Outline

Homeworks:
- Homework 9 due Thursday April 27
- Homework 10 due Thursday May 4

Class roadmap:

<table>
<thead>
<tr>
<th>Tuesday, April 18</th>
<th>Games I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thursday, April 20</td>
<td>Games II</td>
</tr>
<tr>
<td>Tuesday, April 25</td>
<td>Reinforcement Learning I</td>
</tr>
<tr>
<td>Thursday, April 27</td>
<td>Reinforcement Learning I</td>
</tr>
<tr>
<td>Tuesday, May 2</td>
<td>Review of RL + Games</td>
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<tr>
<td>Thursday, May 4</td>
<td>Ethics and Trust in AI</td>
</tr>
</tbody>
</table>
Outline

• Introduction to game theory
  – Properties of games, mathematical formulation

• Simultaneous-Move Games
  – Normal form, strategies, dominance, Nash equilibrium
So Far in The Course

We looked at techniques:
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- **Unsupervised**: See data, do something with it. Unstructured.
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- **Planning and Games**: Much more structure.
More General Model

Suppose we have an agent interacting with the world
More General Model

Suppose we have an agent interacting with the world
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Suppose we have an agent interacting with the world.

- Agent receives a reward based on state of the world.
More General Model

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  - Goal: maximize reward / utility
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Suppose we have an **agent interacting** with the **world**

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  - **Goal**: maximize reward / utility ($$$)
  - Note: now **data** consists of actions, observations, and rewards
More General Model

Suppose we have an agent interacting with the world

- Agent receives a reward based on state of the world
  - **Goal**: maximize reward / utility ($$$)
  - Note: now data consists of actions, observations, and rewards
  - Setup for decision theory, reinforcement learning, planning
Games: Multiple Agents

Games setup: *multiple* agents
Games: Multiple Agents

Games setup: **multiple** agents

Player 1
Games: Multiple Agents

Games setup: multiple agents
Games: Multiple Agents

Games setup: **multiple** agents

- Player 1
- World
- Player 3
Games: Multiple Agents

Games setup: multiple agents

Player 1

World

Player 2

Player 3
Games: Multiple Agents

Games setup: *multiple* agents

Player 1 <-> World <-> Player 2
Player 3
Games: Multiple Agents

Games setup: **multiple** agents

Player 1 — World — Player 2

Player 2 — World — Player 3
Games: Multiple Agents

Games setup: multiple agents
Games: Multiple Agents

Games setup: *multiple* agents

- Now: interactions between agents

![Diagram showing multiple agents connected to the world](image-url)
Games: Multiple Agents

Games setup: **multiple** agents

- Now: interactions between agents
- Still want to maximize utility
Games: Multiple Agents

Games setup: **multiple** agents

- Now: interactions between agents
- Still want to maximize utility
- Requires **strategic** decision making.
Modeling Games: Properties

Let’s work through properties of games
Modeling Games: Properties

Let’s work through properties of games

• Number of agents/players
Modeling Games: Properties

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- **Number** of agents/players
- Action space: finite or infinite
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Property 1: **Number** of players

Pretty clear idea: 1 or more players
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- Usually interested in ≥ 2 players
Property 1: **Number** of players

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- Usually interested in $\geq 2$ players
- Typically a finite number of players
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Property 1: **Number** of players

Pretty clear idea: 1 or more players

- Usually interested in \( \geq 2 \) players
- Typically a finite number of players
Property 2: Action Space
Property 2: Action Space

Action space: set of possible actions an agent can choose from.

Can be finite or infinite.

Examples:
- Rock-paper-scissors
- Tennis
Property 3: Deterministic or Random
Property 3: **Deterministic or Random**

• Is there *chance* in the game?
  – Poker
  – Scrabble
  – Chess
Property 3: **Deterministic or Random**

- Is there *chance* in the game?
  - Poker
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Property 4: **Sum of payoffs**
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- Two basic types: zero sum vs. general sum.
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- Zero sum: one player’s win is the other’s loss
  - Pure competition.
  - Example: rock-paper-scissors
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- General sum
  - Example: driving to work, prisoner’s dilemma
Property 5: **Sequential** or **Simultaneous Moves**
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- Simultaneous: all players take action at the same time
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- **Simultaneous**: all players take action at the same time
- **Sequential**: take turns (but payoff only revealed at end of game)
Quiz break:
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Give the properties of the game shown on the right:
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Give the properties of the game shown on the right:

- Number of players?
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Normal Form Game

Mathematical description of simultaneous games.
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- $n$ players $\{1, 2, \ldots, n\}$
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- \( n \) players \( \{1,2,...,n\} \)
- Player \( i \) chooses strategy \( a_i \) from \( A_i \).
Normal Form Game

Mathematical description of simultaneous games.

