an injection. This means there exists $x_1, x_2 \in X$ with $f(x_1) = f(x_2) = y$ but $x_1 \neq x_2$. Let Z be some non empty set, and let $g_1(z) = x_1$ for all $z \in Z$ and $g_2(z) = x_2$ for all $z \in Z$. Then $(f \circ g_1)(z) = y = (f \circ g_2)(z)$ for all $z \in Z$. This means $f \circ g_1 = f \circ g_2$. Because f is a monomorphsim, $g_1 = g_2$. But $g_1(z) = x_1 \neq x_2 = g_2(z)$ for each z. This is a contradiction. Thus f is an injection. □

Now we must show that if E and eq is an equaliser, then eq is a monomorphism. (This holds for any category. Not just **Set**).

Lemma 3. Let E and $eq : E \to X$ be an equaliser of $f : X \to Y$ and $g : X \to Y$. Then eq is a monomorphism.

Proof. Suppose there are morphisms $h_1, h_2 : A \to E$ with $eq \circ h_1 = eq \circ h_2$. We need to show $h_1 = h_2$. Consider $f \circ (eq \circ h_1)$. Since composition is associative and eq is an equaliser we have

$$f \circ (eq \circ h_1) = (f \circ eq) \circ h_1 = (g \circ eq) \circ h_1 = g \circ (eq \circ h_1)$$

That is $eq \circ h_1$ can take on the role of m in the definition of an equaliser. This means there is a unique u such that $eq \circ h_1 = eq \circ u$, but $eq \circ h_1 = eq \circ h_2$. That is, both h_1 and h_2 can take on the role of u. Because u is unique it must be the case that $u = h_1 = h_2$. □

To finish off the ($\Longleftarrow$) direction we just observe that $range(eq) = Eq(f, g)$ is equivalent to $f \circ eq = g \circ eq$.

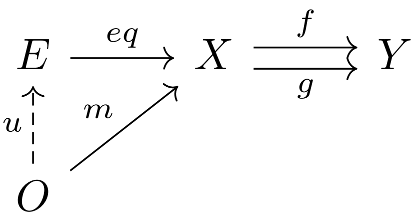
The point of theorem 1 is to say that uniqueness of u in the definition of the equaliser is what ensures that the set E has the same cardinality as $Eq(f, g)$. That is E is exactly the right size.

2.3 Getting out of Set

So we have beaten equalisers in **Set** to death now. It's time to go purely categorical. Fortunately, we don't have to go far. Definition 2.2 directly generalizes to any category. Just replace a mention of set with object and function with morphism.

Definition 2.4. Consider objects X and Y, and two morphisms $f : X \to Y$ and $g : X \to Y$. The *equaliser* of f and g is an object E and a morphism $eq : E \to X$ with $f \circ eq = g \circ eq$, such that for any other object O and morphism $m : O \to X$ with $f \circ m = g \circ m$, then there exists a unique morphism $u : O \to E$ with $m = eq \circ u$.

We also have the same commutative diagram:



Note that this diagram doesn't completely follow the rule of commutative diagrams stated before. Namely, this diagram is not meant to indicate that $f = g$. However, for all other paths which start and end at the same vertex the diagram denotes equal morphisms.

These types of diagrams are very common when defining universal properties. The dashed arrow for u denotes uniqueness. Sometimes you see the addition of a $\forall$ next to m and $\exists$ next to u, to help explain that for any m there is a corresponding u.

The general idea to get from this diagram is the triangle part, which encodes the unique factorization idea. This piece is common to all universal properties.[1] $A \xrightarrow{h} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ does the job of specifying the particular property, and the triangle defines universality. That is, imagine we have a lot of diagrams $O_i \xrightarrow{m_i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ floating around. Such a diagram indicates that O_i and m_i satisfy some pattern. In this case, every O_i and m_i are faux-equalisers. The real equaliser E and eq satisfies the pattern and uniquely factorizes all the O_i's and m_i's. It is totally possible to understand how some O_i and m_i is a faux-equaliser by going through E and eq. That is, every diagram $O_i \xrightarrow{m_i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ can be constructed by the factorizer u_i and $E \xrightarrow{eq} X \underset{g}{\overset{f}{\rightrightarrows}} Y$.

Make note of another fact that will come up when learning about adjunctions, the set of m_i's and u_i's are in a one-to-one correspondence. For every m_i there is a unique u_i, and if you give me a u_i I can give you an m_i by $eq \circ u_i$.

3 More than the equaliser

Like I said, the general idea to get from the equaliser is the triangle. The diagram $E \xrightarrow{eq} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ was only relevant for the case of the equaliser. We can use definition 2.4 as a schema for other mathematical objects, and just use different diagrams.

