Here's a hard level and power problem that is a lot like the problem in the 6.6 and 6.7 lecture notes.

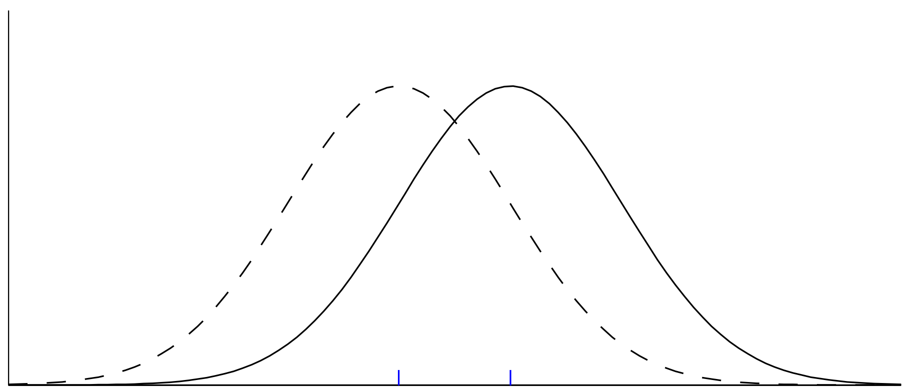
e.g. (p. 260 #7, revised) A tire company claims its tire lifetimes average 50000 miles, with standard deviation 5000. You sample 100 tires and test the hypothesis that the mean is 50000 against the alternative that it is less.

a. State the null and alternate hypotheses.

H_0: $\underline{\mu = 50000}$ vs. H_1: $\underline{\mu < 50000}$

In (b)-(d), below, let "power" refer to $P(\text{reject } H_0 | H_0 \text{ is false because } \mu = 49500)$.

b. If we reject H_0 if the sample mean is less than 49400, find the level and power.



- level $\alpha = P(\text{reject } H_0 | H_0 \text{ is true})$

  Under $H_0, \bar{X} \sim N(50000, (5000/\sqrt(100))^2) = N(50,000, 500^2)$

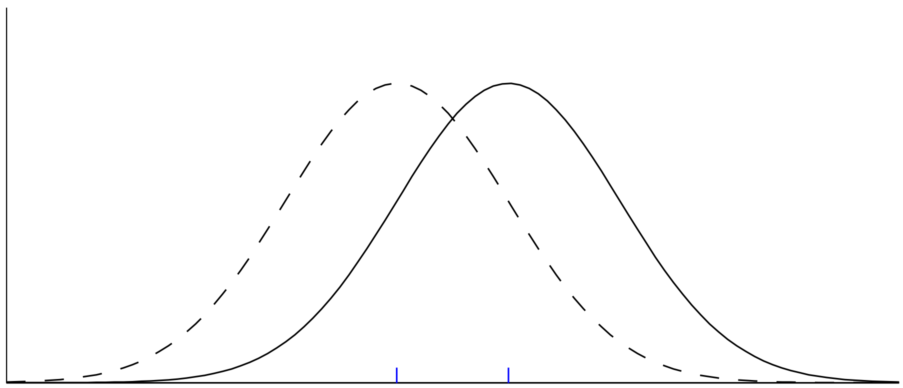
  $\implies \alpha = P(\text{reject } H_0 | H_0 \text{ is true}) = P(\bar{X} < 49400 | \mu = 50000) = P(Z < (49400 - 50000)/500) = P(Z < -1.2) = .1151$

- power $= P(\text{reject } H_0 | H_0 \text{ is false because } \mu = \mu_1)$:

  $P(\text{reject } H_0 | H_0 \text{ is false because } \mu = 49500) = P(\bar{X} < 49400 | \mu = 49500) = P(Z < (49400 - 49500)/500) = P(Z < -.2) = .4207$

Add to picture (of H_0 and H_1 distributions of $\bar{X}$): write "50000" under center of solid curve and "49500" under center of dashed curve; draw vertical line at $x = 49400$; shade $\alpha = .1151$ = area under solid curve left of line; shade power = .4207 = area under dashed curve left of line

c. If the test is made at the 5% level, what is the power?



- level $\alpha = .05 \implies z = -1.645$

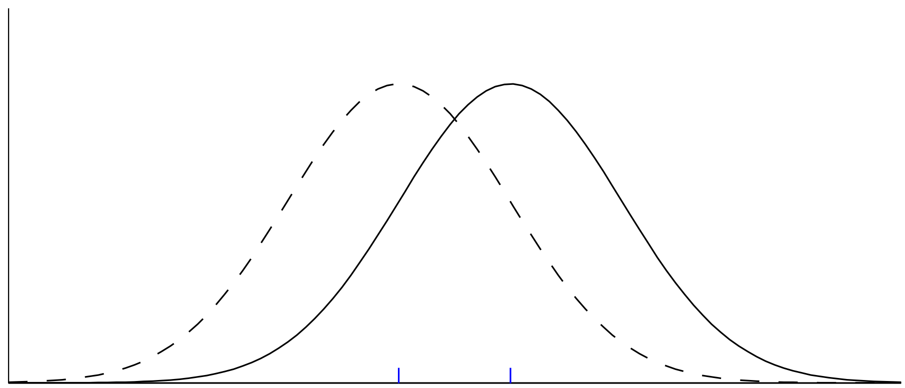
  $\implies (\bar{x} - 50000)/500) = -1.645$

  $\implies \bar{x} = 500 * (-1.645) + 50000 = 49177.5 =$ critical value

- power $= P(\text{reject } H_0 | H_0 \text{ is false because } \mu = 49500) = P(\bar{X} < 49177.5 | \mu = 49500) = P(Z < (49177.5 - 49500)/500) = P(Z < -.65) = .2578$

Add to picture: write "50000" under center of solid curve and "49500" under center of dashed curve; draw vertical line at $\bar{x} = 49177.5$; shade $\alpha = .05 =$ area under solid curve left of line; shade power $= .2578 =$ area under dashed curve left of line

d. At what level should the test be conducted so that the power is .80?

