CS 540 Introduction to Artificial Intelligence

Reinforcement Learning II

University of Wisconsin-Madison

Spring 2023

Announcements

Assignments:

- Homework 10 due Thursday May 4
- Complete course evaluations by Friday May 5

Class roadmap:

Thursday, April 27	Reinforcement Learning I
Tuesday, May 2	Advanced Search
Thursday, May 4	Ethics and Trust in AI

Final Exam: May 12 5:05 - 7:05 pm

Outline

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- Review of reinforcement learning setting.

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  - MDPs, value functions, Q-learning
- Bellman equations and dynamic programming
- From dynamic programming to Q-learning

Key Ideas in Reinforcement Learning

Define RL Problem

States, Actions, Transitions, Rewards, Markov property, discounting

Value Functions

Bellman Equation

Writing the value of one state in terms of successor states.

Using values to choose optimal actions.

Q-learning

Value Iteration

Exploration vs. Exploitation

Back to Our General Model

We have an **agent** **interacting** with the **world**

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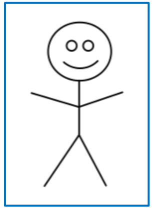

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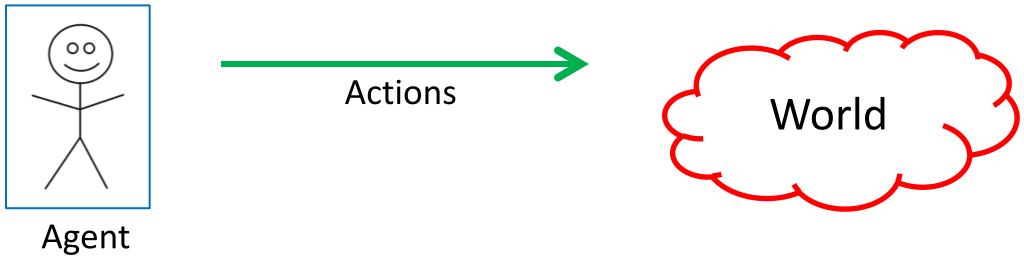

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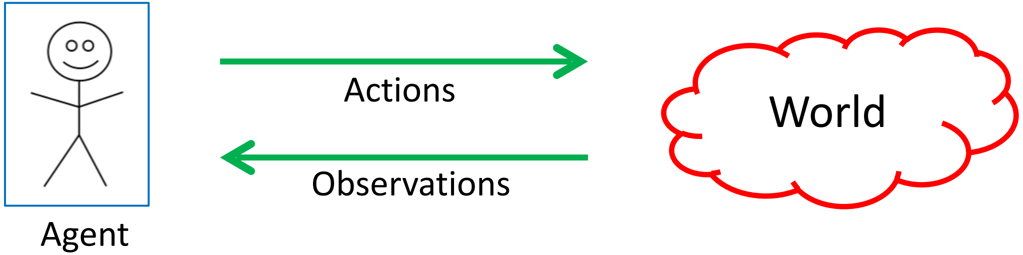

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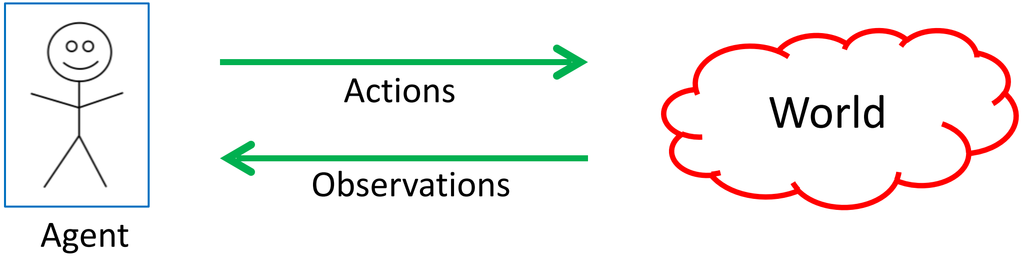