- $n$ players \{1,2,...,n\}
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- Strategy profile: $a = (a_1, a_2, ..., a_n)$
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  - **Note**: reward depends on other players!
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  – **Note**: reward depends on other players!

• We consider the simple case where all reward functions are common knowledge.
Example of Normal Form Game

Ex: Prisoner’s Dilemma
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Example of Normal Form Game

**Ex:** Prisoner’s Dilemma

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- 2 players, 2 actions: yields 2x2 payoff matrix
Example of Normal Form Game

**Ex:** Prisoner’s Dilemma

- 2 players, 2 actions: yields 2x2 payoff matrix
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Strictly Dominant Strategies
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Let’s analyze such games. Some strategies are better than others!
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- Strictly dominant strategy: if \( a_i \) strictly better than \( a_i' \) regardless of what other players do, \( a_i \) is strictly dominant
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  - I.e., $u_i(a_i, a_{-i}) > u_i(b, a_{-i}), \forall b \neq a_i, \forall a_{-i}$
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All of the other entries of $a$ excluding $i$
Strictly Dominant Strategies

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  \[ u_i(a_i, a_{-i}) > u_i(b, a_{-i}), \forall b \neq a_i, \forall a_{-i} \]

- Sometimes a dominant strategy does not exist!
Strictly Dominant Strategies Example

Back to Prisoner’s Dilemma
Strictly Dominant Strategies Example

Back to Prisoner’s Dilemma

• Examine all the entries: betray strictly dominates
Strictly Dominant Strategies Example

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• Check:
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\( a^* \) is a (strictly) dominant strategy equilibrium (DSE), if all players have a strictly dominant strategy \( a_i^* \).
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Dominant Strategy: Absolute Best Responses
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Player i’s best response to strategy

\( a_{-i}: BR(a_{-i}) = \arg \max_{a} u_i(a, a_{-i}) \)
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\[ BR(\text{player 2 = silent}) = \text{betray} \]

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\[
\begin{align*}
BR(\text{player2=silent}) &= \text{betray} \\
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\end{align*}
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\[ a^*_i \] is the dominant strategy for player i, if

\[ a^*_i = BR(a_{-i}), \quad \forall \ a_{-i} \]
Dominant Strategy Equilibrium

Dominant Strategy Equilibrium does not always exist.
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Nash Equilibrium

\( a^* \) is a Nash equilibrium if no player has an incentive to unilaterally deviate
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\[
u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i
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Nash Equilibrium: Best Response to Each Other
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\[ \forall i, \forall b \in A_i : u_i(a^*_i, a^*_{-i}) \geq u_i(b, a^*_{-i}) \]
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• Equivalently, for each player $i$:

$$a^*_i \in BR(a^*_{-i}) = \arg\max_b u_i(b, a^*_{-i})$$
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- Compared to DSE (a DSE is a NE, the other way is generally not true):
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- Pure Nash equilibrium:
  - A \textbf{pure strategy} is a deterministic choice (no randomness).
  - Later: we will consider \textbf{mixed} strategies
Nash Equilibrium: Best Response to Each Other

\( a^* \) is a Nash equilibrium:

\[ \forall i, \forall b \in A_i : u_i(a^*_i, a^*_{-i}) \geq u_i(b, a^*_{-i}) \]

(no player has an incentive to unilaterally deviate)

- Pure Nash equilibrium:
  - A pure strategy is a deterministic choice (no randomness).
  - Later: we will consider mixed strategies
  - In pure Nash equilibrium, players can only play pure strategies.
Finding (pure) Nash Equilibria by hand

- As player 1: For each column, find the best response, underscore it.
Finding (pure) Nash Equilibria by hand

- As player 1: For each column, find the best response, underscore it.

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<tr>
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<th>L</th>
<th>R</th>
</tr>
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<tbody>
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<td></td>
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<tr>
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- As player 2: for each row, find the best response, upper-score it.

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Finding (pure) Nash Equilibria by hand

- Entries with both lower and upper bars are pure NEs.

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Pure Nash Equilibrium may not exist

So far, pure strategy: each player picks a deterministic strategy. But:
Pure Nash Equilibrium may not exist

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Mixed Strategies
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Can also randomize actions: “mixed”
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• Player $i$ assigns probabilities $x_i$ to each action
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$$x_i(a_i), \text{ where } \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0$$
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- Now consider expected rewards

\[ u_i(x_i, x_{-i}) = E_{a_i \sim x_i, a_{-i} \sim x_{-i}} u_i(a_i, a_{-i}) = \sum_{a_i} \sum_{a_{-i}} x_i(a_i)x_{-i}(a_{-i})u_i(a_i, a_{-i}) \]
Mixed Strategy Nash Equilibrium
Mixed Strategy Nash Equilibrium

Consider the mixed strategy \( x^* = (x_1^*, ..., x_n^*) \)
Mixed Strategy Nash Equilibrium

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- This is a **Nash equilibrium** if
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Better than doing anything else, "best response"
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Space of probability distributions over strategies.
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\]

Better than doing anything else, “best response”

Space of probability distributions over strategies.

