CS 540 Introduction to Artificial Intelligence

Perceptron

University of Wisconsin-Madison

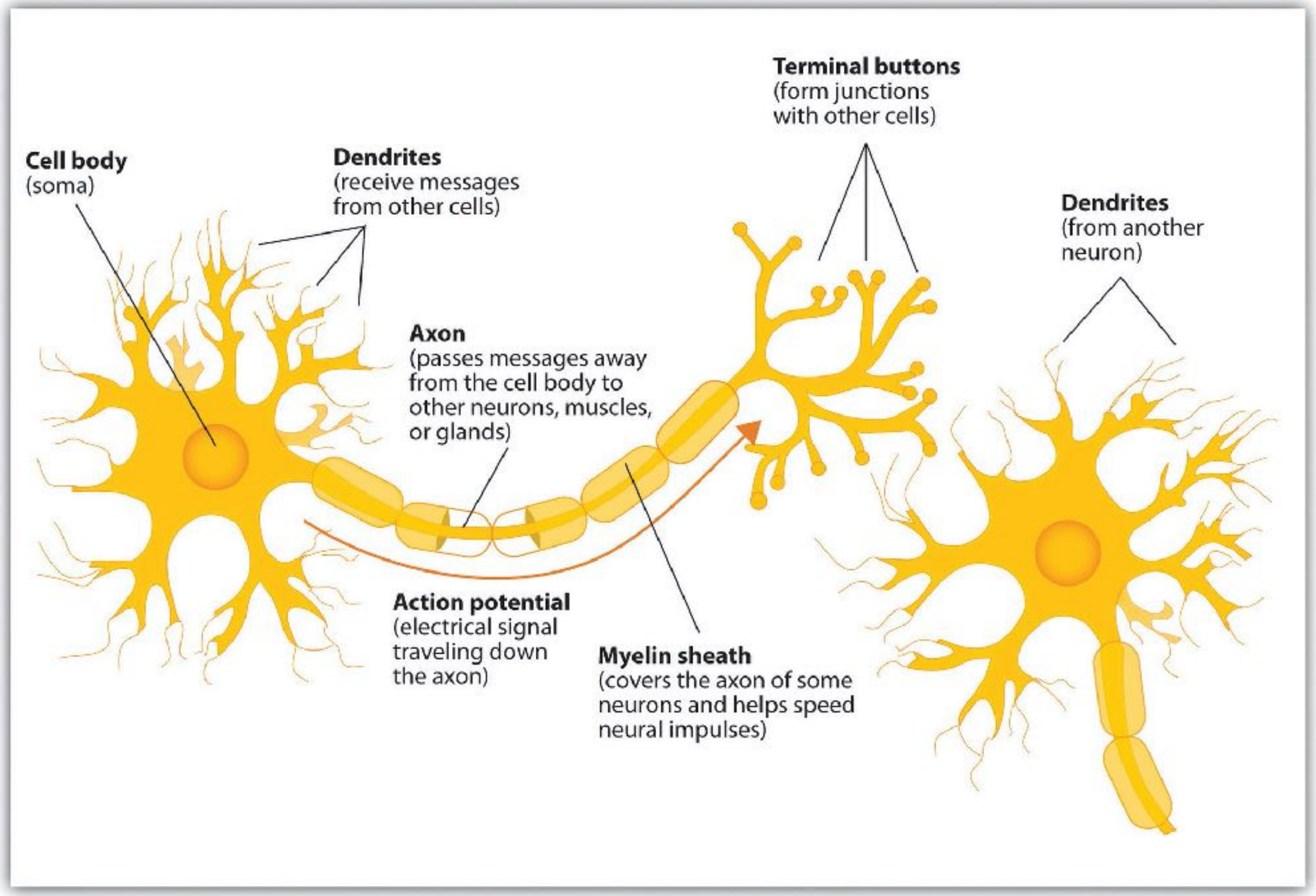
Fall 2023

Part I: Single-layer Neural Network

Inspiration from neuroscience

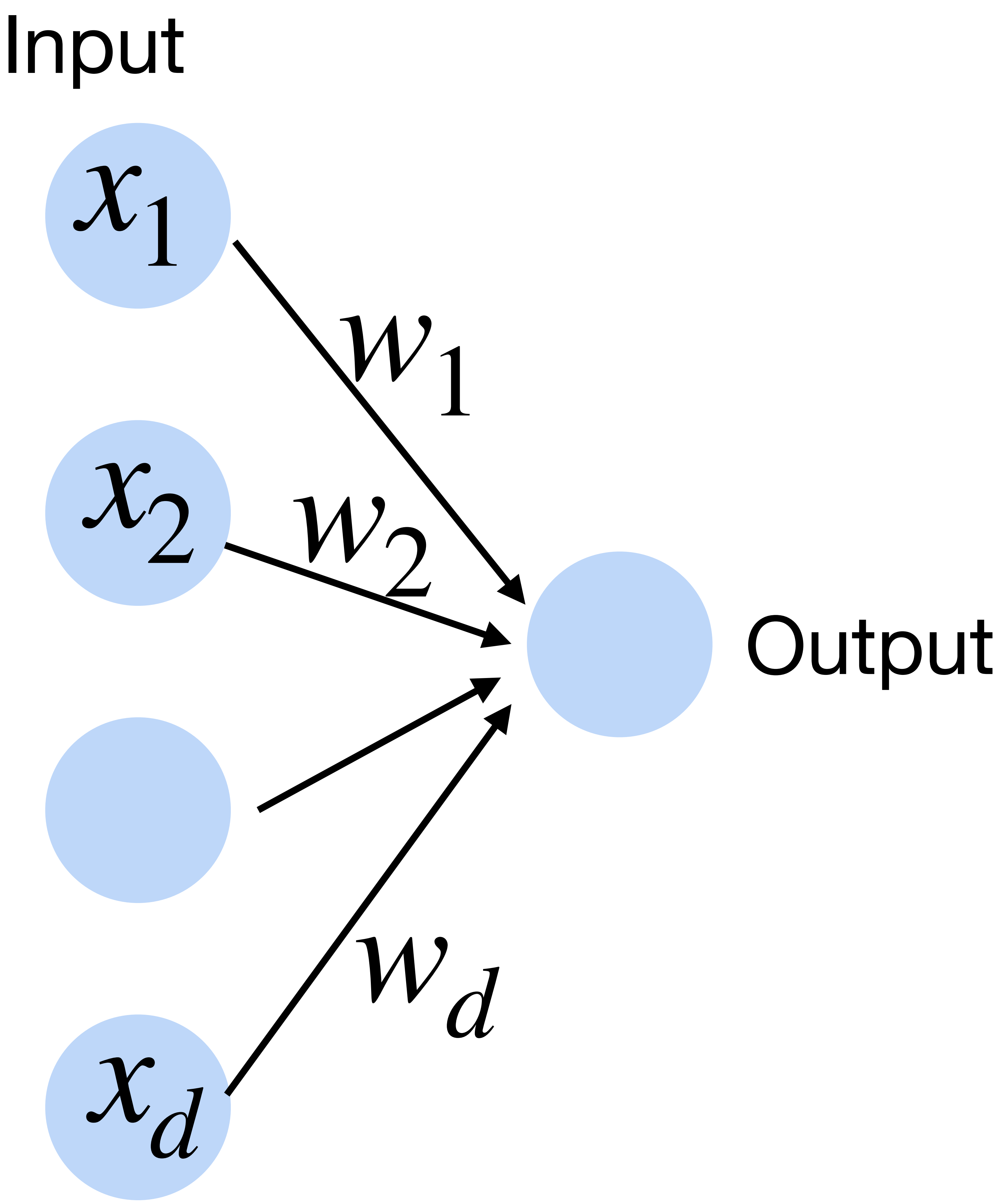
- Inspirations from human brains
- Networks of simple and homogenous units

(wikipedia)



Perceptron

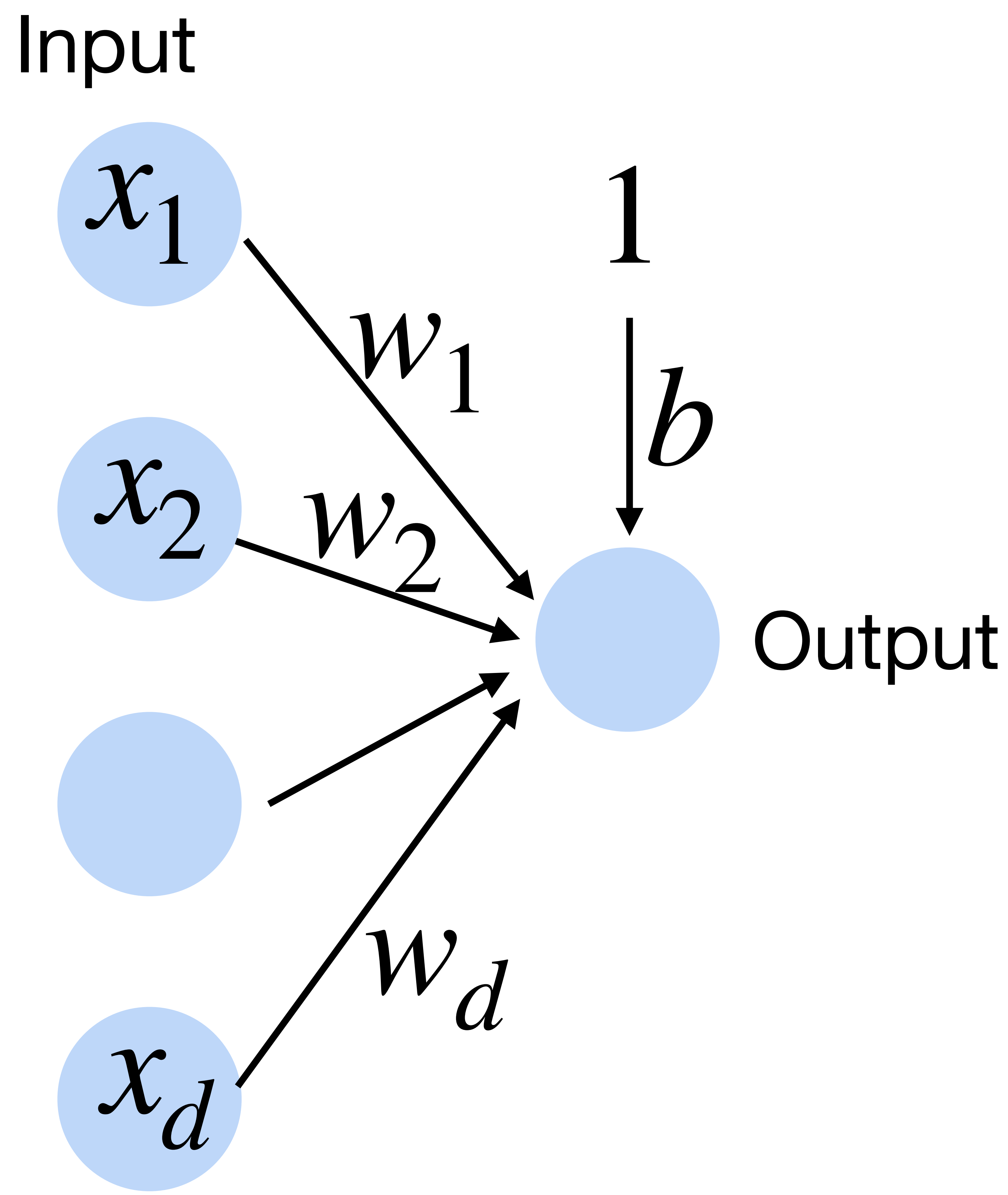
Cats vs. dogs?



Linear Perceptron

- Given input $\mathbf{x}$, weight $\mathbf{w}$ and bias b, perceptron outputs:

$$f = \langle \mathbf{w}, \mathbf{x} \rangle + b$$



Cats vs. dogs?

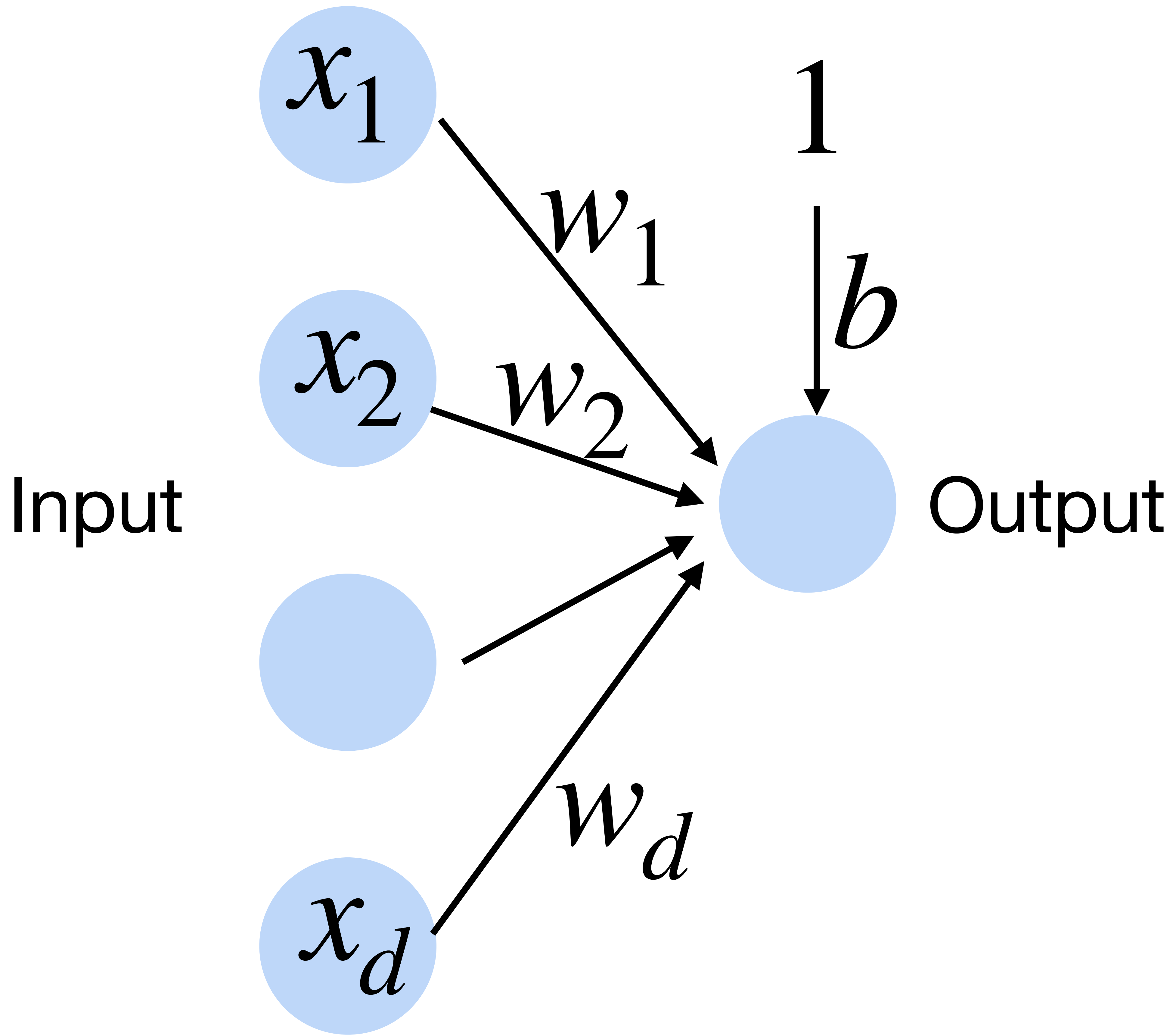
Perceptron

- Given input $\mathbf{x}$, weight $\mathbf{w}$ and bias b, perceptron outputs:

$$o = \sigma\left(\langle \mathbf{w}, \mathbf{x} \rangle + b\right) \qquad \sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Activation function