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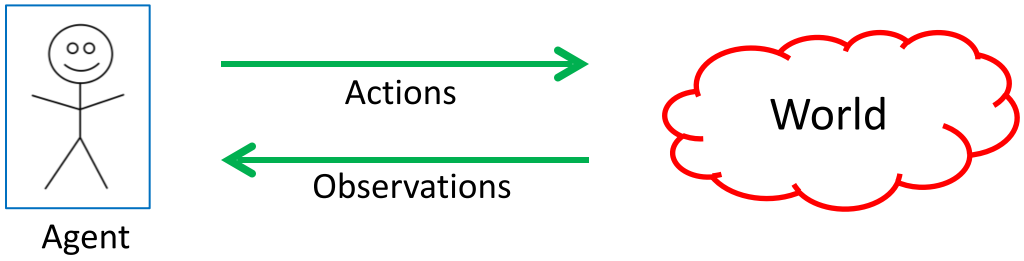

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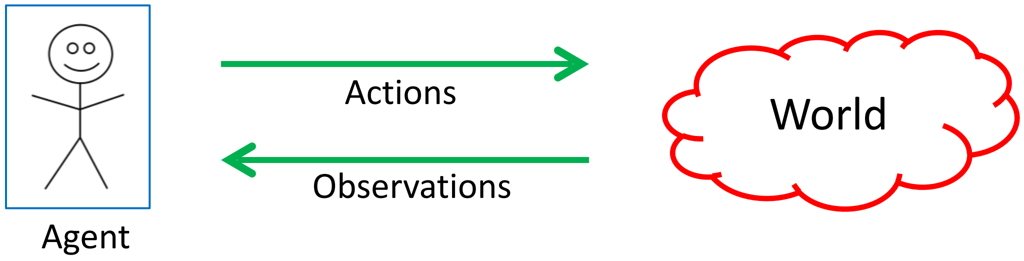

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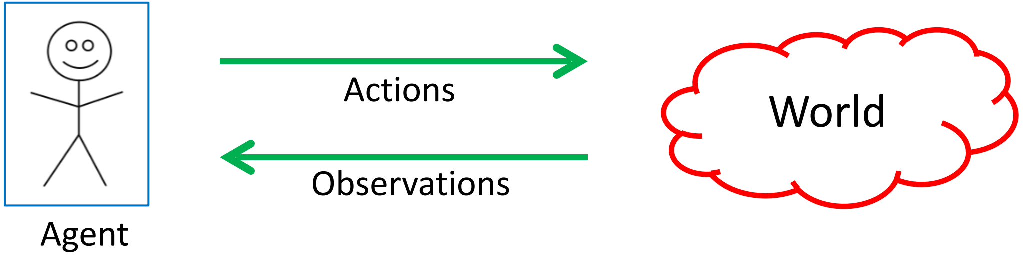

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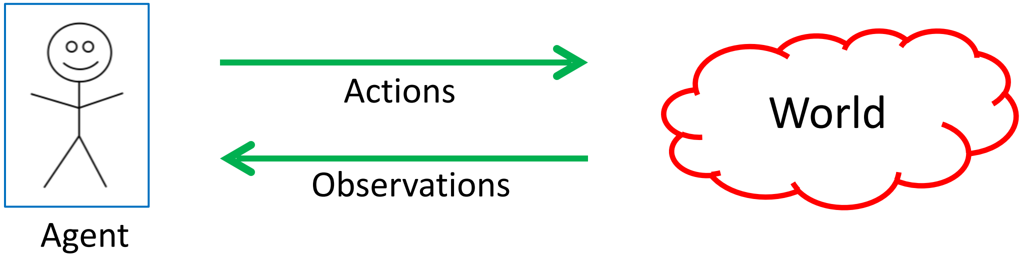

- Agent receives a reward based on state of the world
  - **Goal**: maximize reward / utility **($$$)**
  - Note: **data** consists of actions & observations
    - Compare to unsupervised learning and supervised learning

Markov Decision Process (MDP)

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$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \dots$$

Defining the Optimal Policy

For policy π, **expected utility** over all possible state sequences from s_0 produced by following that policy:

$$V^{\pi}(s_0) = \sum_{\substack{\text{sequences} \\ \text{starting from } s_0}} P(\text{sequence})U(\text{sequence})$$

Called the **value function** (for π, s_0)