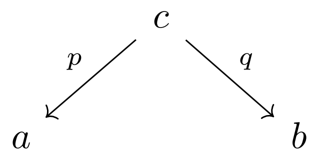
I'm now going to give some more categorical examples before I give a general definition. I want to make note of something before we get started. It is the case that for *any* two functions $f : X \to Y$ and $g : X \to Y$ there is an equaliser in the category **Set**. Such a category is said to "have equalisers". However, this does not need to hold in general. We talk about the equaliser for particular morphisms $f : X \to Y$ and $g : X \to Y$, but that doesn't mean that an equaliser exists say for any morphisms $f' : X' \to Y'$ and $g' : X' \to Y'$. A universal property is with respect to a *particular* diagram (particular objects and morphisms), not with respect to every pattern that fits a diagram. If we are in a situation for which *every* diagram of a certain pattern has a satisfying universal property, then we have an adjunction, but we won't talk about that for a little while.

3.1 Product

Like I said, we can use the equaliser definition as a schema to define other universal constructions.

Consider the following diagram:

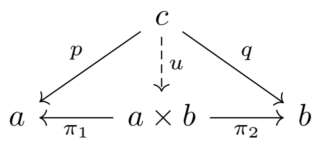
[1] They might look a little different for other examples, but it's there.



The associated definition for this diagram is the categorical product.

Definition 3.1. Consider objects a and b. The *product* of a and b is an object $a \times b$ and two morphisms $\pi_1 : a \times b \to a$ and $\pi_2 : a \times b \to b$, such that for any other object c and morphisms $p : c \to a$ and $q : c \to b$, then there exists a unique morphism $u : c \to a \times b$ with $p = \pi_1 \circ u$ and $q = \pi_2 \circ u$.

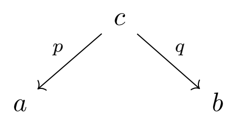
The following diagram usually accompanies the above definition.



There are a few differences between this definition and the one for the equaliser, but the idea is the same. The product $a \times b$ is the product because it has projections π_1 and π_2 and because it uniquely factorizes all other faux-products.

Let me highlight some of the differences between the definition of the equaliser and the product. For an equaliser, we have one object E and one morphism $eq : E \to X$, whereas a product consists of an object $a \times b$ and two morphisms π_1 and π_2. As a slight aside, it is imperative that you remember what a particular universal property is being satisfied by. Even though we colloquially talk about the equaliser as being a morphism eq and a product as an object. Formally, an equaliser is a morphism eq *and* an object E, and a product is an object $a \times b$ *and* two morphisms π_1 and π_2. Because there is no internal structure in category theory, saying an object $a \times b$ is the product of a and b without having projections is a meaningless statement. On the other hand, the pieces not explicitly given when stating something satisfies a universal property are usually implied from context.

Another difference between the equaliser and the product is that morphism of the equaliser eq is required to satisfy an additional property. That is the diagram $O \xrightarrow{m} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ indicates $f \circ m = g \circ m$, where as the diagram



doesn't denote any equalities. Thus, the morphisms of the product don't have any additional properties that they need to satisfy other than existing.

These differences are handled in a general framework with the general definition of a limit, but the use of functors and multiple categories is required. We will not consider this general situation quite yet.

3.1.1 Examples

In the category **Set**, the product of two sets X and Y is the cartesian product of X and Y, with $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = y$ for $(x, y) \in X \times Y$. Consider a set Z and functions $p : Z \to X$ and $q : Z \to Y$. The unique factorizer $u : Z \to X \times Y$ is then

$$u(z) = (p(z), q(z))$$

In the category **Grp**, the product of two groups $(G, *)$ and (H, Δ), is $(G \times H, <*, \Delta>)$, where $G \times H$ is the cartesian product of G and H, and the group operation $<*, \Delta>$ is

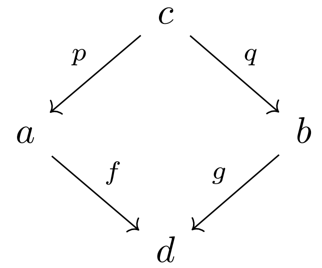
$$(g_1, h_1) <*, \Delta> (g_2, h_2) = (g_1 * g_2, h_1 \Delta h_2)$$

The projections π_1 and π_2 operate on elements the same as in **Set**. It remains to check that $(G \times H, <*, \Delta>)$ is a group, and $\pi_1 : (G \times H, <*, \Delta>) \to (G, *)$ and $\pi_2 : (G \times H, <*, \Delta>) \to (H, \Delta)$ are group homomorphisms.

For another example, recall how a partial order $(P, \leq)$ can be viewed as a category. Objects of the category are the elements of P, and there is a morphism between objects x and y if $x \leq y$. In such a category there is at most one morphism between objects. In this setting the product between two objects a and b is simply the meet $a \sqcap b$. To see why just trace the definition of product. Since $a \sqcap b$ is a product there are morphisms from $a \sqcap b$ to a and b. For this category that means $a \sqcap b \leq a$ and $a \sqcap b \leq b$. Furthermore, for any other c with morphisms to a and b, we have a morphism from c to $a \sqcap b$. This just means that for any c with $c \leq a$ and $c \leq b$ then $c \leq a \sqcap b$. Thus, the statement that $a \sqcap b$ is the product of a and b is exactly the statement that $a \sqcap b$ is the greatest lower bound of a and b.