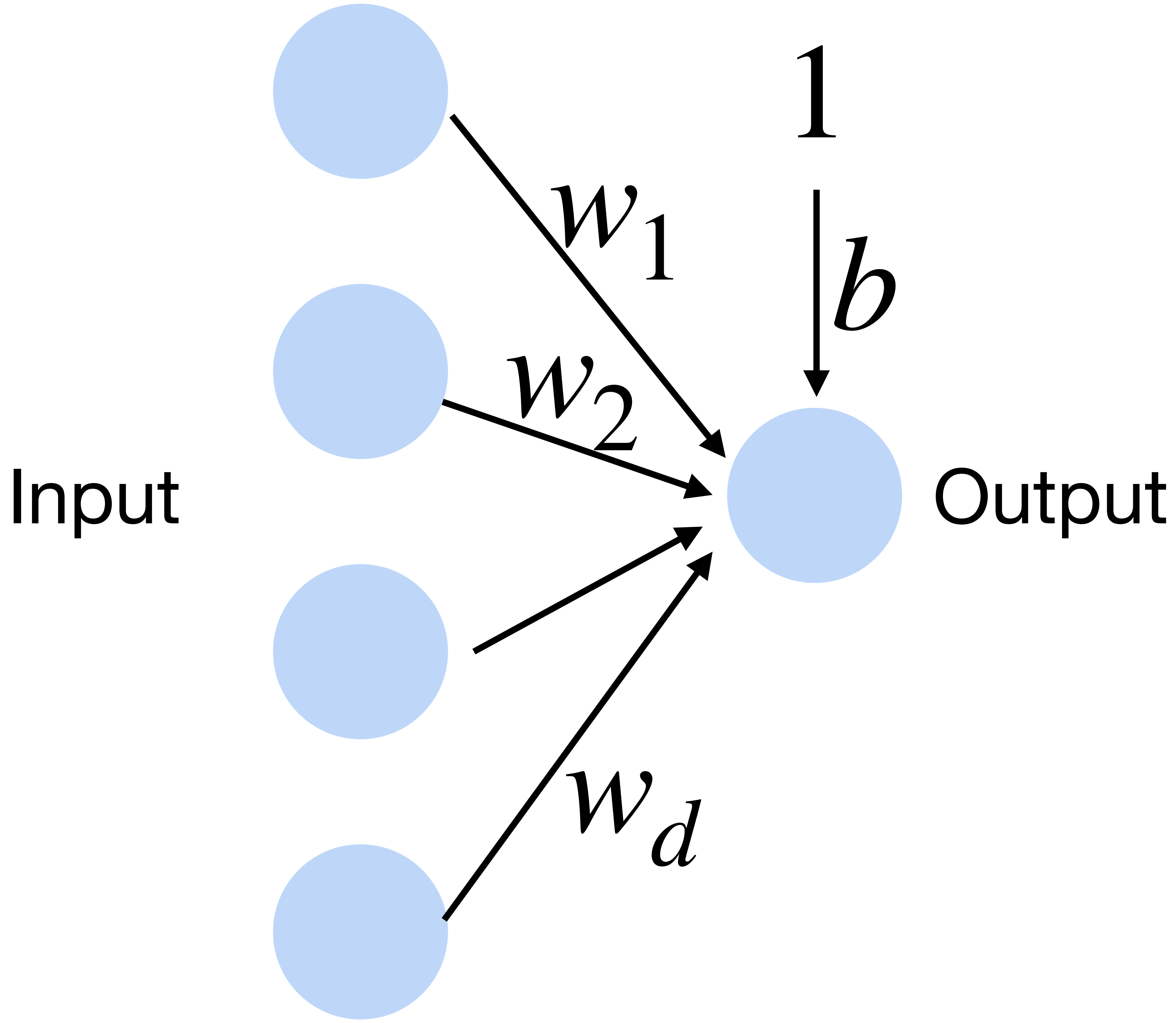
Cats vs. dogs?



Perceptron

- Goal: learn parameters $\mathbf{w} = \{w_1, w_2, \ldots, w_d\}$ and b to minimize the classification error

Cats vs. dogs?



Training the Perceptron

x augmented with dimension of constant 1

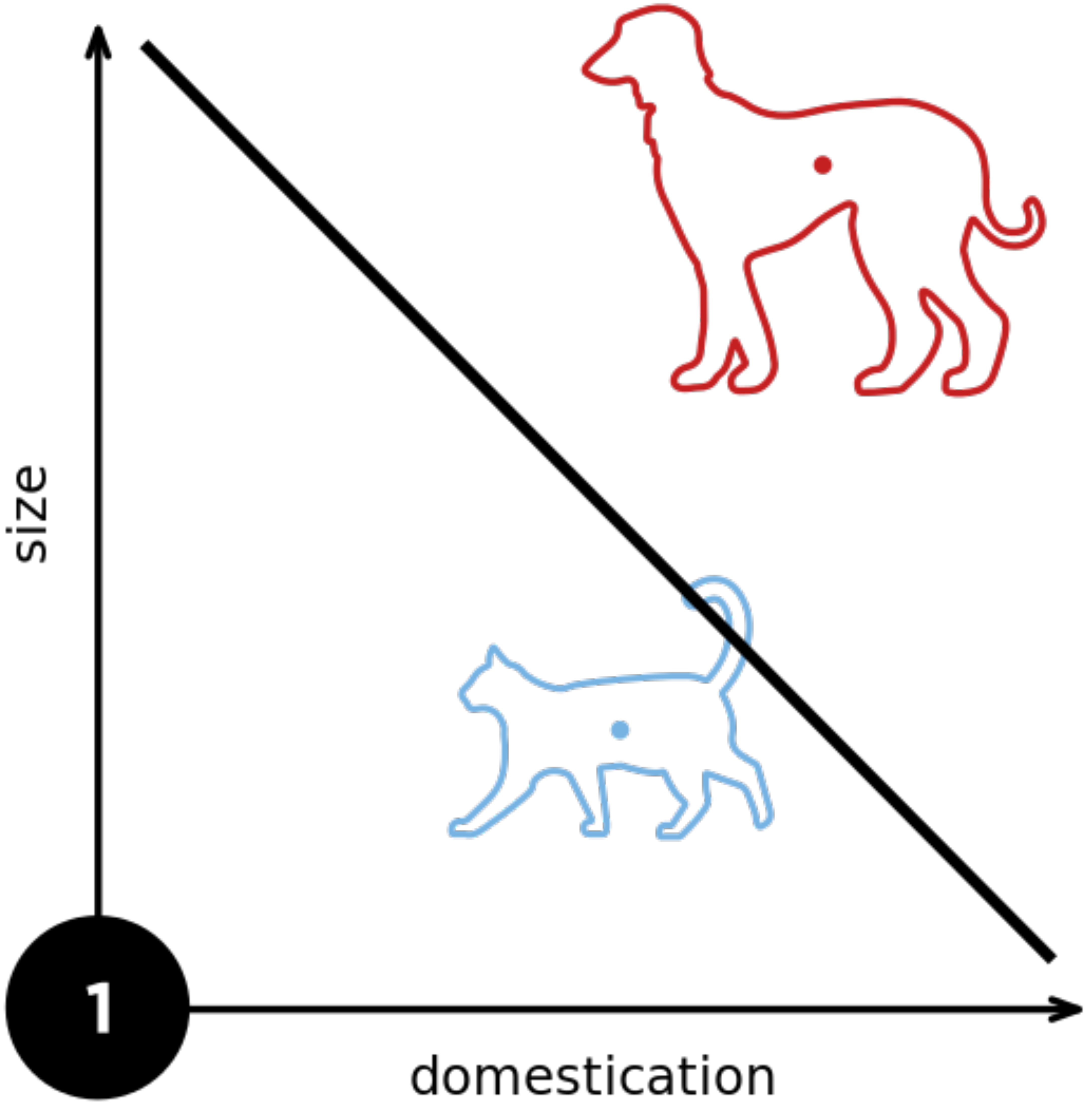
$$o = \sigma\left(\langle \mathbf{w}, \mathbf{x} \rangle\right)$$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Perceptron Algorithm

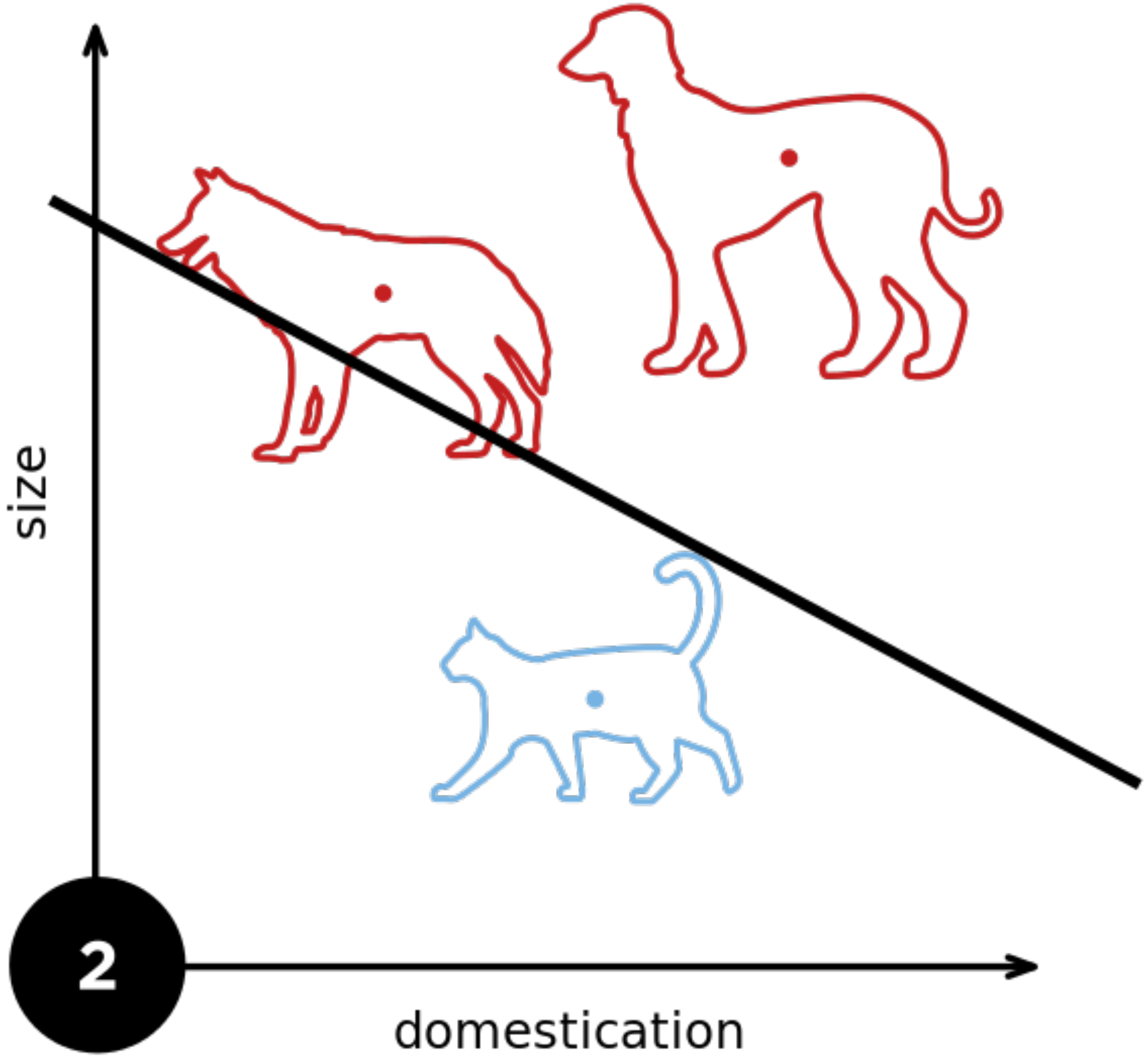
```
Initialize w⃗ = 0⃗                       // Initialize w⃗. w⃗ = 0⃗ misclassifies everything.
while TRUE do                           // Keep looping
    m = 0                               // Count the number of misclassifications, m
    for (x_i, y_i) ∈ D do               // Loop over each (data, label) pair in the dataset, D
        if o_i ≠ y_i                    // If the pair (x⃗_i, y_i) is misclassified
            w⃗ ← w⃗ + x_i if y_i = 1,   w⃗ ← w⃗ − x_i if y_i = 0
            m ← m + 1                   // Counter the number of misclassification
        end if
    end for
    if m = 0 then                       // If the most recent w⃗ gave 0 misclassifications
        break                           // Break out of the while-loop
    end if
end while                               // Otherwise, keep looping!
```

Perceptron



From wikipedia

Perceptron



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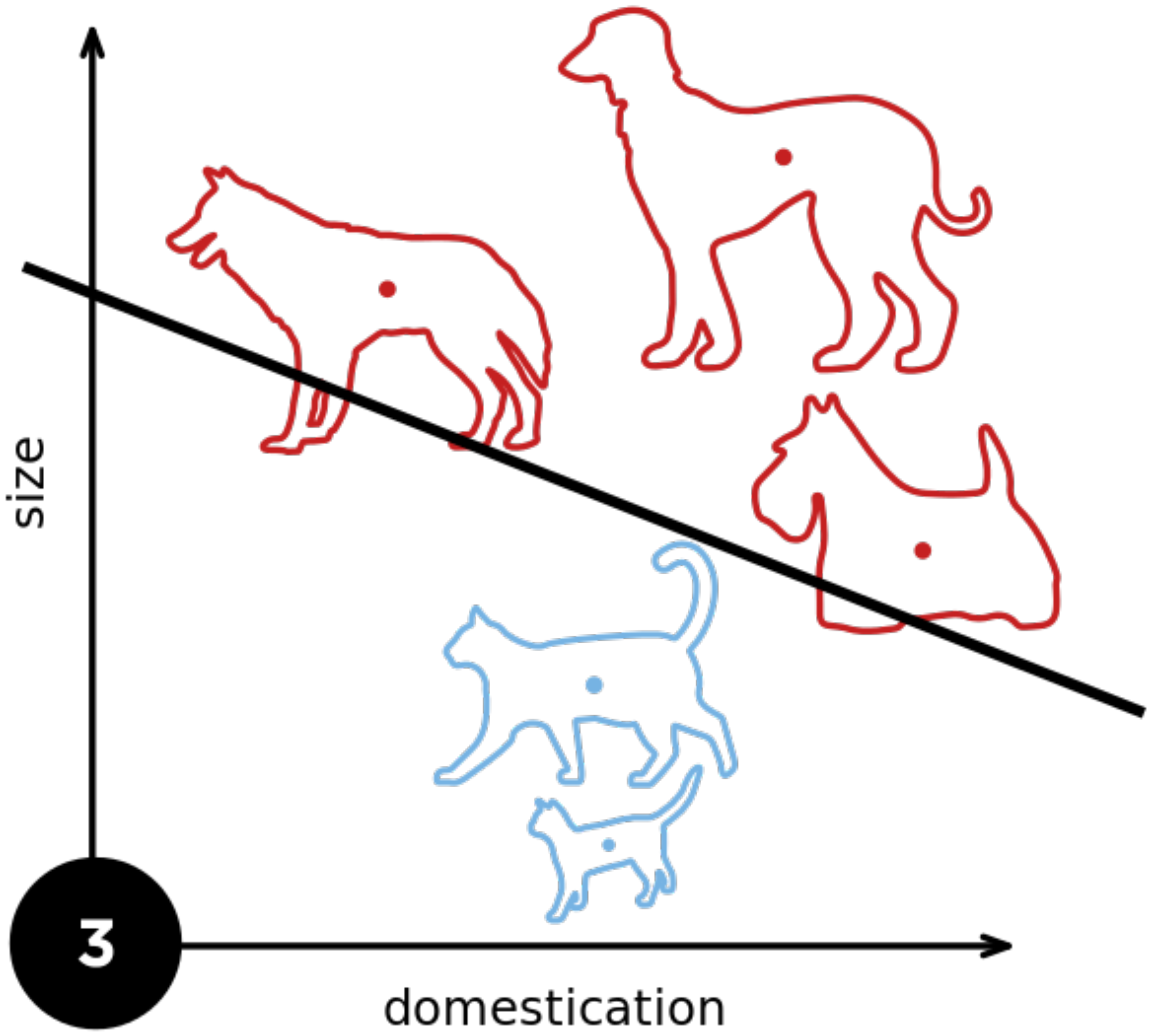