Discounting Rewards

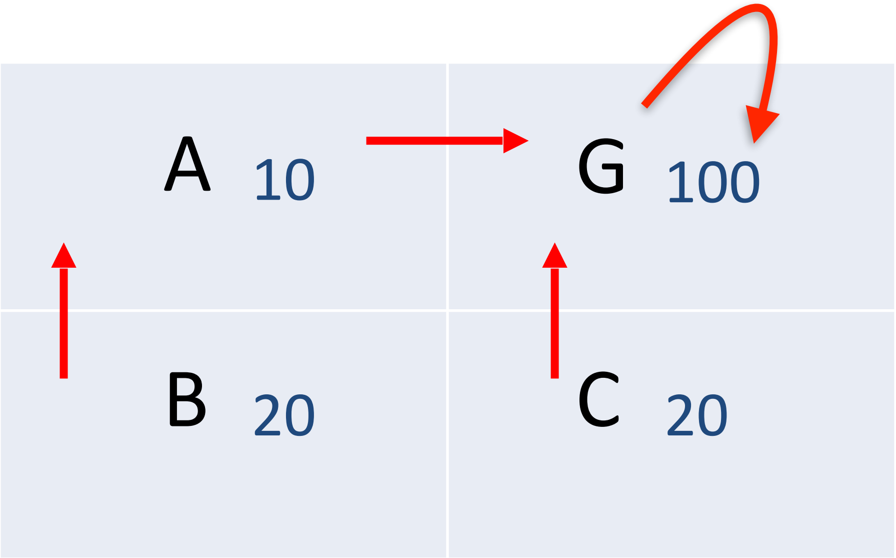
One issue: these are infinite series. **Convergence**?

- Solution

$$U(s_0, s_1 \ldots) = r(s_0) + \gamma r(s_1) + \gamma^2 r(s_2) + \ldots = \sum_{t \geq 0} \gamma^t r(s_t)$$

- Discount factor γ between 0 and 1
  - Set according to how important **present** is VS **future**
  - Note: has to be less than 1 for convergence

Example



Deterministic transitions; $\gamma = 0.8$; policy shown with red arrows.

Values and Policies

- Now that $V^{\pi}(s_0)$ is defined what a should we take?
  - First, set V*(s) to be expected utility for **optimal** policy from s
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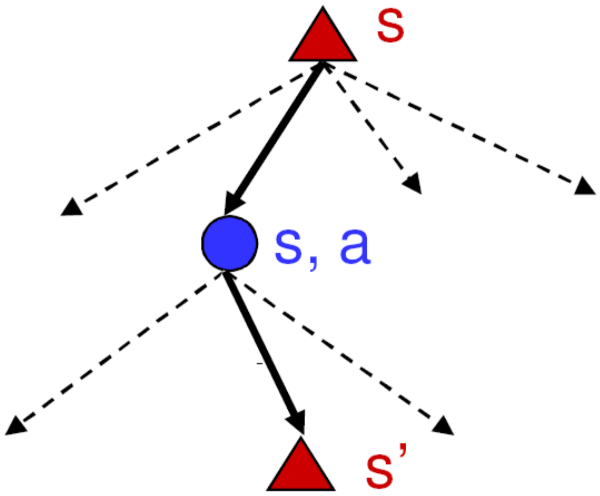

Credit L. Lazbenik

Bellman Equations

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Current state reward

Discounted expected future **rewards**

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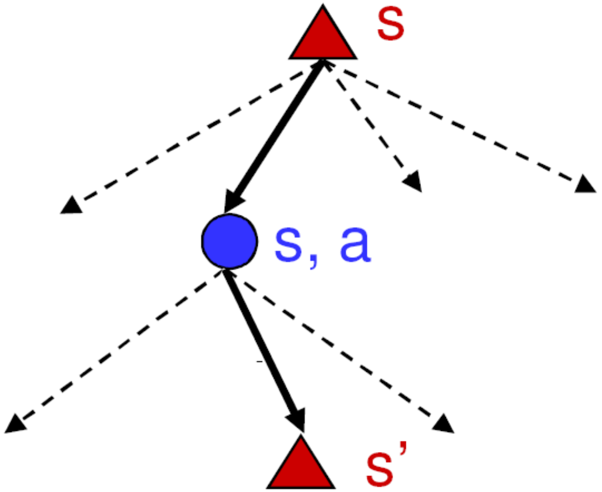

