7.3 Small-Sample Inferences on the Difference Between Two Means

As in §5.4 and §6.4, we don't have elementary small-sample methods for ___________ populations.

However, if $X_1, \cdots, X_{n_X}$ and $Y_1, \cdots, Y_{n_Y}$ are (possibly small) independent random samples from ____________ populations with means μ_X and μ_Y and standard deviations σ_X and σ_Y, then

$$\bar{X} - \bar{Y} \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}\right)$$

(This "$\bar{X} - \bar{Y} \sim N(\dots)$" statement is exact. We figured out an approximate version of it via the CLT for large samples in §7.1. The same reasoning, without ____________ but with the assumption of ________________________, leads to §7.3's statement.)

But we don't know σ_X or σ_Y, and, with small samples, the approximations $\sigma_X \approx s_X$ and $\sigma_Y \approx s_Y$ are ____________. We still use them to standardize $\bar{X} - \bar{Y}$, but, as in the one-small-sample case, we get a ______ statistic instead of a ______ statistic. Recall that the t_ν distributions look like ____________, but are ____________ with ___________________.

Experts say $T = \dfrac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}} \sim t_\nu$ ($\approx$), where $\boxed{\nu = \dfrac{\left(\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}\right)^2}{\frac{(s_X^2/n_X)^2}{n_X - 1} + \frac{(s_Y^2/n_Y)^2}{n_Y - 1}}}$, rounded _____.

Small-Sample Confidence Interval for the Difference of Two Means

Recall that many confidence intervals have the form

$$\text{(point estimate)} \pm \text{(margin of error)} = \hat{\theta} \pm \text{(table value for confidence)} \times \sigma_{\hat{\theta}}$$

To get a $(100\%)(1 - \alpha)$ confidence interval for $\mu_X - \mu_Y$, start with $t_{\nu,\alpha/2}$ such that

$$P(-t_{\nu,\alpha/2} < T < t_{\nu,\alpha/2}) = 1 - \alpha$$

Unstandardize T to get

$$P\left(-t_{\nu,\alpha/2} < \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}} < t_{\nu,\alpha/2}\right) = 1 - \alpha$$

Solve for $(\mu_X - \mu_Y)$ in the middle:

$$P\left((\bar{X} - \bar{Y}) - t_{\nu,\alpha/2}\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}} < (\mu_X - \mu_Y) < (\bar{X} - \bar{Y}) + t_{\nu,\alpha/2}\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}\right) = 1 - \alpha$$

That is, $\boxed{(\bar{X} - \bar{Y}) \pm t_{\nu,\alpha/2}\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}}$ contains ____________ for a proportion ______ of samples.

Small-Sample Hypothesis Test for the Difference of Two Means

Recall that many hypothesis tests use test statistics of the form

$$\frac{\text{(point estimate)} - \text{(parameter value under } H_0\text{)}}{\text{(estimated or true) standard deviation of point estimate}}$$

which tells how far the estimate is from the parameter, in standard deviations.

Let $X_1, \cdots, X_{n_X}$ and $Y_1, \cdots, Y_{n_Y}$ be independent random samples from *normal* populations with means μ_X and μ_Y and standard deviations σ_X and σ_Y. To test $H_0 : \mu_X - \mu_Y = \Delta_0$,

1. State null and alternative hypotheses, H_0 and H_1
2. Check assumptions
3. Find the test statistic $t = \dfrac{(\bar{x} - \bar{y}) - \Delta_0}{\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}}$
4. Find the degrees of freedom, $\nu = \dfrac{\left(\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}\right)^2}{\frac{(s_X^2/n_X)^2}{n_X - 1} + \frac{(s_Y^2/n_Y)^2}{n_Y - 1}}$, rounded down
5. Find the P-value, which is an area under the t_ν curve depending on H_1:

   $H_1 : \mu_X - \mu_Y > \Delta_0 \implies P\text{-value} = P(T > t)$, the area right of t

   $H_1 : \mu_X - \mu_Y < \Delta_0 \implies P\text{-value} = P(T < t)$, the area left of t

   $H_1 : \mu_X - \mu_Y \neq \Delta_0 \implies P\text{-value} = P(|T| > |t|)$, the sum of the two tail areas
6. Draw a conclusion

Examples

e.g. (Like p. 292 #17) While working an avalanche control route at Jackson Hole, Eric says his overhand spiral bomb throws penetrate the snowpack better, and blow a bigger hole, than Tyler's underhand lobs. Tyler says throwing style doesn't matter (and thinks extra penetration could even shrink the hole). To test $H_0 : \mu_{\text{spiral}} - \mu_{\text{lob}} = 0$ vs. $H_1 : \mu_{\text{spiral}} - \mu_{\text{lob}} \neq 0$, each man throws five bombs, and they measure the resulting hole diameters (in meters):

Tyler (spiral):	2.13	2.14	2.10	2.09	2.07
Eric (lob):	2.09	2.15	2.07	2.13	2.07

Can Tyler conclude that hole diameters are different for overhand spiral throws? The argument's loser must ____________________ for the entire patrol.

e.g. A study on logging in Borneo counted the number of tree species in 12 randomly chosen unlogged forest plots and in 9 similar plots logged 8 years earlier.

	#Tree species											
Unlogged	22	18	22	20	15	21	13	13	19	13	19	15
Logged	17	4	18	14	18	15	15	10	12			

Does logging reduce the mean number of species in a plot after 8 years? Use $\alpha = .10$.

e.g. Find a 90% confidence interval for the difference in mean number of species between unlogged and logged plots.

Caution

Statistics programs offer the option to find an interval or run a test assuming $\sigma_X = \sigma_Y$. This option is ____________ when the assumption is incorrect, and should usually be ____________.