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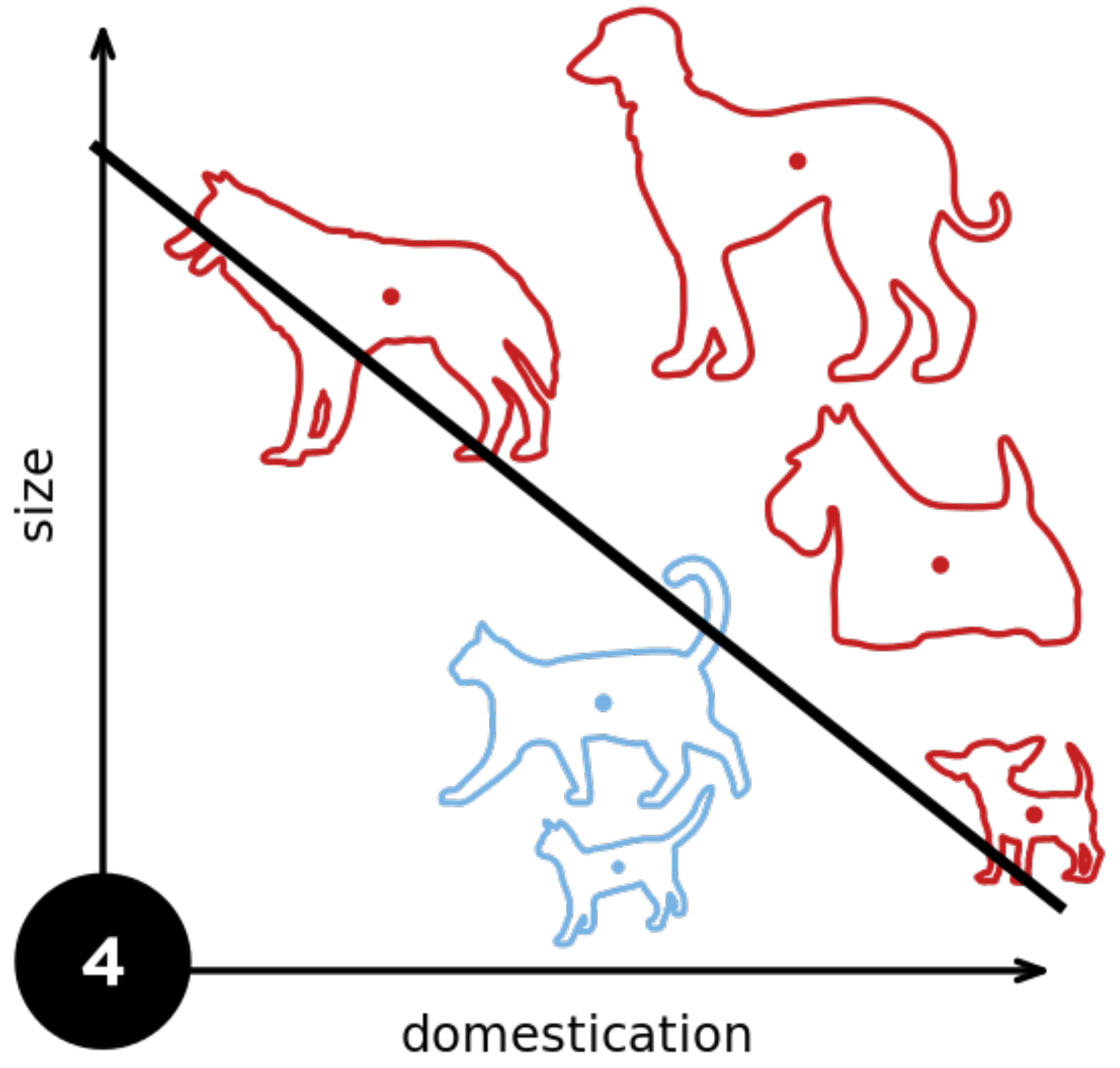
Perceptron



From wikipedia

Learning AND function using perceptron

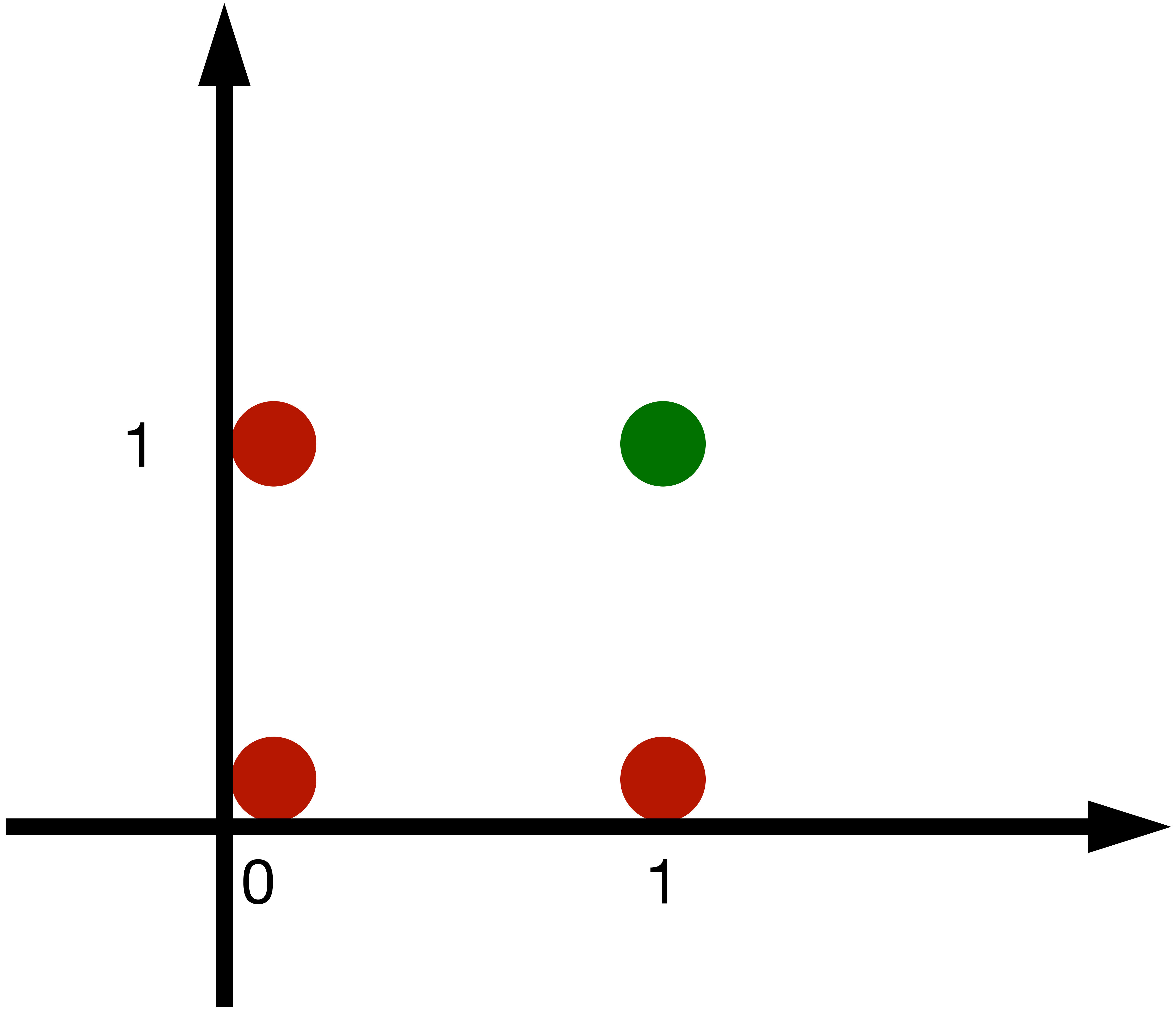
The perceptron can learn an AND function

$x_1 = 1, x_2 = 1, y = 1$

$x_1 = 1, x_2 = 0, y = 0$

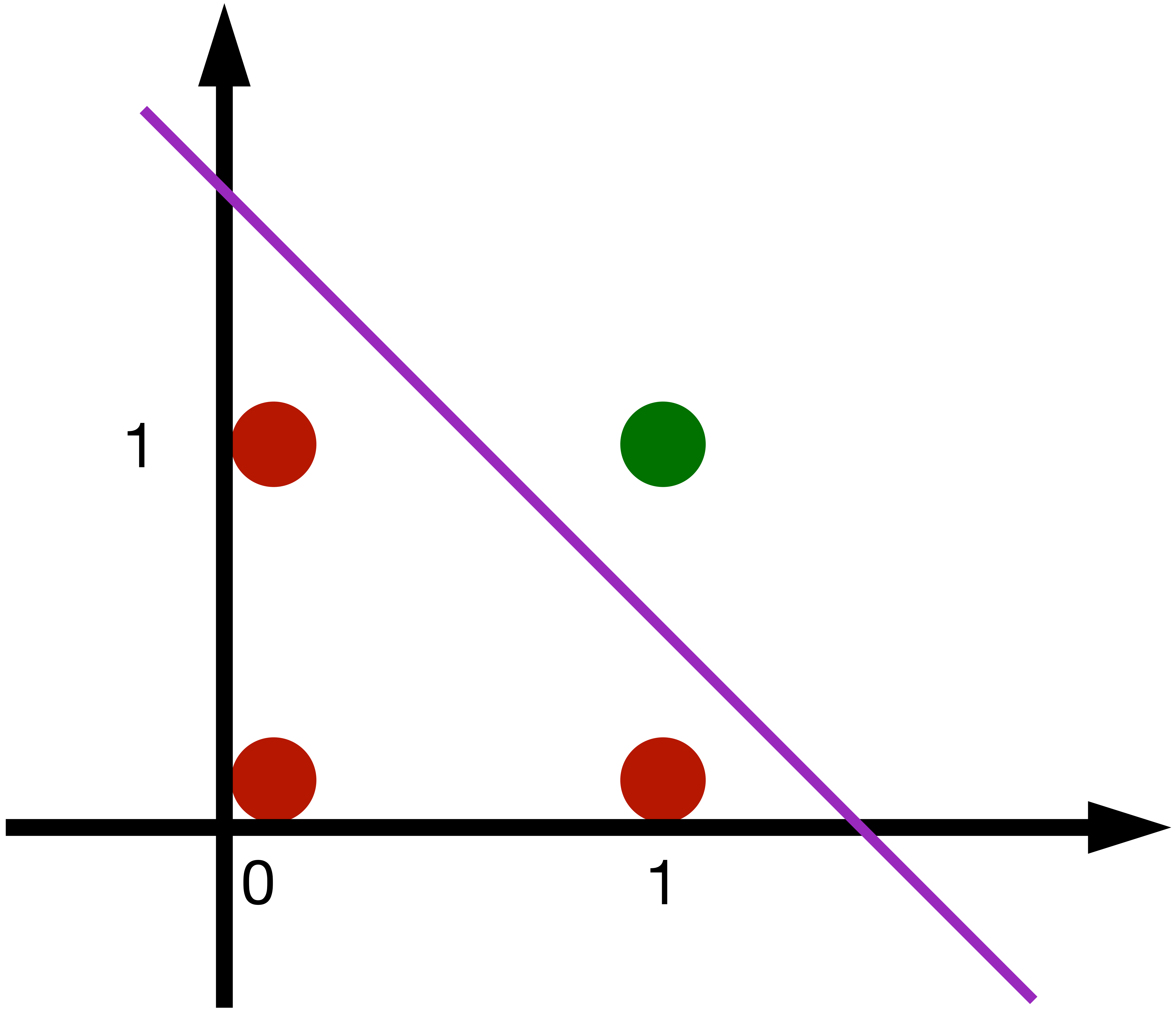
$x_1 = 0, x_2 = 1, y = 0$

$x_1 = 0, x_2 = 0, y = 0$



Learning AND function using perceptron

The perceptron can learn an AND function



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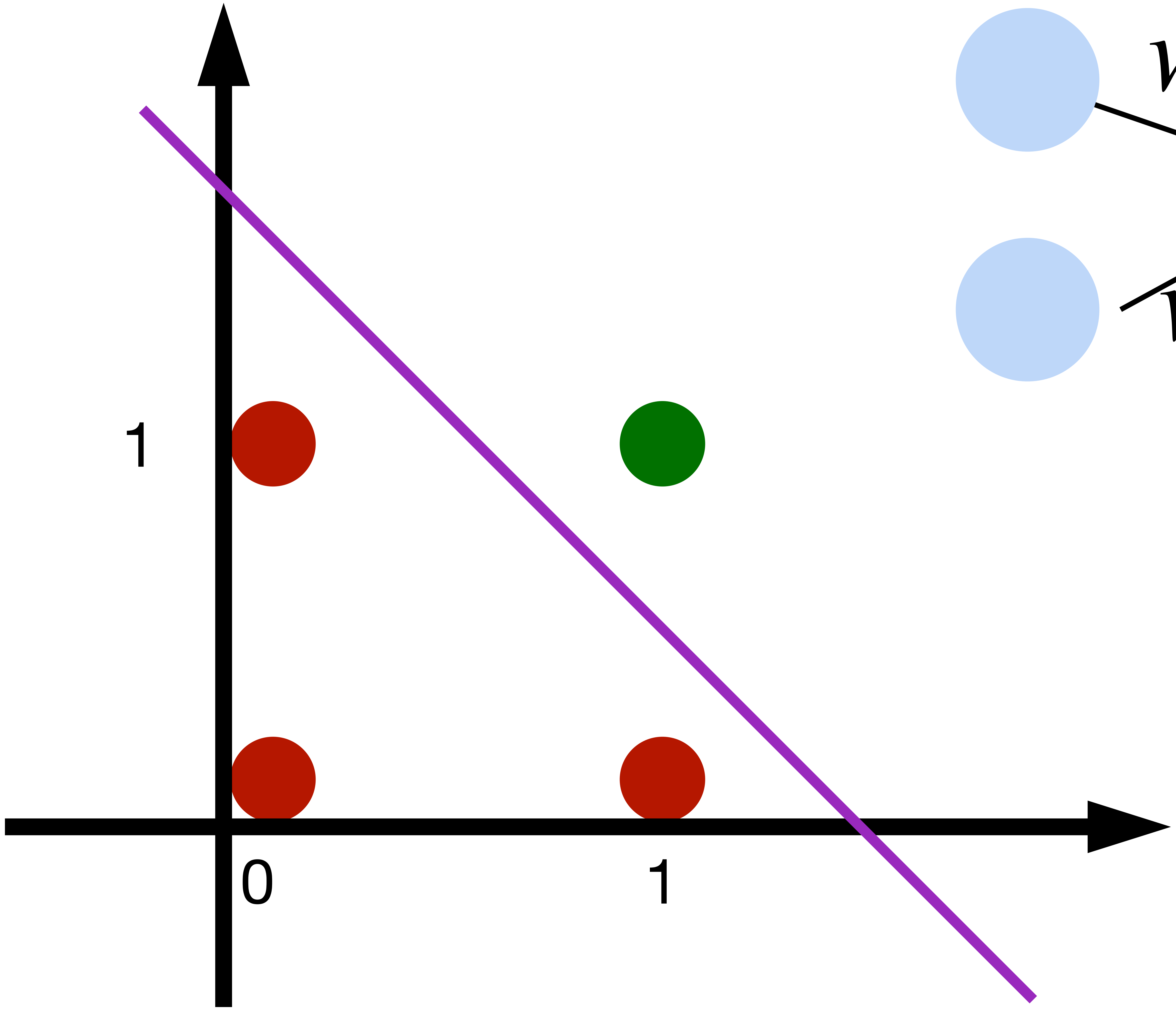
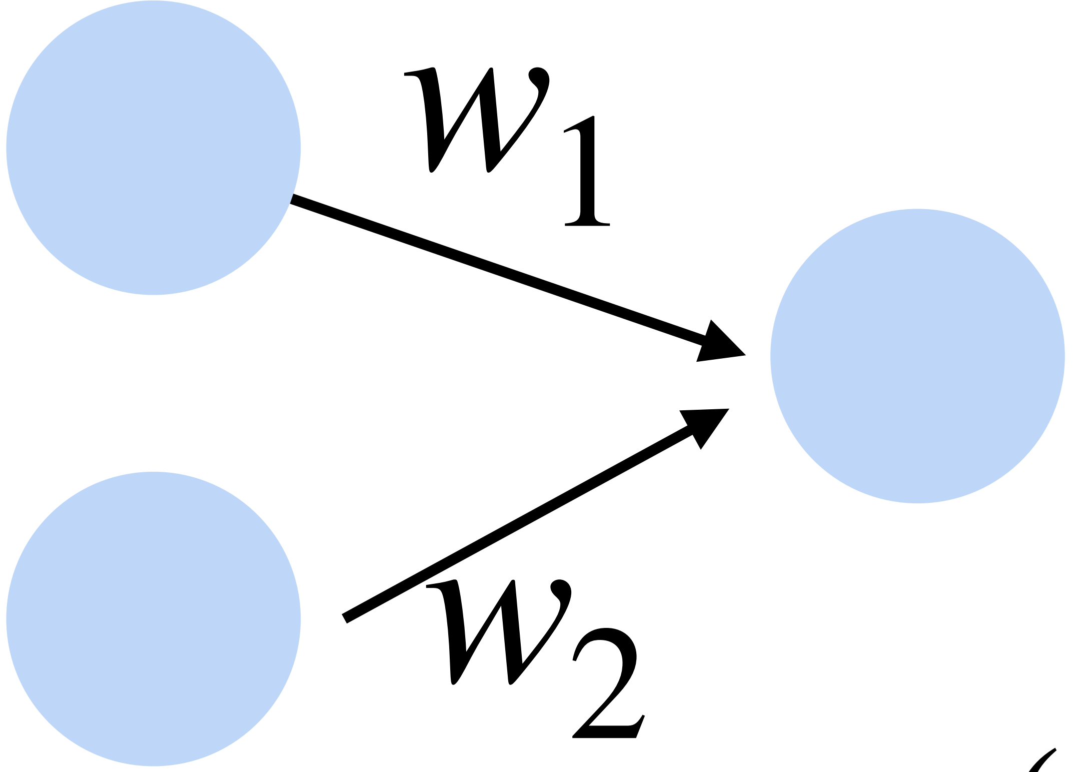




