**CS/Math 240: Introduction to Discrete Mathematics** 6/21/2007

# Homework 1

Instructor: Jeff Kinne TA: Mike Kowalczyk

This homework is due at the beginning of class on Thursday June 28, 2007. Mike will hold a review session at 12:45-1:45 on June 28 to discuss the solutions to these problems (that is after you have handed in the homework).

## Problem 1

Recall the "string strategy" that was given in the first lecture for navigating a maze. There exist mazes for which the string strategy spends an exponential amount of time in the maze, but it performs well on some types of mazes.

Show that for mazes that do not have any free-standing internal structures, the string strategy is equivalent to the breadcrumbs strategy (and is therefore efficient).
*Hint: what can you show about mazes that have no free-standing structures?*

## Problem 2

The basic logical operators we gave in class are: not, and, or, and implication. We remarked that $p \rightarrow q$ is equivalent to $\neg p \vee q$, so implication is not necessary to express logical statements (though it is convenient to have). Show that the NOR logical operator alone is sufficient to express each of our basic logical operators, where

$$NOR(p, q) = \neg(p \vee q).$$

*Hint: you may use truth tables to prove the equivalences just as we did for $p \rightarrow q$ and $\neg p \vee q$.*

## Problem 3

Prove that for any two real numbers $x$ and $y$, if $x$ is irrational and $y$ is rational then $x + y$ is irrational.

You may wish to use the following facts, which you should prove if you use them. For any real numbers $x$ and $y$

$$\text{if } x \text{ and } y \text{ are both rational then } x + y \text{ is rational,} \tag{1}$$

$$\text{if } x \text{ is rational then so is } -x. \tag{2}$$

*Hint: use proof by contradiction.*
*Note: you do not need to give a "line proof". You can just argue using plain English, but you may use the rules of inference.*

## Problem 4

In class we gave a proof "by picture" of the equality $|A \cup B| = |A| + |B| - |A \cap B|$, for finite sets $A$ and $B$. This is called the *inclusion-exclusion* rule.

### Part a

Give a more formal proof of the inclusion-exclusion rule for two sets, using the definitions of $\cup$ and $\cap$.
*Hint: First prove that $A \cup B$ can be partitioned into the sets $A - B$, $B - A$, and $A \cap B$: prove that for each $x \in A \cup B$, $x$ is in exactly one of the aforementioned sets.*

### Part b

The inclusion-exclusion rule can be generalized to 3 sets, 4 sets, etc. Show that for any sets $A, B, C$,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

You may either give a proof using Venn diagrams or set-builder notation, whichever you prefer.

## Problem 5

Let $f$ be a function from $A$ to $B$, for finite sets $A$ and $B$ with $|A| = |B|$. Using the definitions of 1-1 and onto, show that $f$ is 1-1 if and only if $f$ is onto.

## Problem 6

### Part a

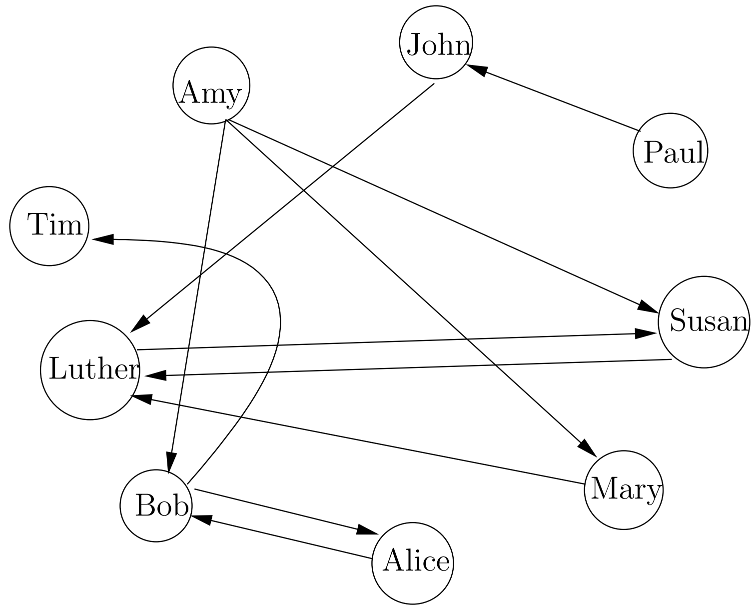
We mentioned in class that a graph $G$ is bipartite if and only if it is *2-colorable*. A graph is 2-colorable if we can color the vertices with 2 colors so that any vertices with a common edge are colored by different colors.

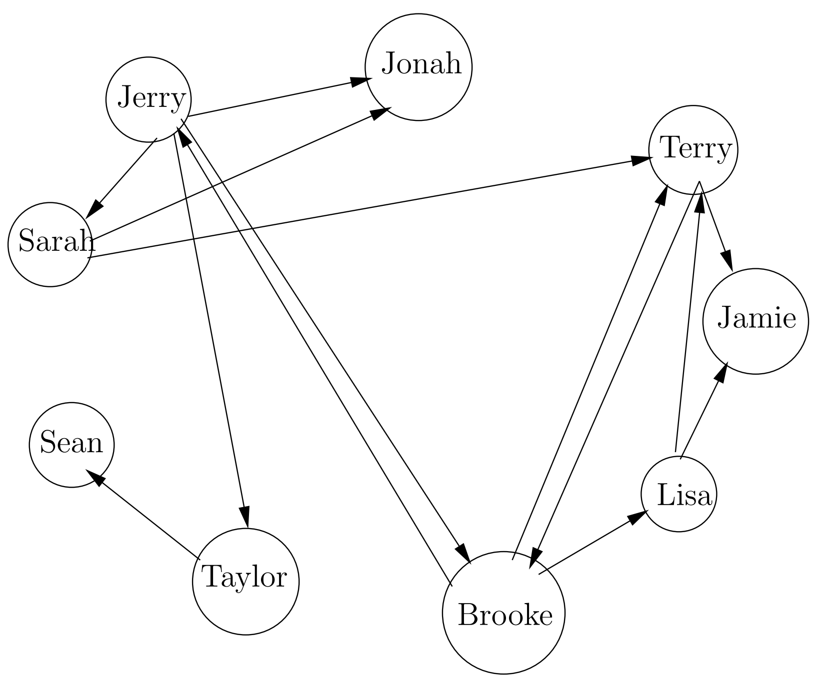
Use this result to design an algorithm to test whether a graph is bipartite or not. You may describe the algorithm in paragraph form, or give pseudo code.

### Part b

The following graphs represent "enemy lists" for a group of people. There is a directed edge from one vertex to another if the first person considers the other person an enemy. If the graph is bipartite, then we can split the group into two halves where there are no enemies within each half. We want to do this to avoid fighting.

Use your algorithm from Part a to determine if each graph is bipartite. For each, describe why/how your algorithm from Part a has decided that the graph is bipartite or not bipartite.


John
Amy
Paul
Tim
Susan
Luther
Mary
Bob
Alice


Jonah
Jerry
Terry
Sarah
Jamie
Sean
Lisa
Taylor
Brooke