4.2 The Poisson Distribution

4.5 The Exponential Distribution

4.6 Some Other Continuous Distributions

4.2 The Poisson Distribution

The Binomial Distribution is ________________ for Large n

Consider a binomial random variable ($X \sim \text{Bin}(n, p)$) where n is large and p is small. For example,

- (p. 132 #3) Suppose $p = .2\%$ of diodes in a certain application fail within the first month of use. Let $X = $ #diodes in a random sample of $n = 1000$ that fail within a month.
- $Y = $ #cars crossing a bridge in a five-minute period in a town of $n = 100000$ cars, each of which has probability $p = \frac{1}{10000}$ of crossing in the period
- $Z = $ #chocolate chips in a ____________ made from a randomly-selected T of 100 T of dough containing $n = 300$ chips; each chip has probability $p = $ __________ of being selected

e.g. Since $X \sim \text{Bin}(1000, .002)$, $P(X = 4) =$

The Poisson Distribution Approximates Bin(n, p) for Large n and Small p

A random variable X has the *Poisson distribution* with parameter $\lambda > 0$, if

$$p(x) = P(X = x) = \begin{cases} \left(e^{-\lambda}\right) \frac{\lambda^x}{x!}, \text{ if } x \text{ is a nonnegative integer} \\ 0, \text{ otherwise} \end{cases}$$

If n is large and p is small, and we let $\lambda =$ ______, then it can be shown that $\binom{n}{x} p^x (1-p)^{(n-x)} \approx \left(e^{-\lambda}\right) \frac{\lambda^x}{x!}$. That is, $P(X_{\text{Bin}(n,p)} = x) \approx P(X_{\text{Poisson}(\lambda = np)} = x)$.

e.g. For the diodes, $\lambda =$ ________________________, so $P(X = 4) \approx$ ________________________.

The Poisson Distribution in Nature

More importantly, for processes (like those above) in which events occur with a fixed __________________ λ and __________________ of the time since the last event, the Poisson distribution models the ________________________ that occur in a fixed time interval (or area or volume).

Experts tell us that $\mu_X =$ ______ and $\sigma^2_X =$ ______.

Examples

e.g. (p. 132 #5) The number of hits on a website is Poisson($\lambda = 4$ / minute).

a. What's the probability that 5 hits occur in a minute?

b. What's the probability that 9 hits occur in 1.5 minutes?

c. What's the probability that fewer than 3 hits occur in 30 seconds?

4.5 The Exponential Distribution

For events occurring independently at constant average rate $\lambda > 0$, the *exponential* distribution with parameter λ models the ____________________ before an event occurs. e.g. the time until

- a light bulb fails
- a car crosses a bridge
- a Geiger counter clicks due to radioactive decay of an atom

The density function of $X \sim \text{Exp}(\lambda)$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, \text{ for } x > 0 \text{ (draw)} \\ 0, \text{ for } x \leq 0 \end{cases}$$

Integrate the density $f(x)$ to find the cumulative distribution function $F(x)$. For $x > 0$, we need $F(x) = \int_{-\infty}^{x} f(t)\,dt =$

X's cumulative distribution function is therefore $F(x) = P(X \leq x) = \begin{cases} \underline{\qquad\qquad\qquad\qquad}, \text{ for } x > 0 \\ 0, \text{ for } x \leq 0 \end{cases}$

The mean and variance of X are (by integration by parts, omitted) $\mu_X =$ ______ and $\sigma_X^2 =$ ______.

e.g. (p. 151 #4) The distance D between flaws on a long cable is exponentially distributed with mean 12 m.

a. Find the probability that the distance between two flaws is greater than 15 m.

b. Find the probability that the distance between two flaws is between 8 and 20 m?

c. Find the median distance.

d. Find the standard deviation of the distances. $\sigma_D =$

e. Find the 65^{th} percentile, p, of the distances.

Lack of Memory Property

A cool feature of Exp(λ) is it's "lack of memory." For $T \sim \text{Exp}(\lambda)$, suppose we've ____________ ____________________ seconds. Then the probability of waiting ____________________ seconds is

$$\begin{aligned} P(T > t + s | T > s) &= \frac{P[(T > t + s) \cap (T > s)]}{P(T > s)} \\ &= \frac{P(T > t + s)}{P(T > s)} \\ &= \frac{1 - P(T \leq t + s)}{1 - P(T \leq s)} \\ &= \frac{1 - (1 - e^{-\lambda(t+s)})}{1 - (1 - e^{-\lambda s})} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= 1 - P(T \leq t) \\ &= P(T > t) \end{aligned}$$

So for exponential waiting times, it __________________________ that we've already waited s seconds; the probability of waiting another t seconds is the same as waiting t seconds from the beginning. e.g. These behaviors are approximately "memoryless:" the next click on Geiger counter, the next car across a bridge, and the next __________________________ in the middle of the road.

The Poisson Process

A *Poisson process* with rate parameter λ has

- ____________________ numbers of events in disjoint time intervals
- _____________ average rate λ
- $X \sim \text{Poisson}(\lambda t)$, where X is the ____________________ that occur in an interval of length t

The random waiting time, T, from _____________ starting point until the next event in a Poisson process with rate parameter λ has distribution $T \sim \text{Exp}(\lambda)$.

4.6 Some Other Continuous Distributions

The Uniform Distribution

The probability density of the continuous *uniform* distribution with parameters a and b is

$$f(x) = \begin{cases} \frac{1}{b-a}, \text{ for } a < x < b \text{ (draw)} \\ 0, \text{ otherwise} \end{cases}$$

If $X \sim U(a, b)$, then

- $\mu_X =$

- $\sigma_X^2 = \int_a^b (x - \frac{a+b}{2})^2 \frac{1}{b-a} dx = \cdots = \frac{(b-a)^2}{12}$

e.g. Computer programming languages often offer functions to generate random numbers from an interval $[a, b]$ according to $U(a, b)$.

(We'll omit §4.6's Gamma and Weibull distributions.)