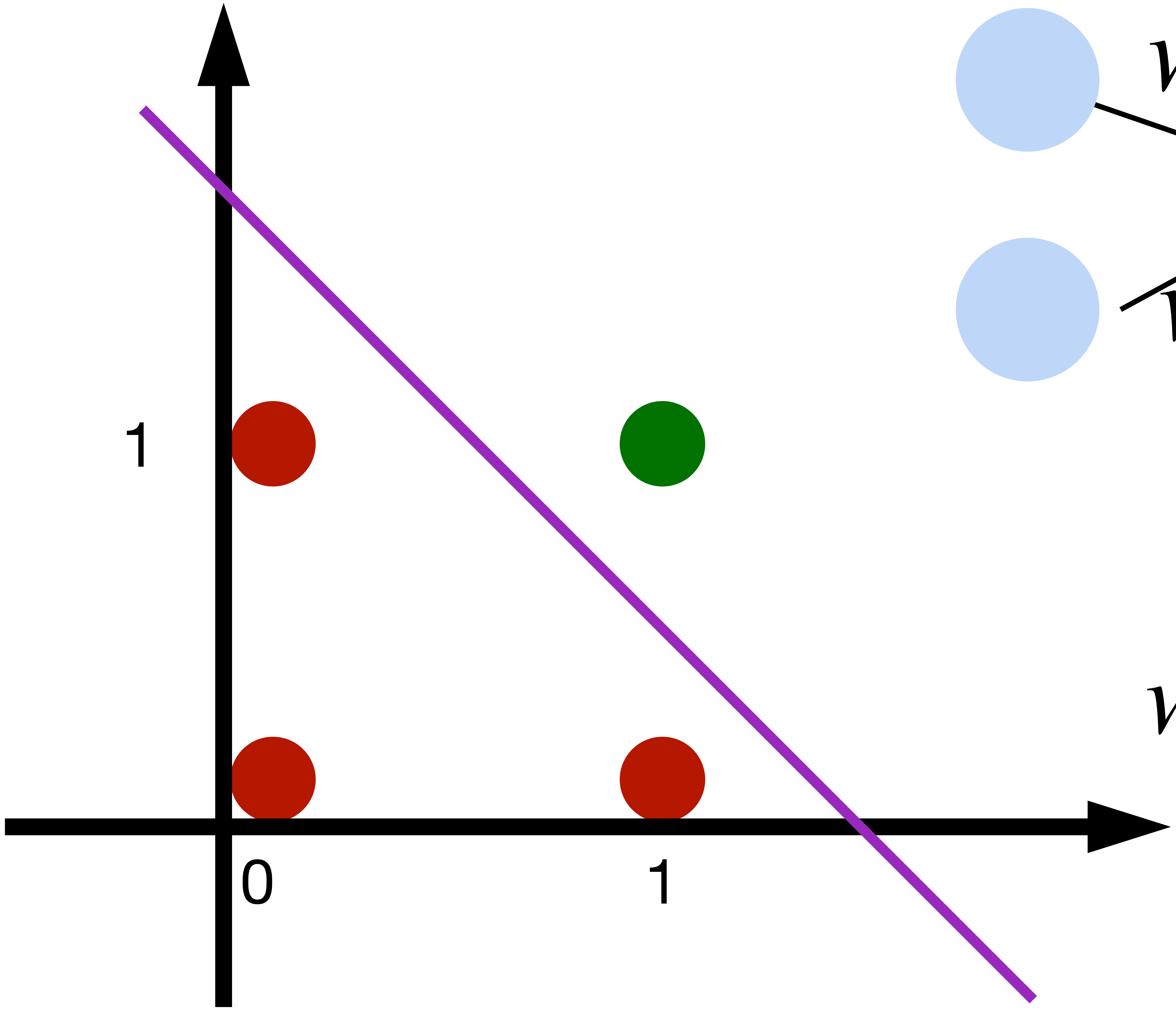
Output $\sigma(x_1 w_1 + x_2 w_2 + b)$

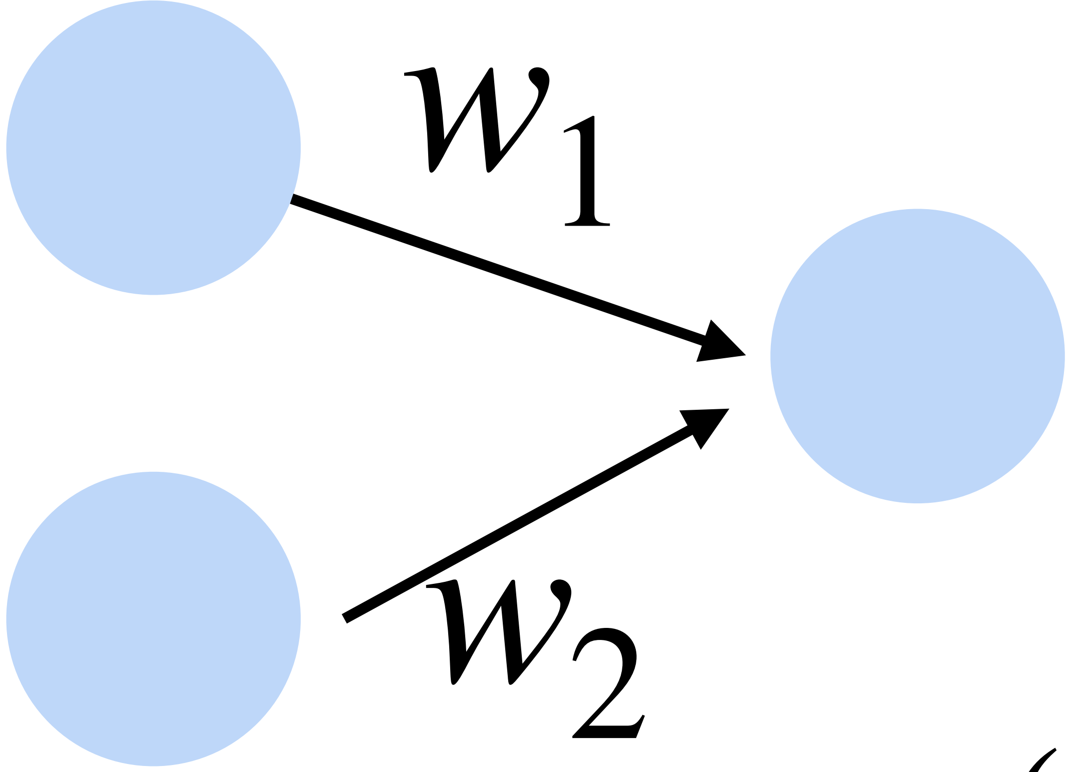
$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

What's w and b?

Learning AND function using perceptron

The perceptron can learn an AND function





Output $\sigma(x_1 w_1 + x_2 w_2 + b)$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1 = 1, w_2 = 1, b = -1.5$$