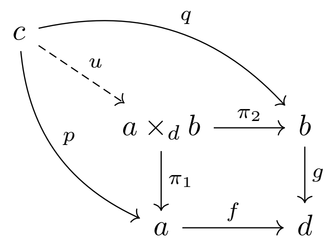
This unification of different concepts starts to show the utility of category theory. The cartesian product of sets and the meet don't seem that related when you take the internal perspective of set theory, but from the external view of category theory they are actually the same exact thing in different categories.

3.2 Pullback



Definition 3.2. Consider morphisms $f : a \to d$ and $g : b \to d$. The *pullback* of f and g is an object $a \times_d b$ and two morphisms $\pi_1 : a \times b \to a$ and $\pi_2 : a \times b \to b$ with $f \circ \pi_1 = g \circ \pi_2$, such that for any other object c and morphisms $p : c \to a$ and $q : c \to b$ with $f \circ p = g \circ q$, then there exists a unique morphism $u : c \to a \times_d b$ with $p = \pi_1 \circ u$ and $q = \pi_2 \circ u$.

Usually we have the following diagram:



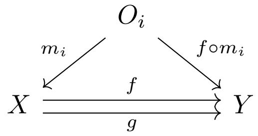
In **Set** the pullback of functions $f : X \to Z$ and $g : Y \to Z$ is the set $X \times_Z Y = \{(x, y) | f(x) = g(y)\}$. The projections π_1 and π_2 are the same as the cartesian product.

As can be seen the pullback is very related to the product and the equaliser. In fact, if a category has products and equalisers it necessarily has pullbacks. That is pullbacks can be constructed from products and equalisers. Actually, a much more general thing is true. It turns out if a category has products and equalisers then a wide variety of universal constructions necessarily exist. We will not investigate this presently.

4 Limits

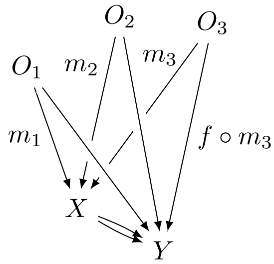
I think we're ready to approach a general definition. All the examples we've been talking about so far are examples of universal properties, but they are also examples of a more specific type of universal property. This more specific universal property is a *limit*.

Recall what I said about how the equaliser is the most universal among diagrams of the form $O_i \xrightarrow{m_i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Observe that because morphisms compose in a category, for each O_i and m_i there necessarily is a morphism form O_i to Y. In fact, you can write it as $f \circ m_i$ or $g \circ m_i$. Based on the nature of the diagram these two morphisms are necessarily equal. That means instead of talking about $O_i \xrightarrow{m_i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$, we could have been talking about

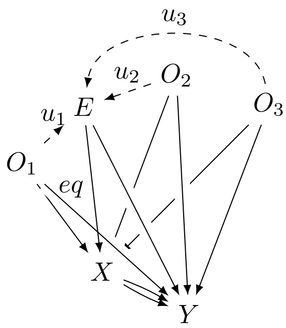


That extra leg of the triangle doesn't add any information, so in the formal definition we don't include it. But for now I want you to visualize it being there.

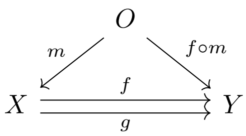
Now it helps to try and visualize this next part. In the case of the equaliser each O_i and m_i has its own associated triangle, and all these triangles share the same base $X \underset{g}{\overset{f}{\rightrightarrows}} Y$. You can try and picture this as a fan around a spoke of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Here's my attempt at drawing this



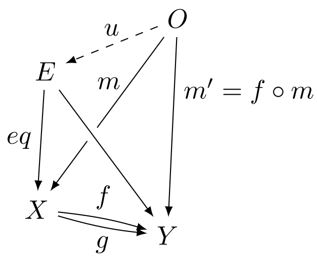
The above diagram is just a rearrangement of $O_i \xrightarrow{m_i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ for $i = 1, 2, 3$. The equaliser E and eq is also part of this picture. But we can also include the unique factorizers u_i.



The above diagram commutes. What define's the equaliser is that for any other spoke of this fan

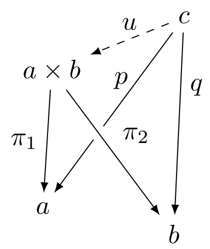


, there is a unique $u : O \to E$, such that the following diagram commutes.



This is the same diagram from definition 2.4, just rearranged slightly. I've also called the right leg of the O triangle m'. It just so happens that because of the commuting condition $m' = f \circ m = g \circ m$.

The above diagram essentially defines the equaliser. Notice that this diagram is the same diagram from the product except that the base $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ no longer has an f or g.



This commuting diagram essentially defines the product.

We can also define the pullback using $a \xrightarrow{f} d \xleftarrow{g} b$ as a base.