- Intuition: nobody can **increase expected reward** by changing only their own strategy.
Mixed Strategy Nash Equilibrium

Example: \( x_1(.) = x_2(.) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)
Mixed Strategy Nash Equilibrium

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Finding Mixed NE in 2-Player Zero-Sum Game

Example: Two Finger Morra. Show 1 or 2 fingers. The “even player” wins the sum if the sum is even, and vice versa.
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Two Finger Morra. Two-player zero-sum game. No pure NE:

Finding Mixed NE in 2-Player 2-action Zero-Sum Game

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Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Suppose odd’s mixed strategy at NE is (q, 1-q), and even’s (p, 1-p)

\[
\begin{array}{c|cc}
\text{odd} & f_1 & f_2 \\
\hline
f_1 & 2, -2 & -3, 3 \\
f_2 & -3, 3 & 4, -4 \\
\end{array}
\]
Finding Mixed NE in 2-Player 2-action Zero-Sum Game

Suppose odd’s mixed strategy at NE is \((q, 1-q)\), and even’s \((p, 1-p)\).

By definition, \(p\) is best response to \(q\): 
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u_1(p, q) \geq u_1(p', q) \forall p'.
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### Table of Game Payoffs

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\]

But \(u_1(p, q) = pu_1(f_1, q) + (1 - p)u_1(f_2, q)\)

\[
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& f_1 & f_2 \\
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\]

odd

even

\(q \quad 1-q\)

\(p \quad 1-p\)
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Average is no greater than components

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We want to find \(q\) such that equality holds.

Then even has no incentive to change strategy.
Finding Mixed NE in 2-Player 2-action Zero-Sum Game

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\[ u_1(f_1, q) = u_1(f_2, q) \]
Finding Mixed NE in 2-Player 2-action Zero-Sum Game

\[ u_1(f_1, q) = u_1(f_2, q) \]

\[ 2q + (-3)(1 - q) = (-3)q + 4(1 - q) \]
Finding Mixed NE in 2-Player 2-action Zero-Sum Game

\[ u_1(f_1, q) = u_1(f_2, q) \]
\[ 2q + (-3)(1 - q) = (-3)q + 4(1 - q) \]
\[ q = \frac{7}{12} \]
Finding Mixed NE in 2-Player 2-action Zero-Sum Game

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Similarly, \( u_2(p, f_1) = u_2(p, f_2) \)

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At this NE, even gets -1/12, odd gets 1/12.

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Properties of Nash Equilibrium

Major result: (John Nash ‘51)
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- Every **finite** (players, actions) game has at least one Nash equilibrium
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  - But not necessarily **pure** (i.e., deterministic strategy)
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- Searching for Nash equilibria: computationally **hard**.
Properties of Nash Equilibrium

Major result: (John Nash ’51)

• Every finite (players, actions) game has at least one Nash equilibrium
  — But not necessarily pure (i.e., deterministic strategy)
• Could be more than one
• Searching for Nash equilibria: computationally hard.
  — Exception: two-player zero-sum games (can be found with linear programming).
Q 2.1: Which of the following is false?

(i) Rock/paper/scissors has a dominant pure strategy
(ii) There is a Nash equilibrium for rock/paper/scissors

• A. Neither
• B. (i) but not (ii)
• C. (ii) but not (i)
• D. Both
Break & Quiz

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(ii) There is a Nash equilibrium for rock/paper/scissors

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- **B. (i) but not (ii)**
- C. (ii) but not (i) (i is false: easy to check that there’s no deterministic dominant strategy)
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Break & Quiz

Q 2.2: Which of the following is true

(i) Nash equilibria require each player to know other players’ strategies
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Q 2.2: Which of the following is true

(i) Nash equilibria require each player to know other players’ strategies
(ii) Nash equilibria require rational play

- A. Neither (See below)
- B. (i) but not (ii) (Rational play required: i.e., what if prisoners desire longer jail sentences?)
- C. (ii) but not (i) (The basic assumption of Nash equilibria is knowing all of the strategies involved)
- D. Both
Summary

• Intro to game theory
  – Characterize games by various properties

• Mathematical formulation for simultaneous games
  – Normal form, dominance, Nash equilibria, mixed vs pure