Learning OR function using perceptron

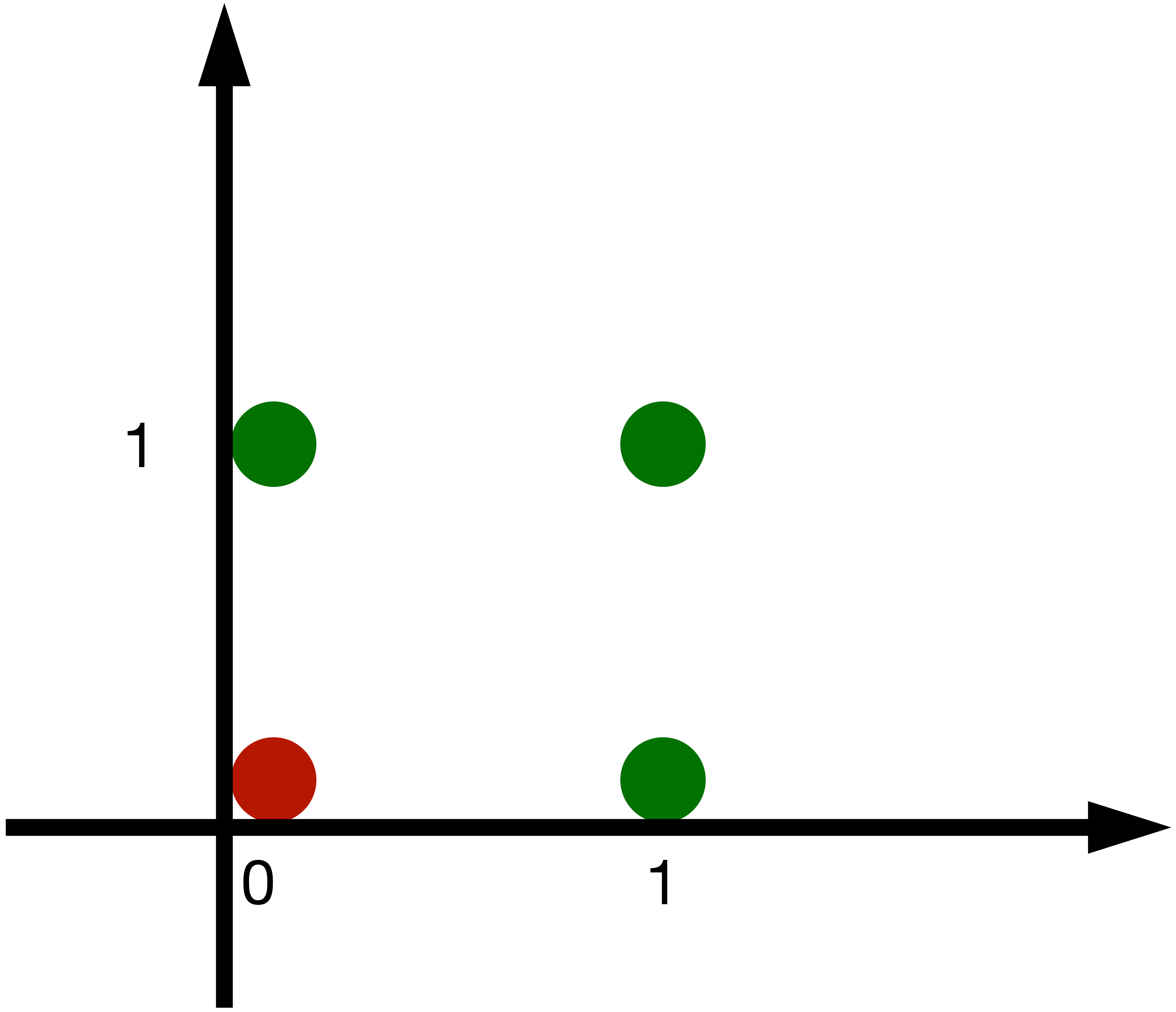
The perceptron can learn an OR function

$x_1 = 1, x_2 = 1, y = 1$

$x_1 = 1, x_2 = 0, y = 1$

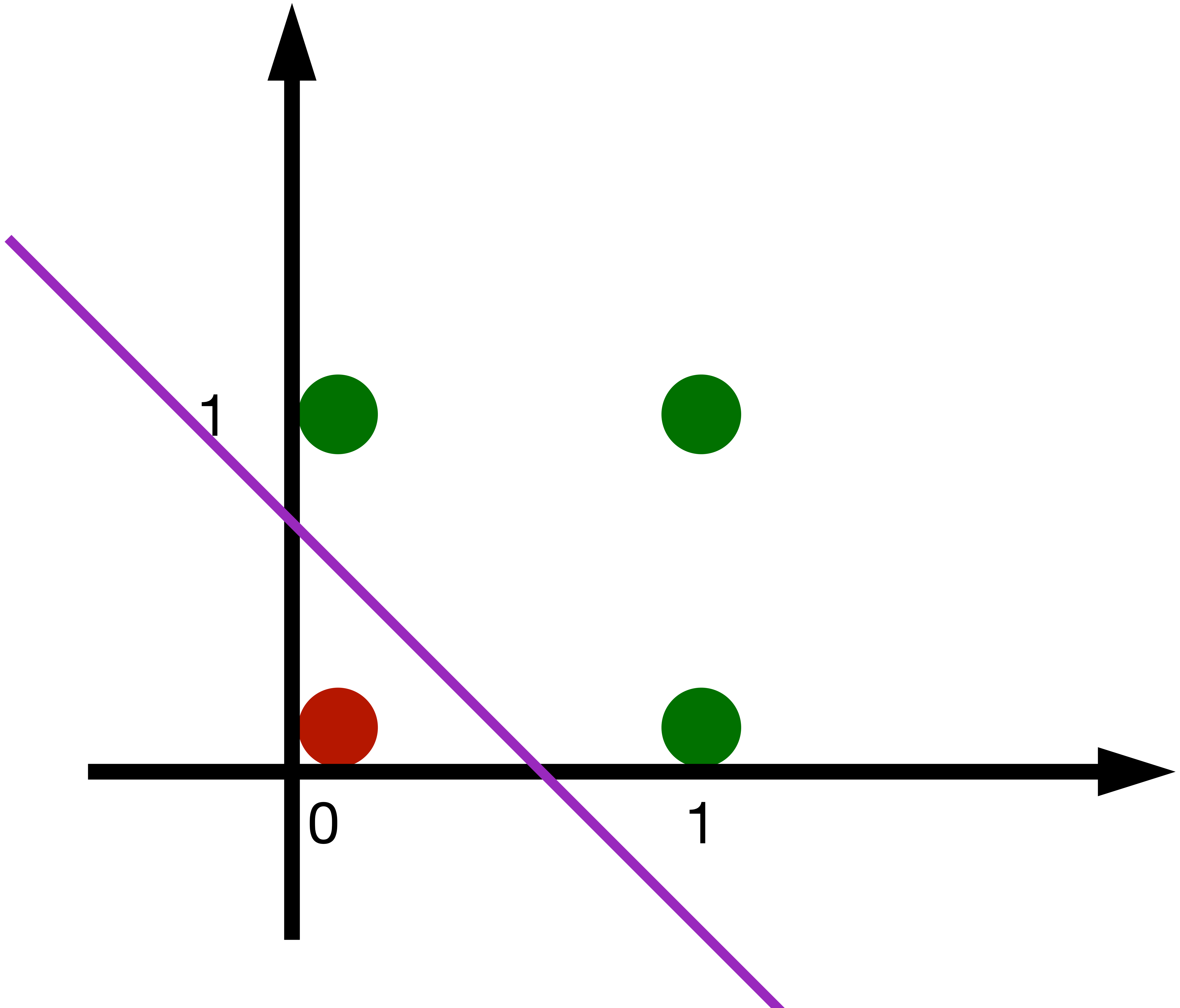
$x_1 = 0, x_2 = 1, y = 1$

$x_1 = 0, x_2 = 0, y = 0$



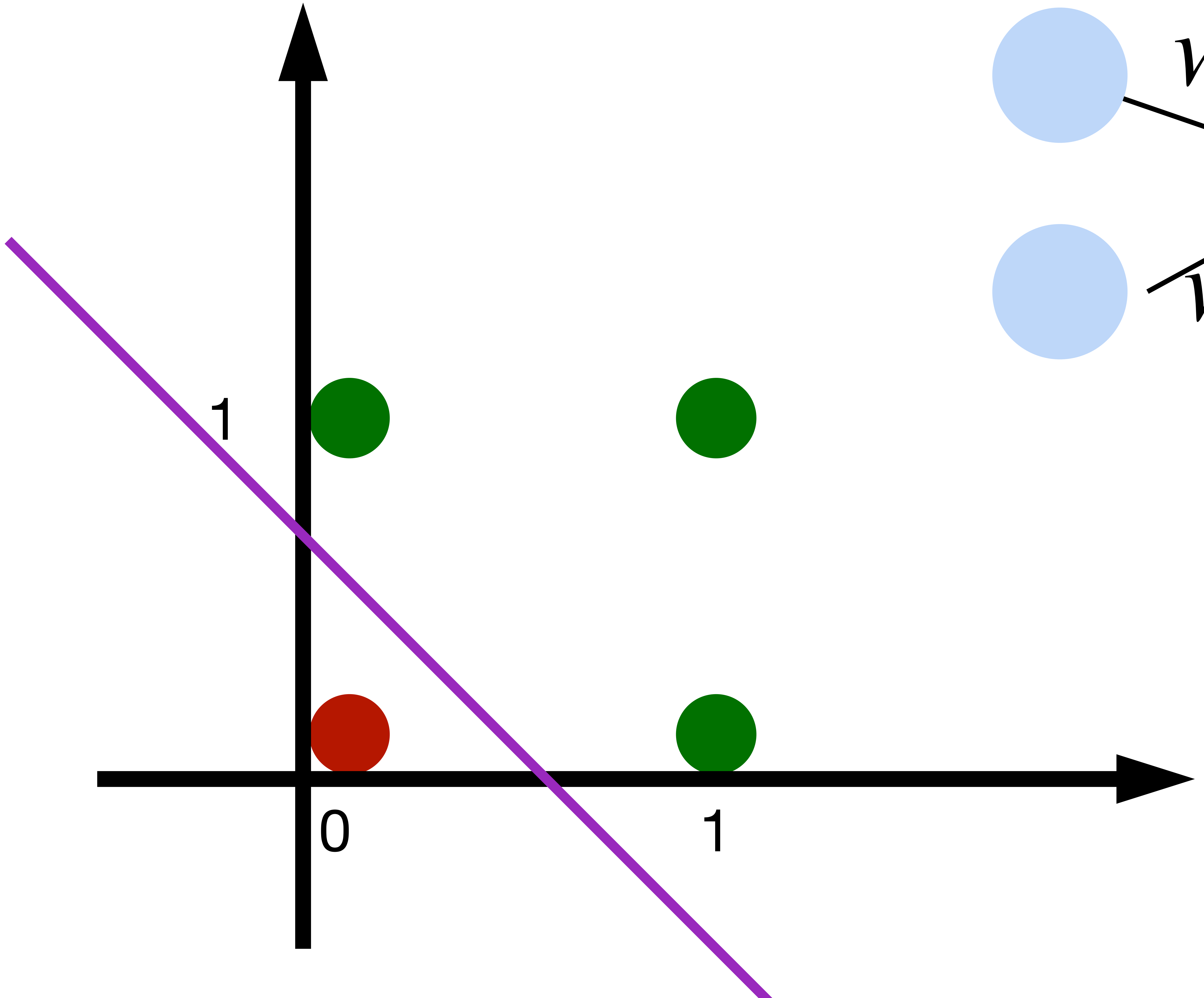
Learning OR function using perceptron

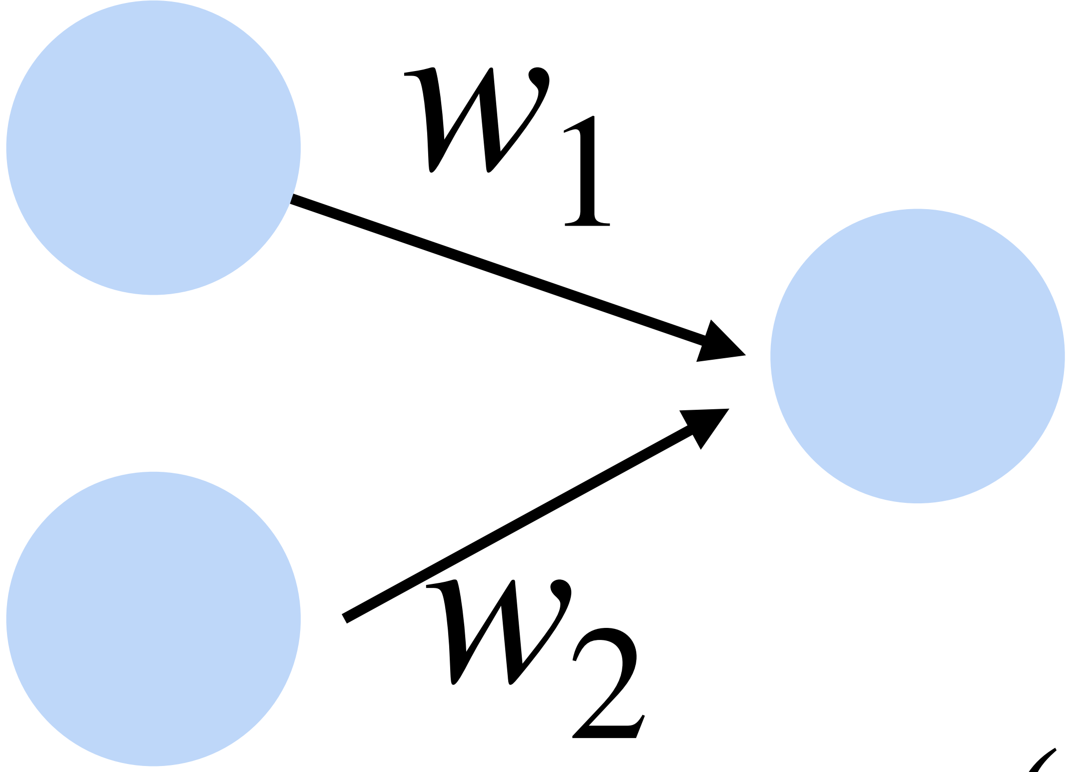
The perceptron can learn an OR function



Learning OR function using perceptron

The perceptron can learn an OR function





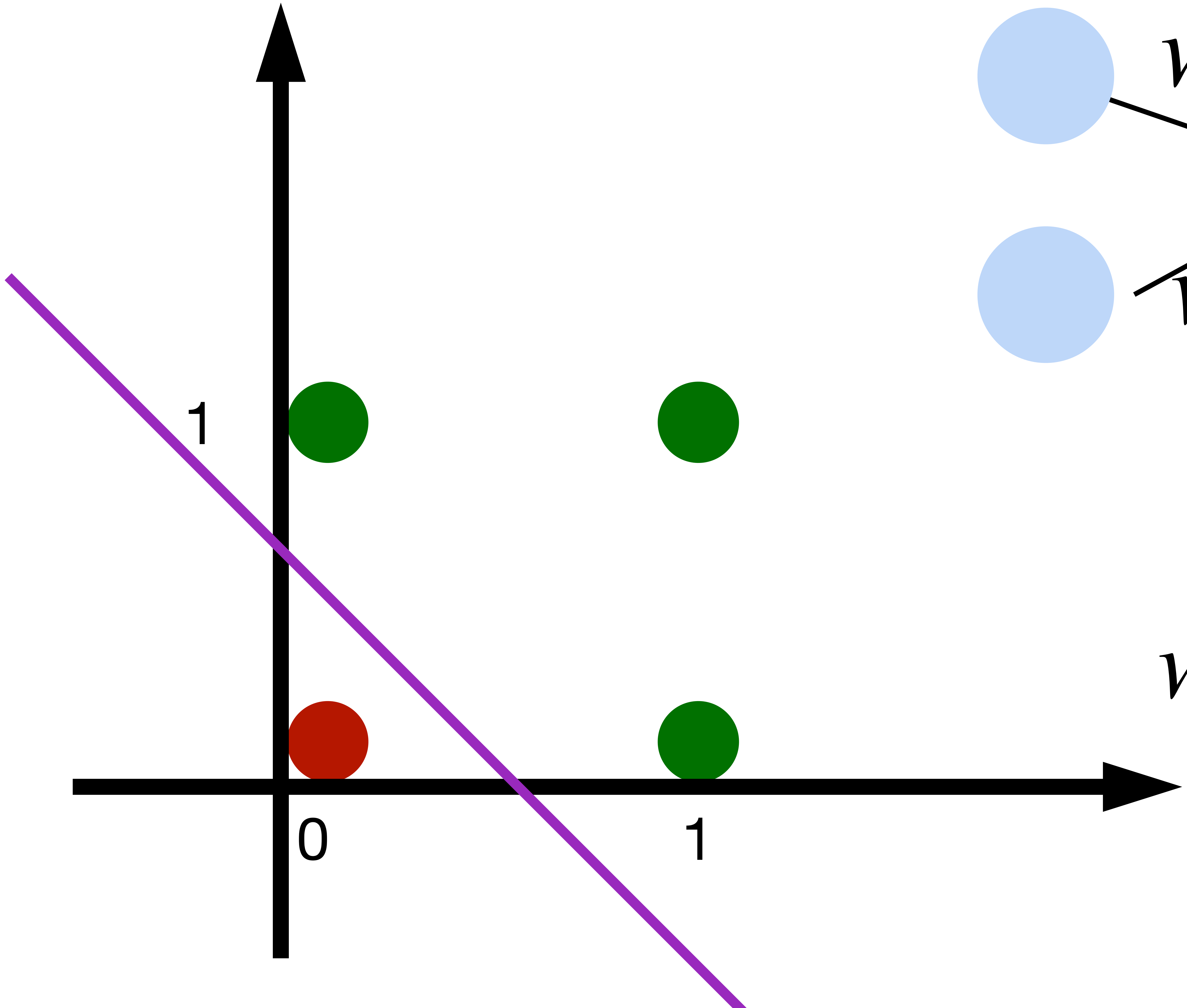
Output $\sigma(x_1 w_1 + x_2 w_2 + b)$

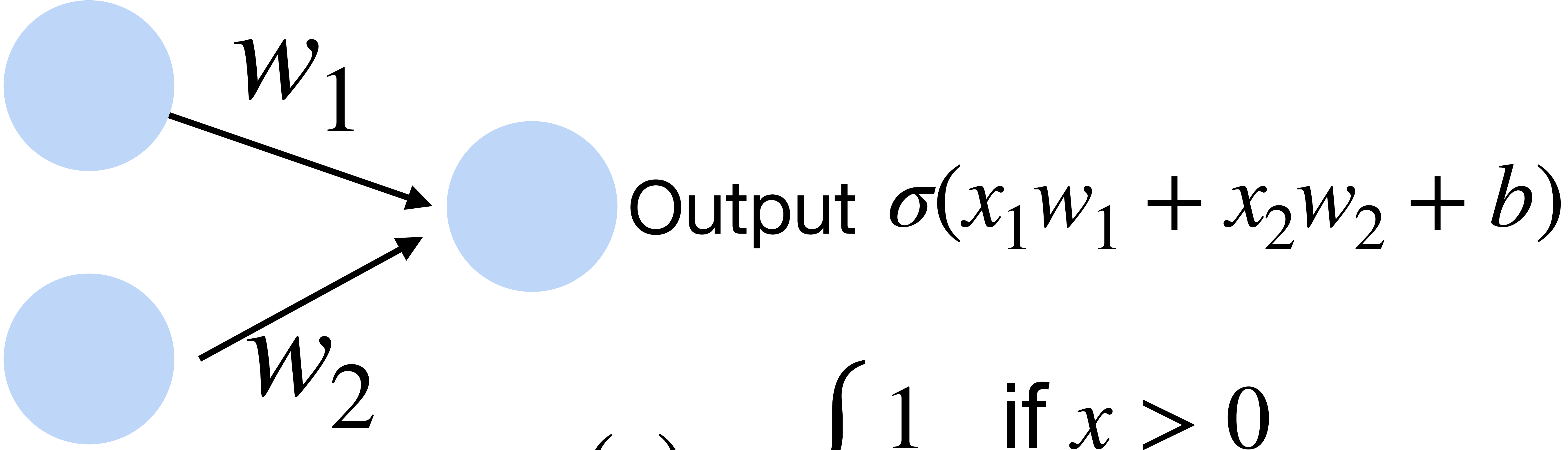
$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

What's w and b?

Learning OR function using perceptron

The perceptron can learn an OR function



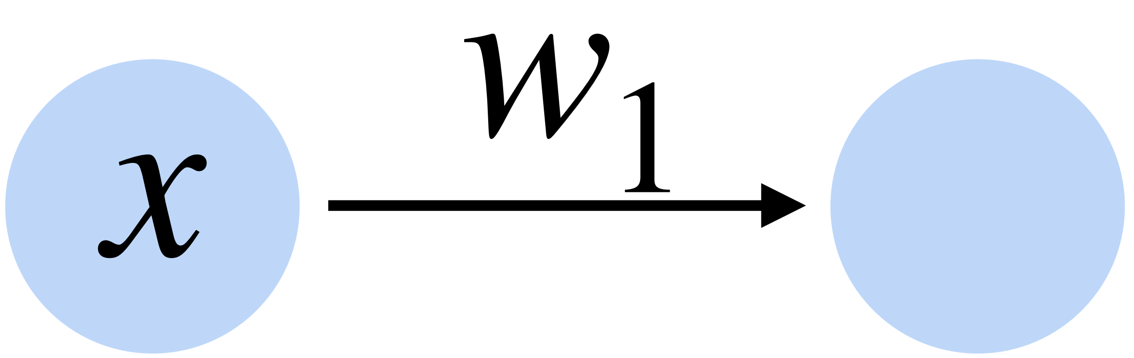


$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1 = 1, w_2 = 1, b = -0.5$$

Learning NOT function using perceptron

The perceptron can learn NOT function (single input)



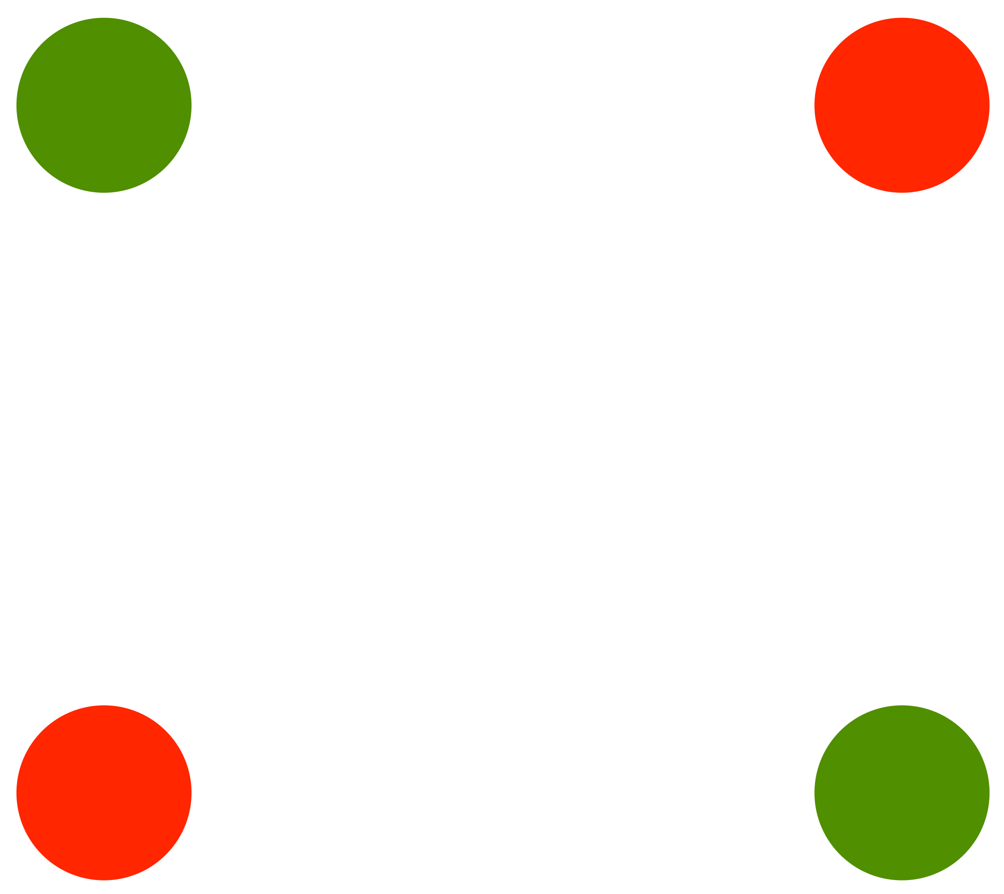
Output $\sigma(xw_1 + b)$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1 = -1, b = 0.5$$

XOR Problem (Minsky & Papert, 1969)

The perceptron cannot learn an XOR function
(it can only generate linear separators)



This contributed to the first AI winter

Quiz Break

Consider the linear perceptron with x as the input. Which function can the linear perceptron compute?