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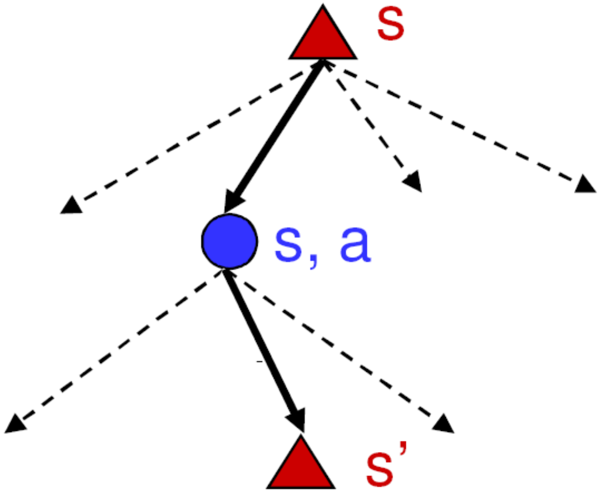

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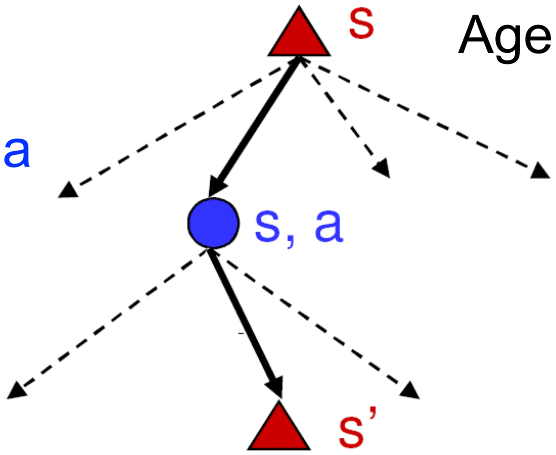
Richard Bellman: Inventor of dynamic programming.

The Bellman equation



Agent receives reward r(s)

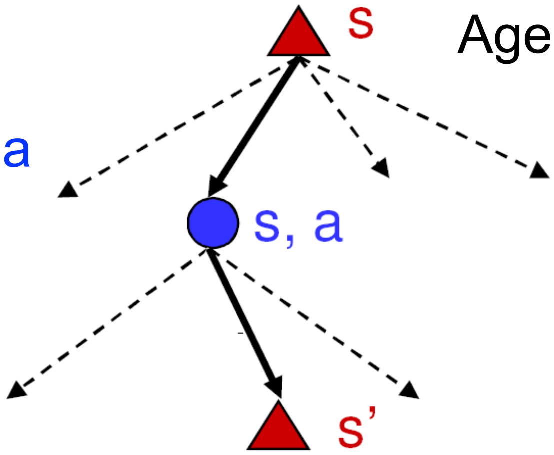
Agent chooses action a

Environment returns
$s' \sim P(\cdot \mid s, a)$

Define state value $V^*(s)$ as the expected sum of discounted rewards if the agent follows an *optimal* policy starting in state s.

Image source: L. Lazbenik

The Bellman equation



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- What is the expected utility of taking action **a** in state **s**?

$$\sum_{s'} P(s'|s,a)V^*(s')$$

Image source: L. Lazbenik

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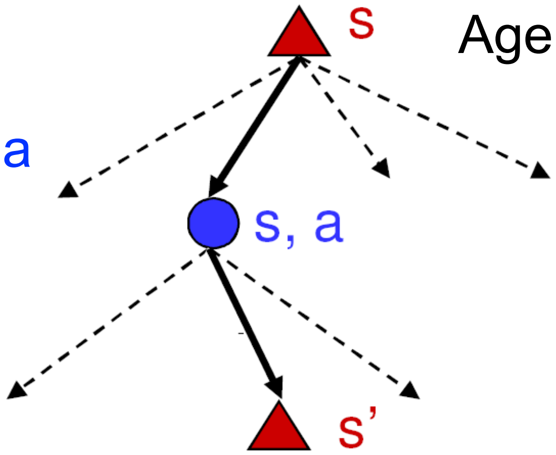

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- What is the recursive expression for $V^*(s)$ in terms of $V^*(s')$ - the utilities of its successors?

$$V^*(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi^*(s))V^*(s')$$

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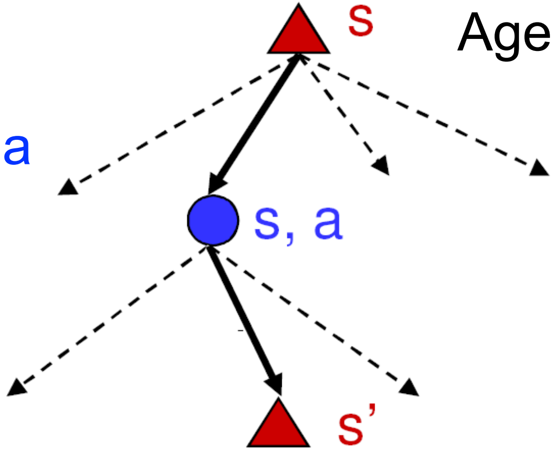

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- How do we choose the action?