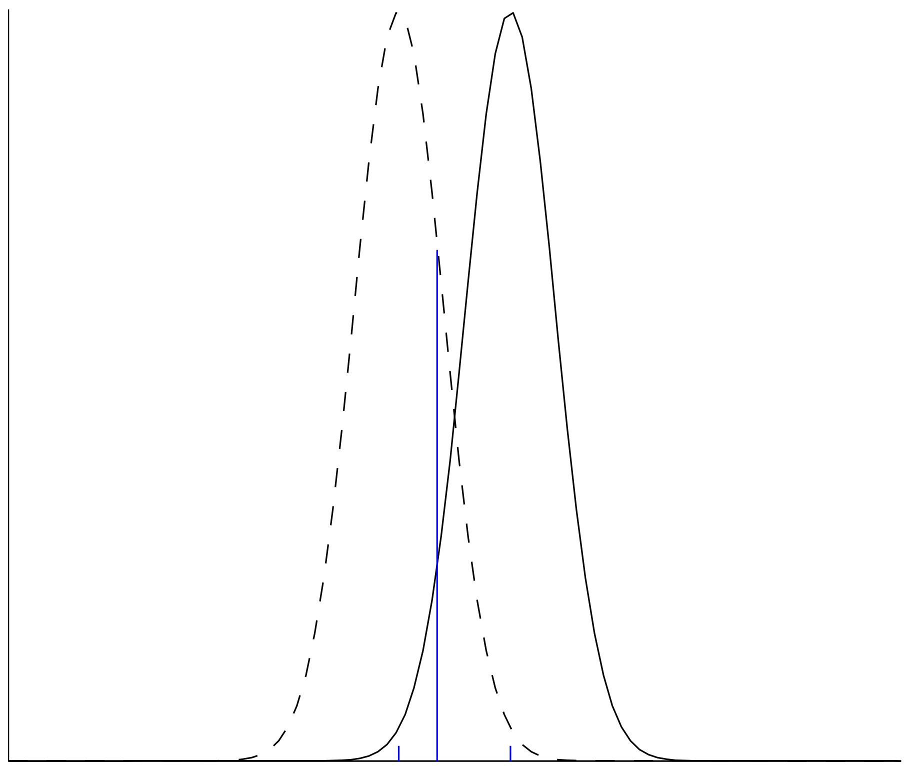


- Find critcal $\bar{x}$ so that Power $= .8 = P(\text{reject } H_0 | H_0 \text{ is false because } \mu = 49500) = P(\bar{X} < \bar{x} | \mu = 49500) = P(Z < (\bar{x} - 49500)/500) = P(Z < z_{1-.8}) = P(Z < z_{.2}) = P(Z < .845) \implies (x - 49500)/500 = .845 \implies x = 500 * .845 + 49500 = 49922.5$

- level $\alpha = P(\text{reject } H_0 | H_0 \text{ is true}) = P(\bar{X} < 49922.5 | \mu = 50000) = P(Z < (49922.5 - 50000)/500) = P(Z < -.155) \approx P(Z < -.16) = .4364$

Add to picture: write “50000” under center of solid curve and “49500” under center of dashed curve; draw vertical line at $\bar{x} = 49922.5$; shade power $= .8 =$ area under dashed curve left of line; shade $\alpha = .4364 =$ area under solid curve left of line

e. You are given the opportunity to sample more tires. How many should be sampled in total so that the power is 0.80 if the test is made at the 5% level?



Use each constraint to find a critcal $\bar{x}$ in terms of the sample size n; then solve the two equations for the two unknowns.

- $.05 = \text{level } \alpha = P(\text{reject } H_0 | H_0 \text{ is true}) = P(\bar{X} < \bar{x} | \mu = 50000) = P(Z < (\bar{x} - 50000)/(5000/\sqrt{n})) = P(Z < -z_{.05}) = P(Z < -1.645) \implies (\bar{x} - 50000)/(5000/\sqrt{n}) = -1.645 \implies \bar{x} = 50000 - 1.645(5000/\sqrt{n})$
- $\text{power} = .80 = P(\text{reject } H_0 | H_0 \text{ is false because } \mu = 49500) = P(\bar{X} < \bar{x} | \mu = 49500) = P(Z < (\bar{x} - 49500)/(5000/\sqrt{n})) = P(Z < z_{.2}) = P(Z < .845) \implies (\bar{x} - 49500)/(5000/\sqrt{n}) = .845 \implies \bar{x} = 49500 + .845(5000/\sqrt{n})$
- set level and power solutions for $\bar{x}$ equal: $50000 - 1.645(5000/\sqrt{n}) = 49500 + .845(5000/\sqrt{n}) \implies 500\sqrt{n} = (1.645 + .845) * 5000 \implies n = [(1.645 + .845) * 5000/500]^2 = 620.01$ (round up) ≈ 621 (book uses 1.64 instead of 1.645 and gets $617.5 \approx 618$)
- (for the sake of the figure) $\implies \bar{x} = 49500 + .845 * 5000/\sqrt{620.01} = 49670$ ($\implies \sigma/\sqrt{n} = 5000/\sqrt{620.01} = 200.8$, which is $s_{\bar{x}}$, standard deviation for the picture)

Add to picture: write "50000" under center of solid curve and "49500" under center of dashed curve; label vertical line as $\bar{x} = 49670$; shade power $= .8 =$ area under dashed curve left of line; shade $\alpha = .05 =$ area under solid curve left of line