(1) $y = \mathrm{ax} + \mathrm{b}$

(2) $y = \mathrm{ax}^2 + bx + c$

A. (1)

B. (2)

C. (1)(2)

D. None of the above

Quiz Break

Consider the linear perceptron with x as the input. Which function can the linear perceptron compute?

(1) $y = ax + b$

(2) $y = ax^2 + bx + c$

A. (1)

B. (2)

C. (1)(2)

D. None of the above

Answer: A. All units in a linear perceptron are linear. Thus, the model can not present non-linear functions.

Quiz Break

Perceptron can be used for representing:

A. AND function
B. OR function
C. XOR function
D. Both AND and OR function

Quiz Break

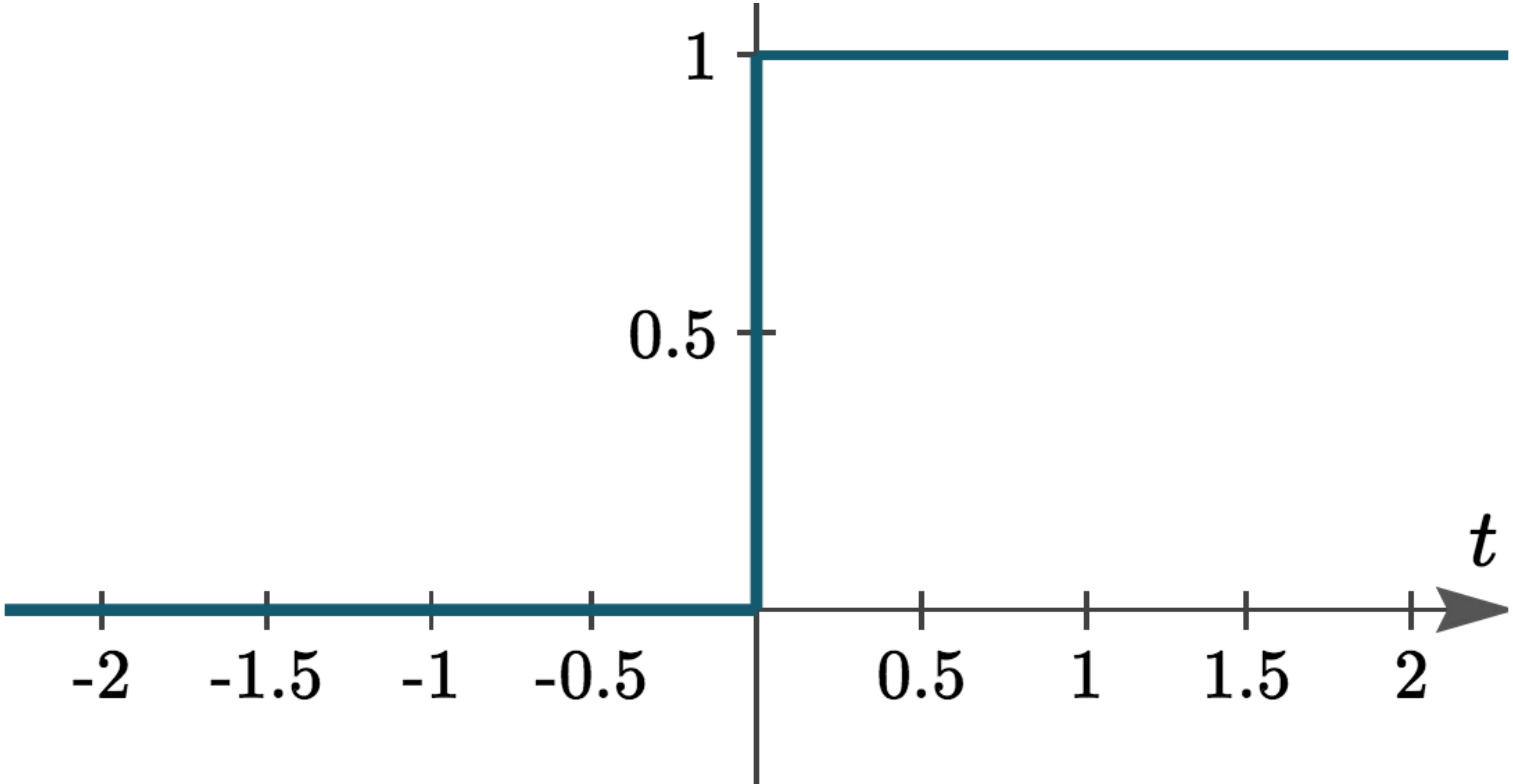
Perceptron can be used for representing:

A. AND function
B. OR function
C. XOR function
D. Both AND and OR function

Step Function activation

Step function is discontinuous, which cannot be used for gradient descent

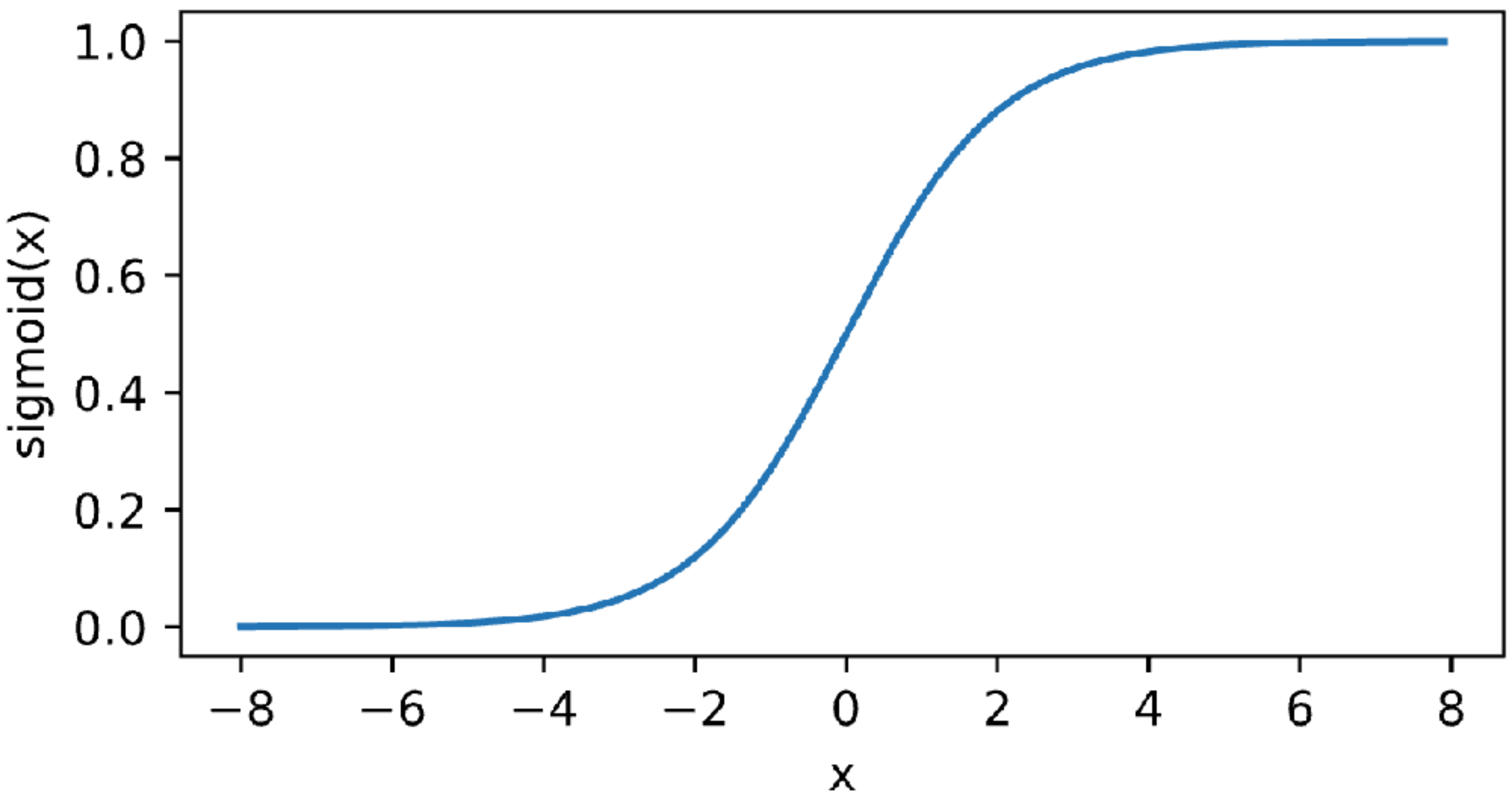
$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$



Sigmoid/Logistic Activation

Map input into [0, 1], a **soft** version of $\sigma(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$

$$\sigma(z) = \text{sigmoid}(z) = \frac{1}{1 + \exp(-z)}$$

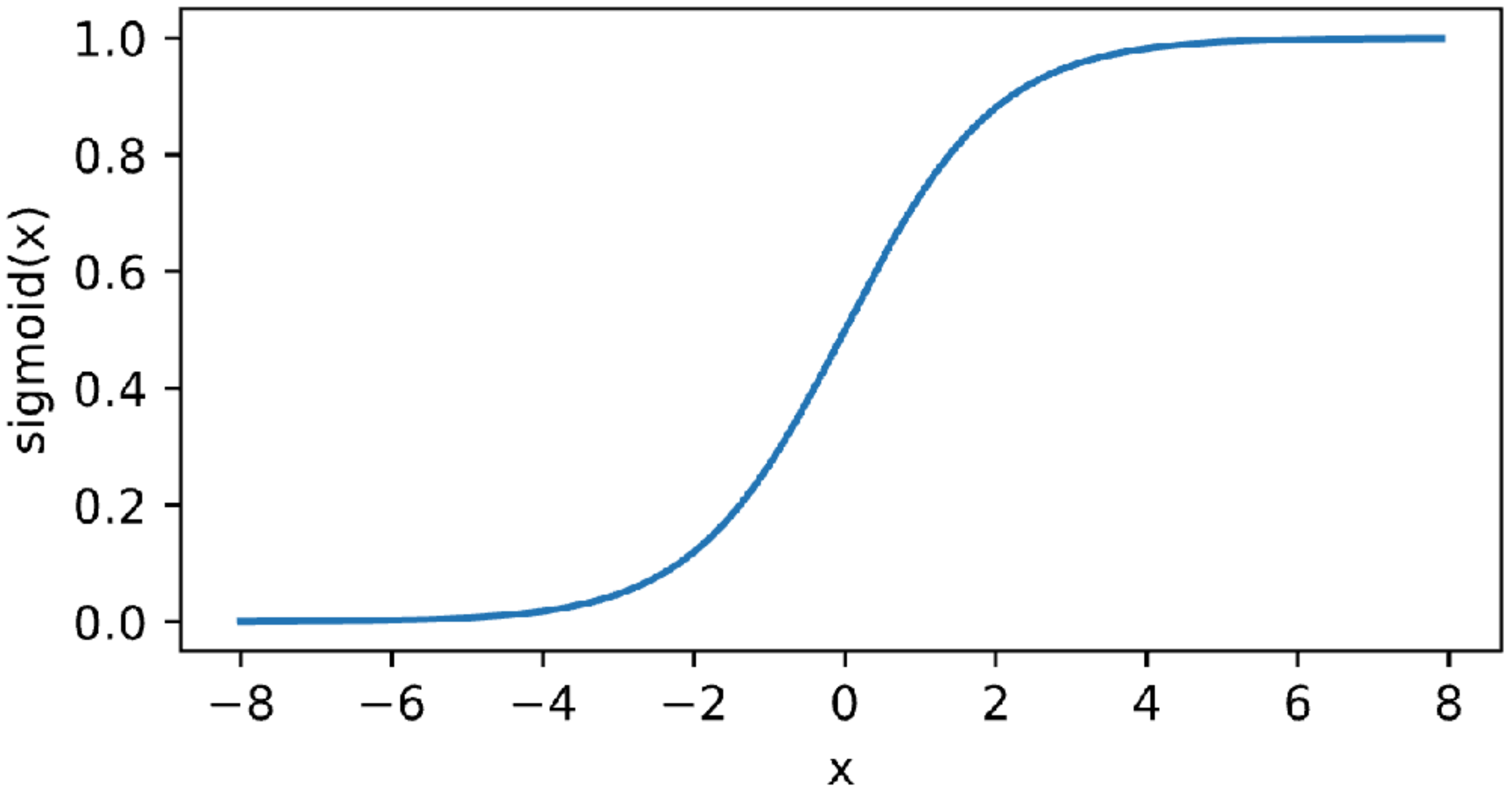


Logistic regression

$\mathbf{x} \in \mathbb{R}^d, y = \{-1, +1\}$

$$p(y = 1 \mid \mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T\mathbf{x})}$$

$$p(y = -1 \mid \mathbf{x}) = 1 - \sigma(\mathbf{w}^T\mathbf{x}) = \frac{1}{1 + \exp(\mathbf{w}^T\mathbf{x})}$$



Logistic regression

Given: $\{(\mathbf{x}_i, y_i)\}_{i=1}^{n}$ $\mathbf{x} \in \mathbb{R}^d, y = \{-1, +1\}$

Training: maximize likelihood estimate (on the conditional probability)

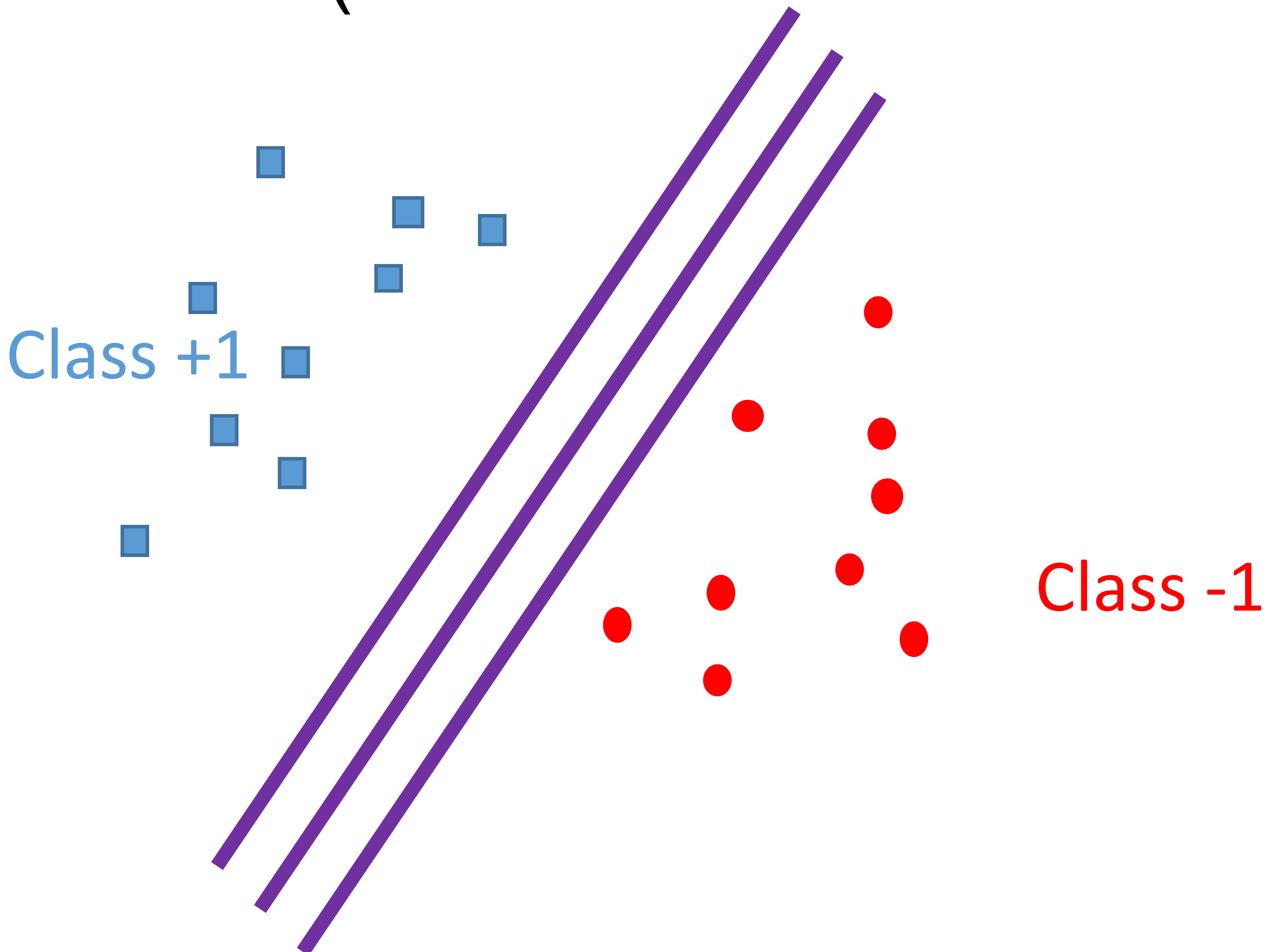
$$\max_{\mathbf{w}} \sum_i \log \frac{1}{1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i)}$$