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s,a) V^*(s')$$

Image source: L. Lazbenik

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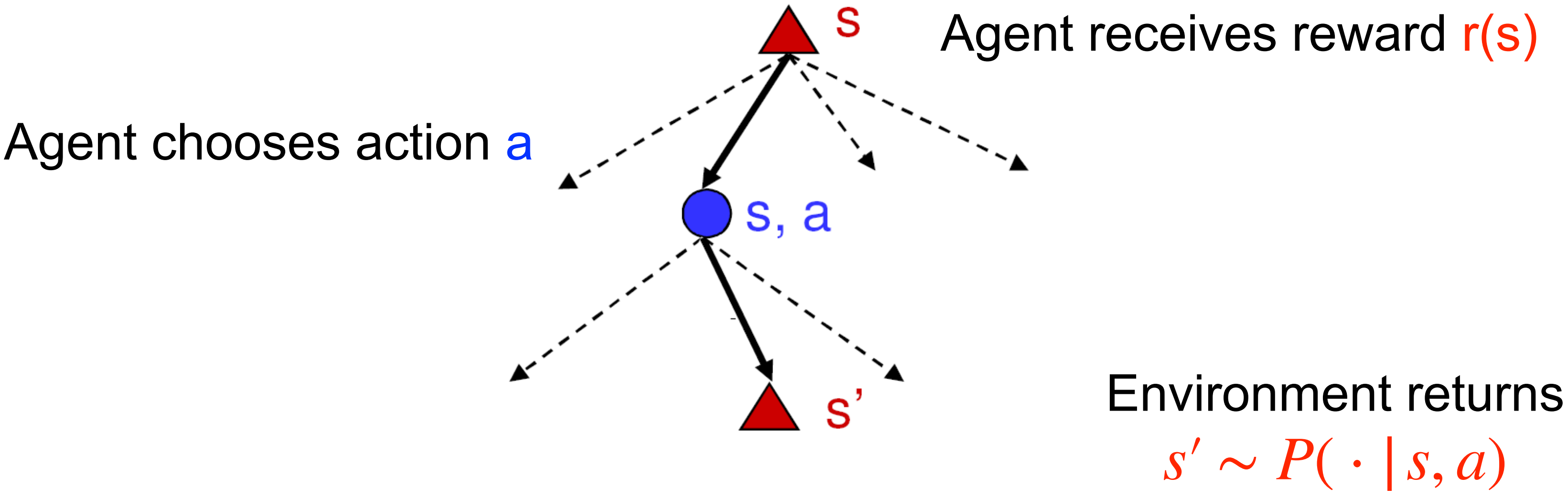

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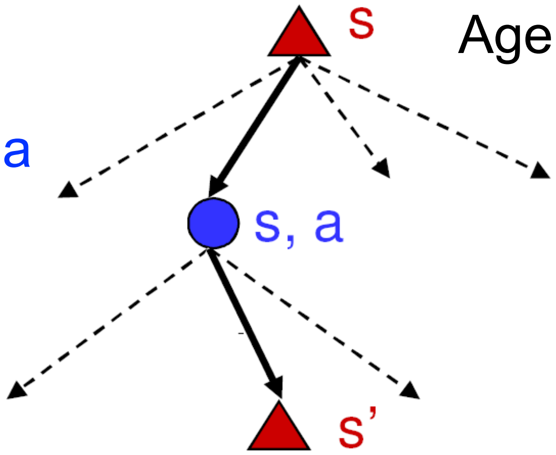