Logistic regression

Given: $\{(\mathbf{x}_i, y_i)\}_{i=1}^{n}$ $\mathbf{x} \in \mathbb{R}^d, y = \{-1, +1\}$

Training: maximize likelihood estimate (on the conditional probability)

When training data is linearly separable, many solutions



Logistic regression

Given: $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ $\quad\quad \mathbf{x} \in \mathbb{R}^d, y = \{-1, +1\}$

Training: maximum A posteriori (MAP)

$$\min_{\mathbf{w}} \sum_i -\log \frac{1}{1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i)} + \frac{\lambda}{2} \|\mathbf{w}\|_2^2$$

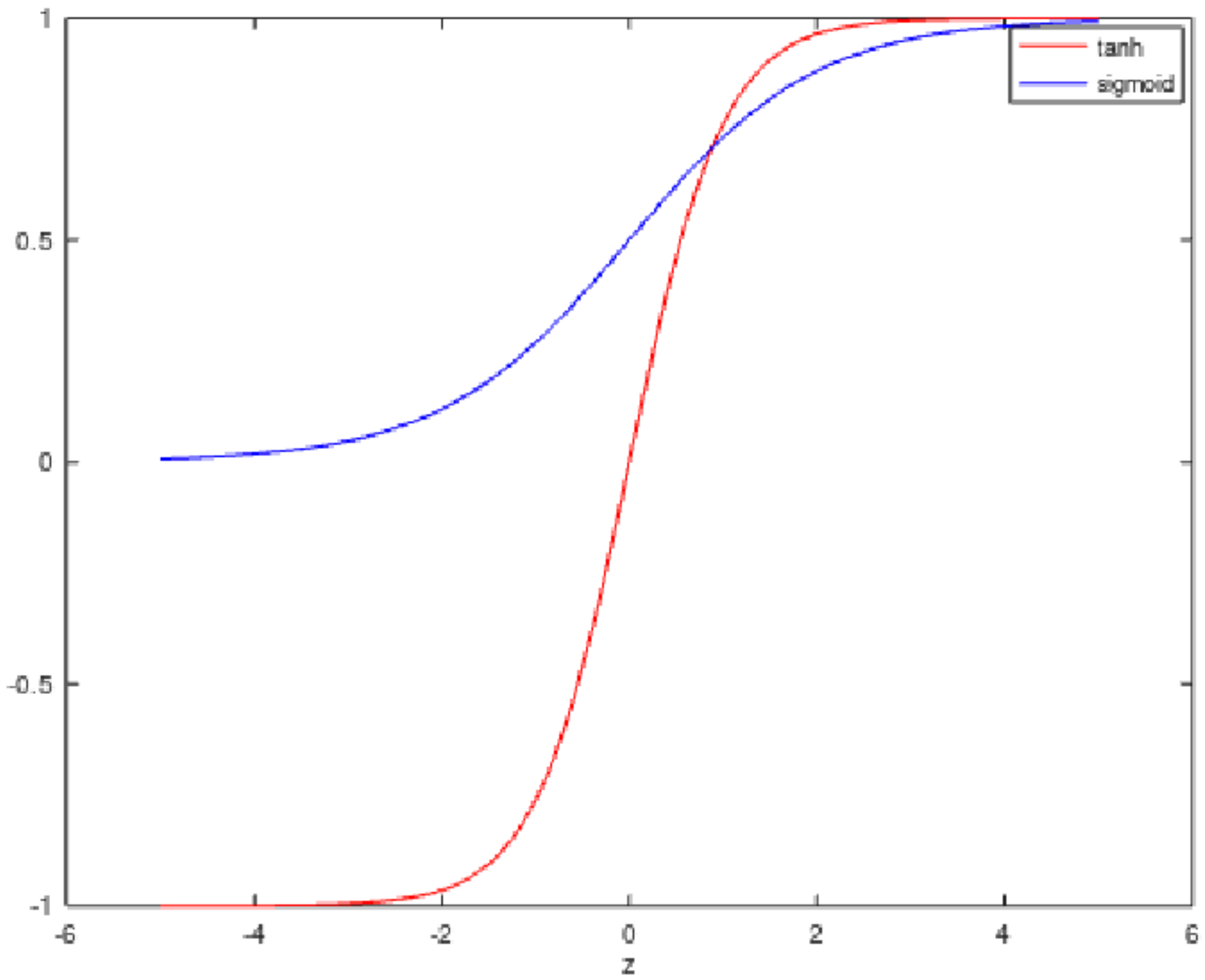
- Convex optimization
- Solve via (stochastic) gradient descent

Tanh Activation

Map inputs into (-1, 1)

$$\sigma(z) = \text{tanh}(z) = \frac{1 - \exp(-2z)}{1 + \exp(-2z)}$$

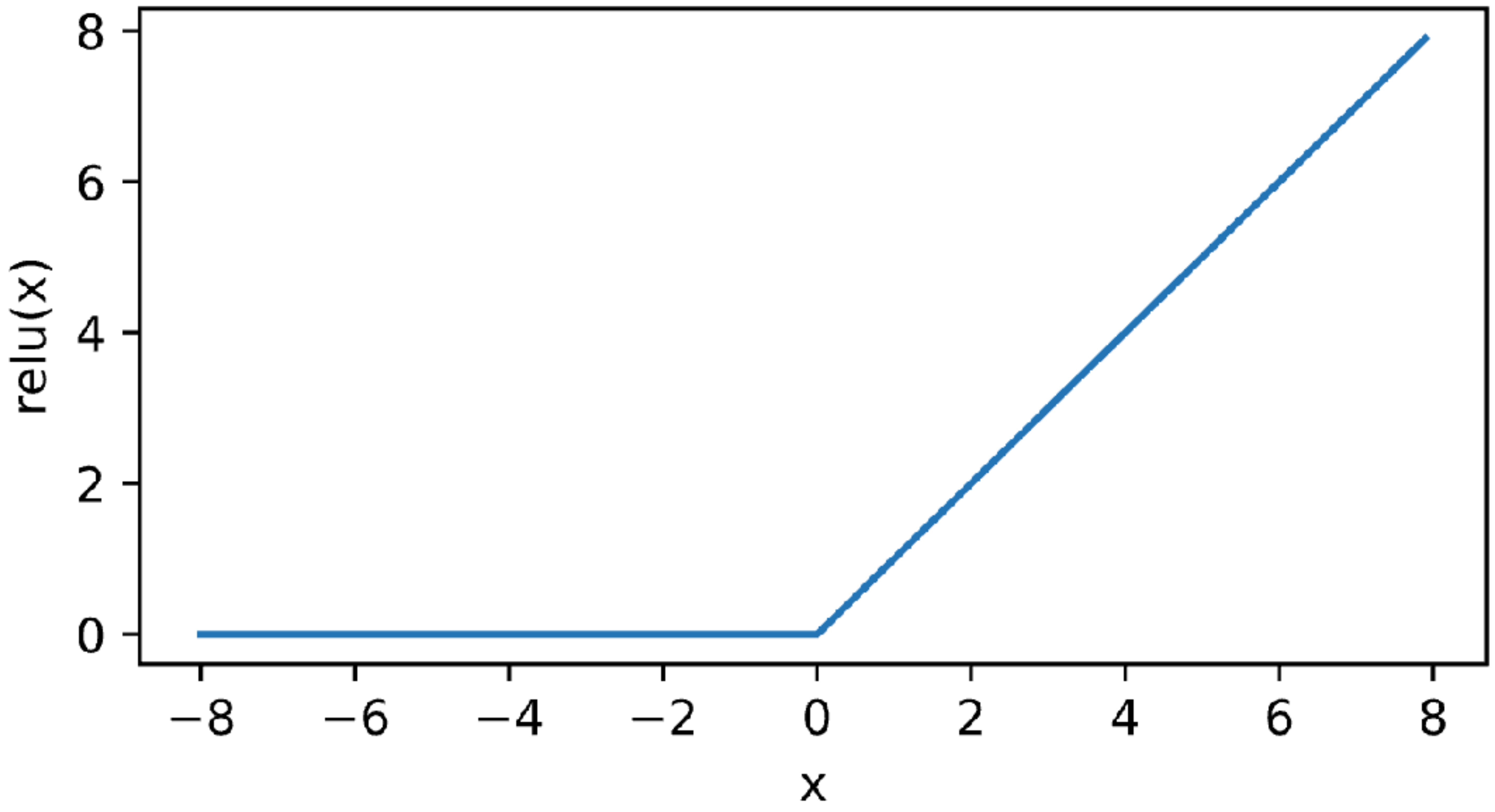
$$tanh(z) = 2sigmoid(2z) - 1$$



ReLU Activation

ReLU: rectified linear unit (commonly used in modern neural networks)

$$\mathrm{ReLU}(x) = \max(x, 0)$$



Quiz Break

Which one of the following is valid activation function

a)Step function

b) Sigmoid function

C) ReLU function

D) all of above

Quiz Break

Which one of the following is valid activation function

a)Step function

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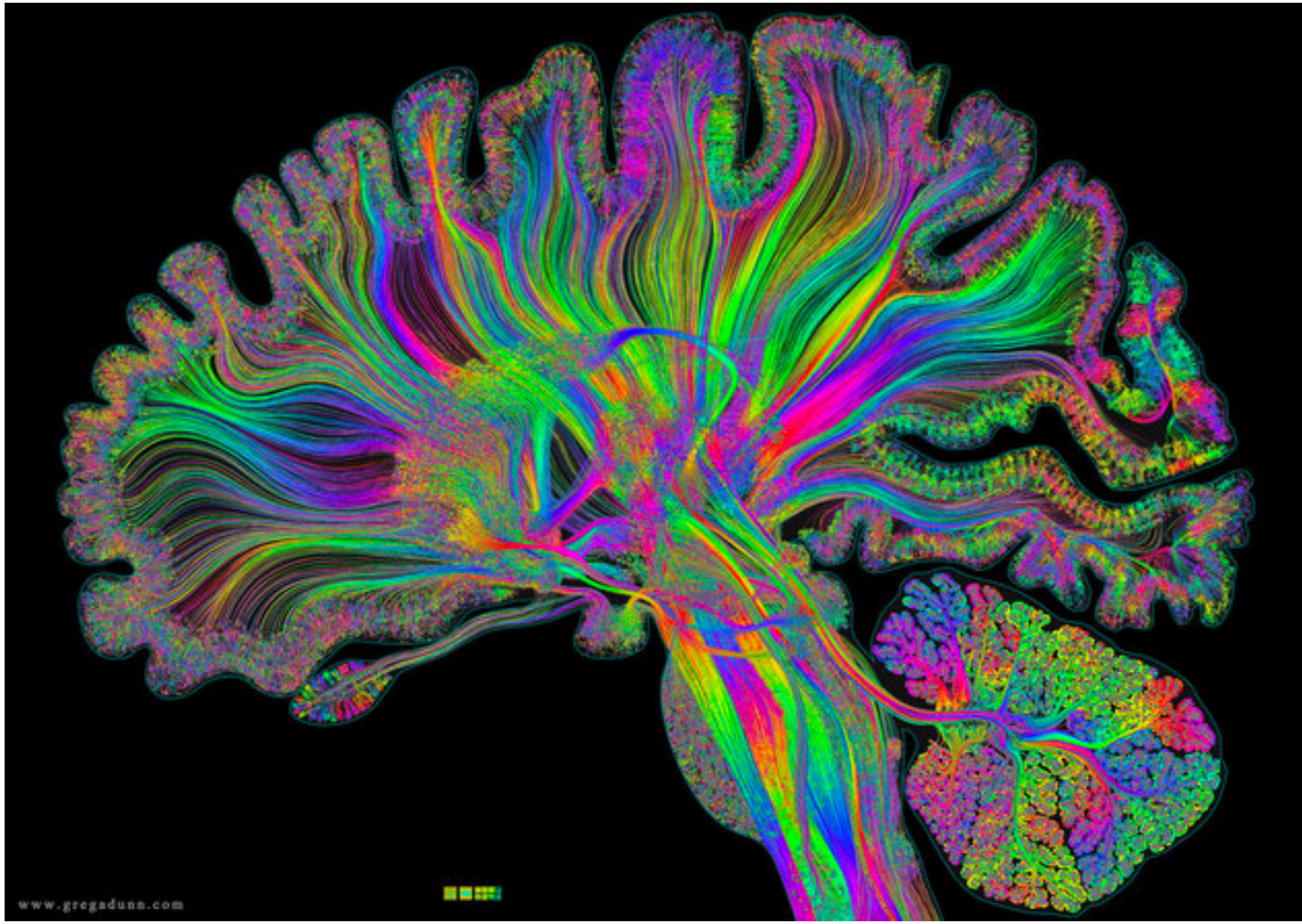
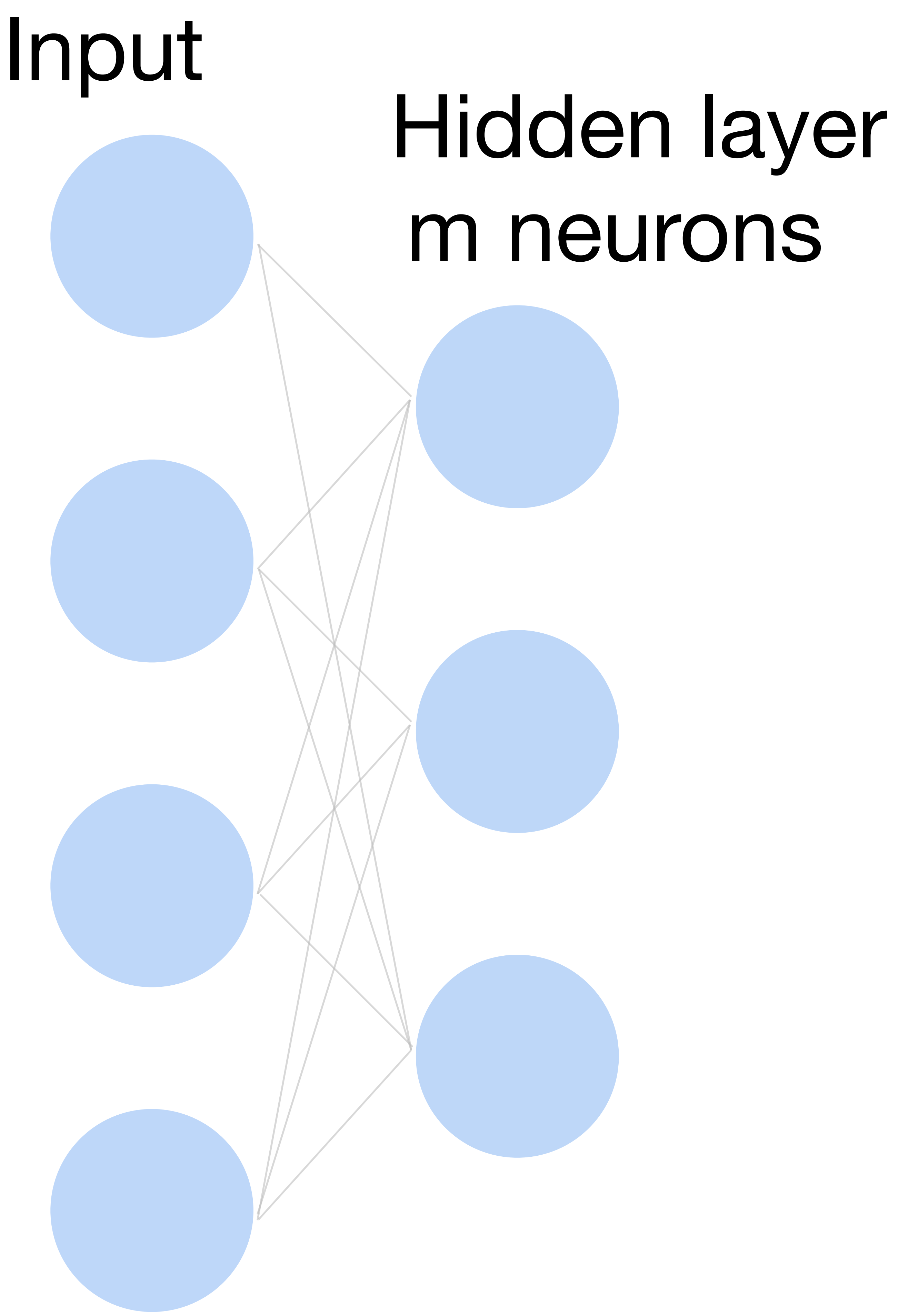
Multilayer Perceptron



Single Hidden Layer

How to classify

Cats vs. dogs?



Single Hidden Layer

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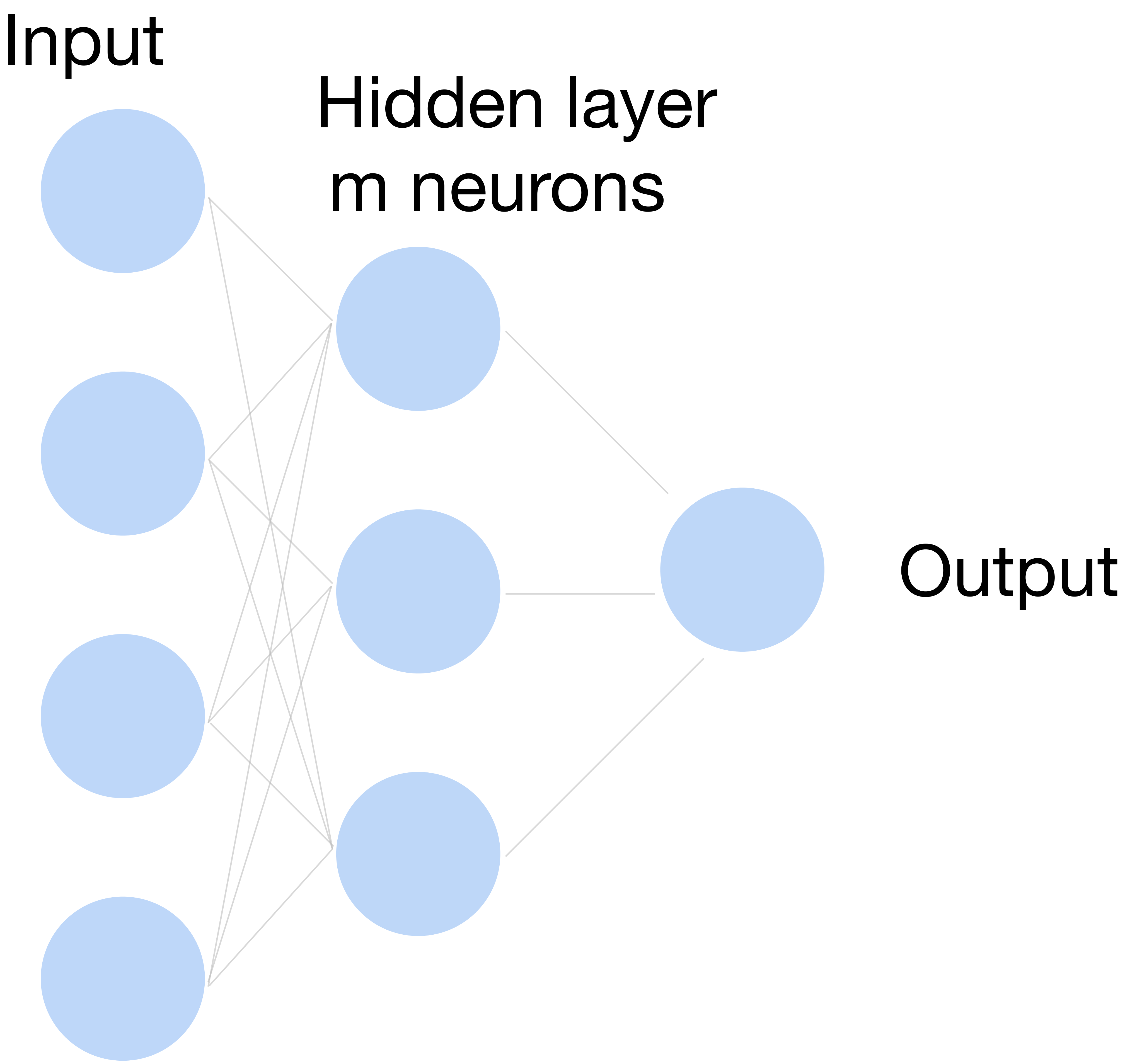
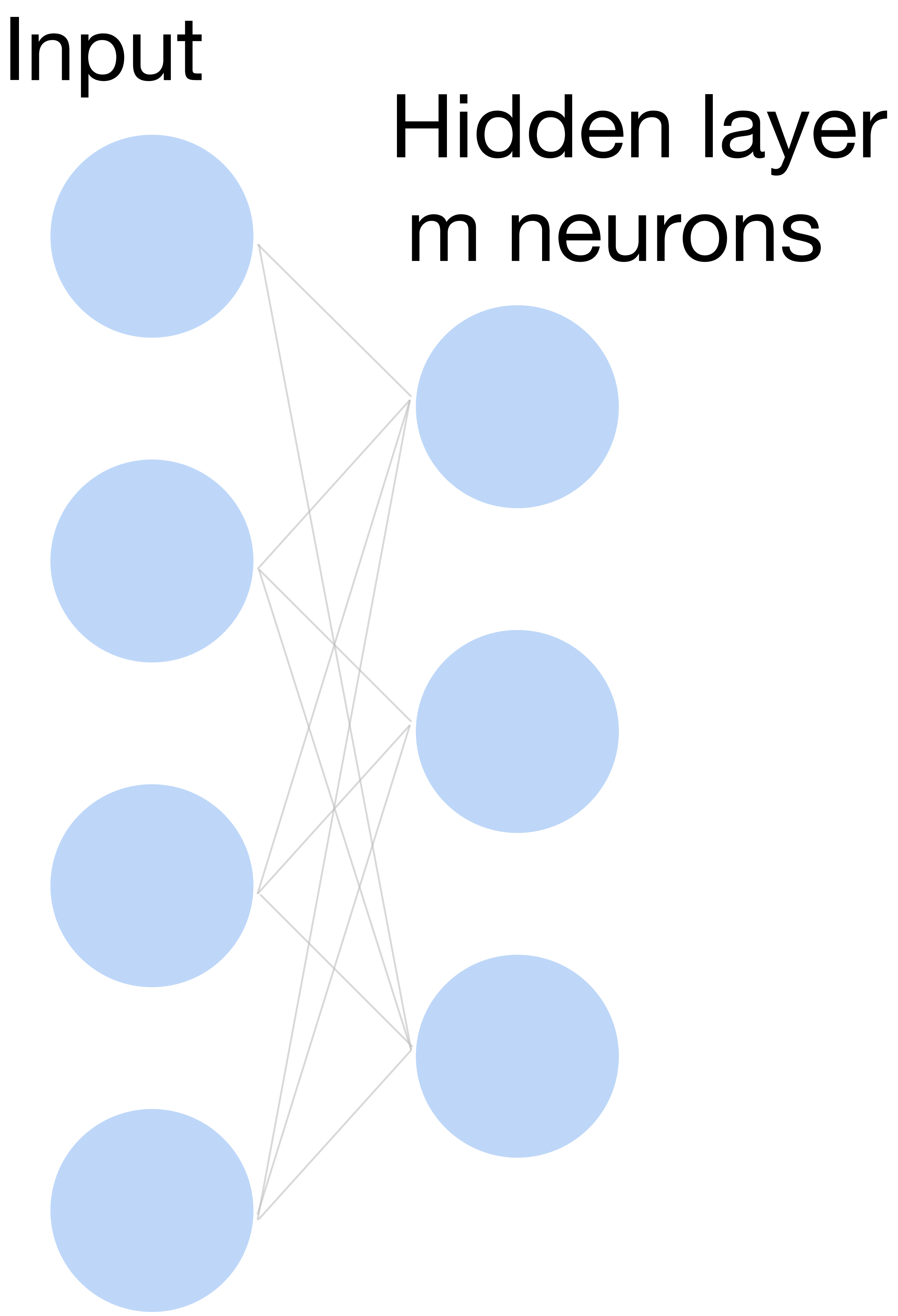


Single Hidden Layer

- Input $\mathbf{x} \in \mathbb{R}^d$
- Hidden $\mathbf{W} \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m$
- Intermediate output

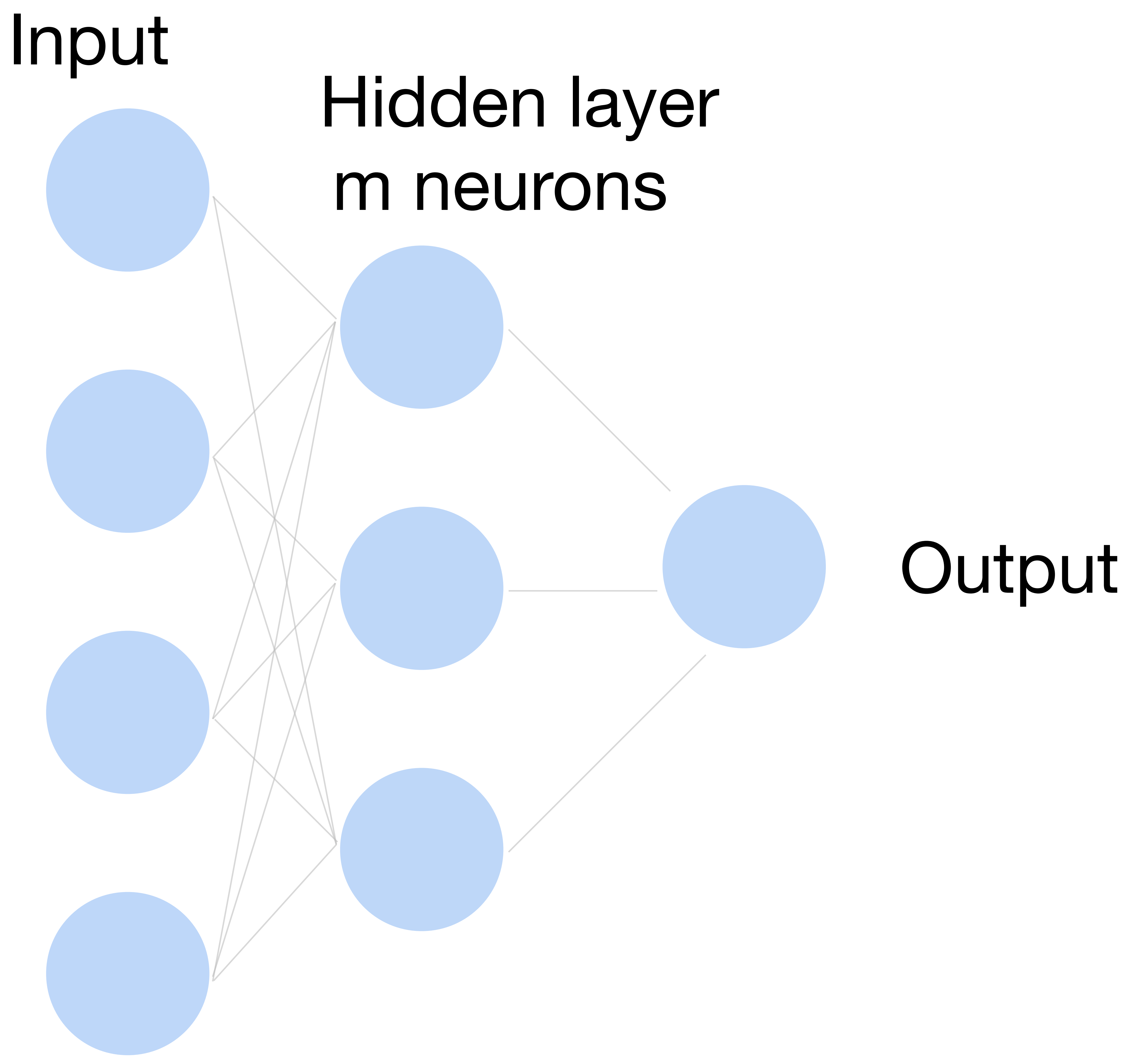
$$\mathbf{h} = \sigma(\mathbf{W}\mathbf{x} + \mathbf{b})$$

σ is an element-wise activation function



Single Hidden Layer

- Output $\mathbf{f} = \mathbf{w}_2^\top \mathbf{h} + b_2$



Quiz Break

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Which of the following functions is NOT an element-wise operation that can be used as an activation function?

A $\mathrm{f}(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

B $\mathrm{f}(x) = \begin{bmatrix} \max(0, x_1) \\ \max(0, x_2) \end{bmatrix}$

C $\mathrm{f}(x) = \begin{bmatrix} \exp(x_1) \\ \exp(x_2) \end{bmatrix}$

D $\mathrm{f}(x) = \begin{bmatrix} \exp(x_1 + x_2) \\ \exp(x_2) \end{bmatrix}$

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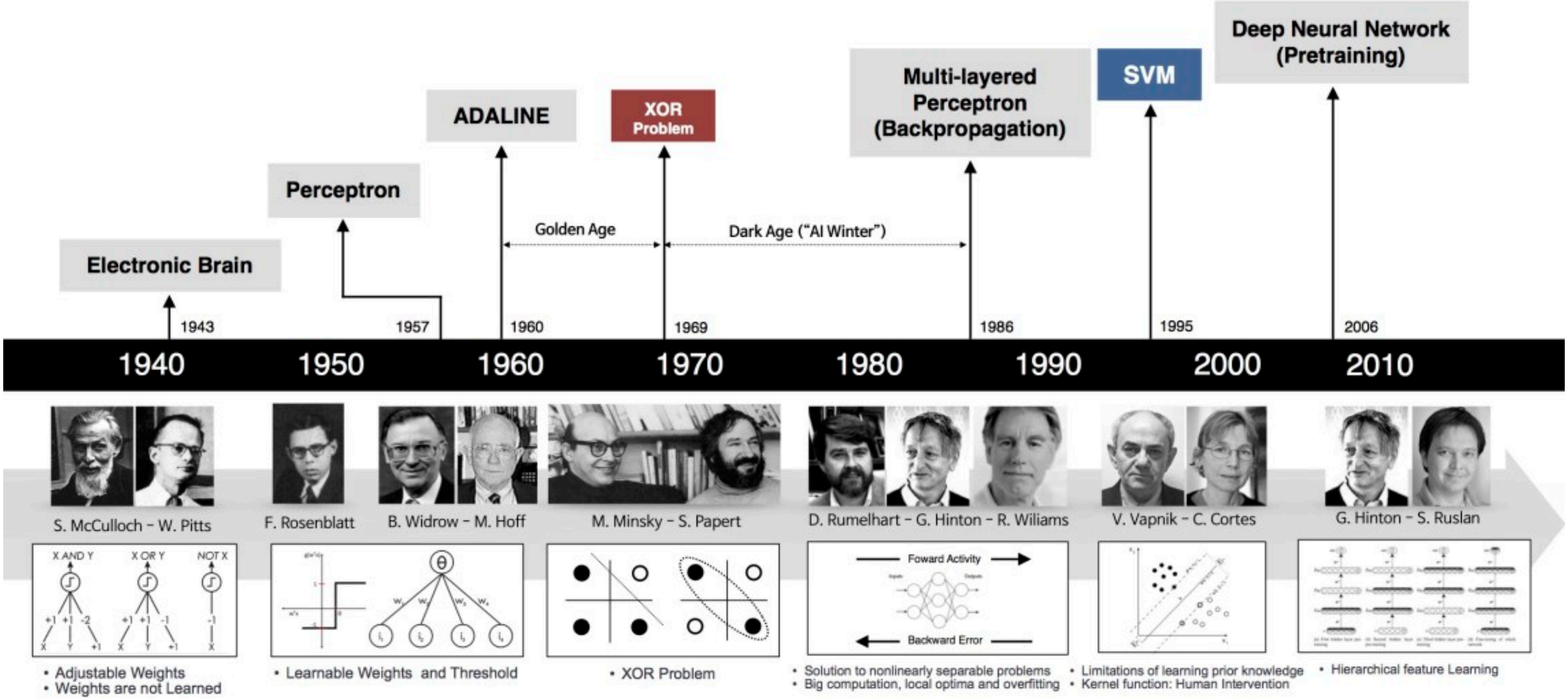
Brief history of neural networks



What we've learned today…

- Single-layer Perceptron
  - Motivation
  - Activation function
  - Representing AND, OR, NOT
- Brief history of neural networks