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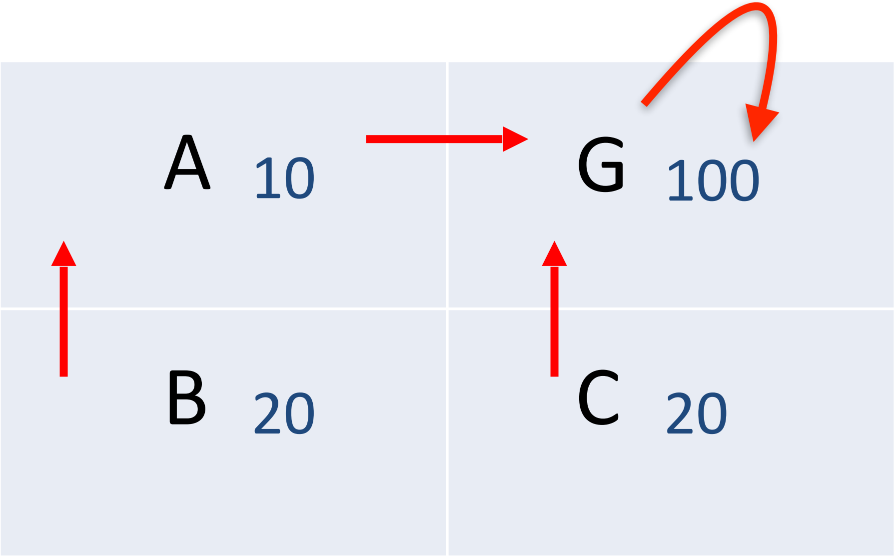
- The same reasoning gives the Bellman equation for a general policy:

$$V^{\pi}(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi(s))V^{\pi}(s')$$

Image source: L. Lazbenik

Example



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A: Use the property. Start with $V_0(s)=0$. Then, update

$$V_{i+1}(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V_i(s')$$

Value Iteration Algorithm

Input: Transition function P, reward function r, precision $\delta > 0$

1. For all states s, set V(s) = 0.
2. $\Delta \leftarrow \infty$
3. While $\Delta > \delta$:
4. Loop for each state s:
5. $V(s) \leftarrow r(s) + \max_a \gamma \sum_{s'} P(s' | s, a) V(s')$
6. $\Delta \leftarrow$ maximum change in $V(s)$ for any state s
7. End Loop
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Here, P and r are known so no need for exploration or interaction with real world.

Value Iteration Demo

https://cs.stanford.edu/people/karpathy/reinforcejs/gridworld_dp.html

Break & Quiz

Q 2.1 Consider an MDP with 2 states {*A, B*} and 2 actions: **"stay"** at current state and "**move**" to other state. Let **r** be the reward function such that **r**(*A*) = 1, **r**(*B*) = 0. Let γ be the discounting factor. Let π: π(*A*) = π(*B*) = **move** (i.e., an "always move" policy). What is the value function $V^\pi(A)$?

- A. 0
- B. 1 / (1 -γ)
- C. 1 / (1 -γ^2)
- D. 1

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- A. 0
- B. $1/(1-\gamma)$
- **C. $1/(1-\gamma^2)$** (States: A,B,A,B,… rewards 1,0, γ^2,0, γ^4,0, …)
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Q-Learning

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- Learns an action-value function Q*(s,a) that tells us the expected value of taking a in state s.

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- Learns an action-value function Q*(s,a) that tells us the expected value of taking a in state s.
  - Note: $V^*(s) = \max_a Q^*(s, a)$.
- Optimal policy is formed as $\pi^*(s) = \arg\max_a Q^*(s, a)$

Q-Learning

Estimate Q*(*s*,*a*) from data $\{(s_t, a_t, r_t, s_{t+1})\}$:

1. Initialize Q(.,.) arbitrarily (eg all zeros)
   1. Except terminal states $Q(s_{terminal},.)=0$
2. Iterate over data until Q(.,.) converges:

$$Q(s_t, a_t) \leftarrow (1-\alpha)Q(s_t, a_t) + \alpha(r_t + \gamma \max_b Q(s_{t+1}, b))$$

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Learning rate

Equivalent update: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r(s_t) + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$

Q-learning Algorithm

Input: step size α, exploration probability ϵ

1. set Q(s,a) = 0 for all s, a.
2. For each episode:
3. Get initial state s.
4. While (s not a terminal state):
5. Perform a = ϵ-greedy(Q, s), receive r, s'
6. $Q(s,a) = (1-\alpha)Q(s,a) + \alpha(r + \gamma \max_{a'} Q(s',a'))$
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Converges to Q*(s,a) in limit if all states and actions visited infinitely often.