CS 540 Introduction to Artificial Intelligence

Logic

University of Wisconsin-Madison

Spring 2023

Announcements

- **Homeworks:**
  - Good luck on HW2!
- Class roadmap:

Thursday, Jan 26	Probability
Tuesday, Jan. 31	Linear Algebra and PCA
Thursday, Feb. 2	PCA, Statistics and Math Review
Tuesday, Feb. 6	**Stats Review, Introduction to Logic**
Thursday, Feb. 8	Natural Language Processing

Fundamentals

Logic & AI

Why are we studying logic?

- **Traditional** approach to AI (’50s-’80s)
  - “Symbolic AI”
  - The Logic Theorist – 1956
    - Proved a bunch of theorems!
- Logic also the language of:
  - Knowledge rep., databases, etc.

Symbolic Techniques in AI

Lots of systems based on symbolic approach:

- Ex: expert systems, planning, more
- Playing great chess

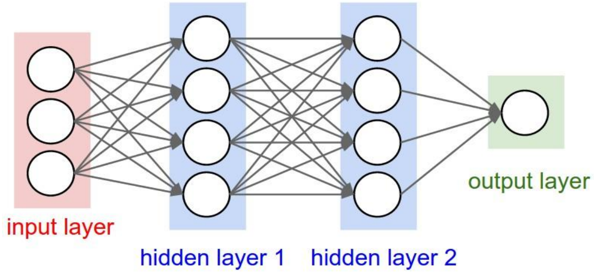
- **Less popular recently**!

J. Gardner

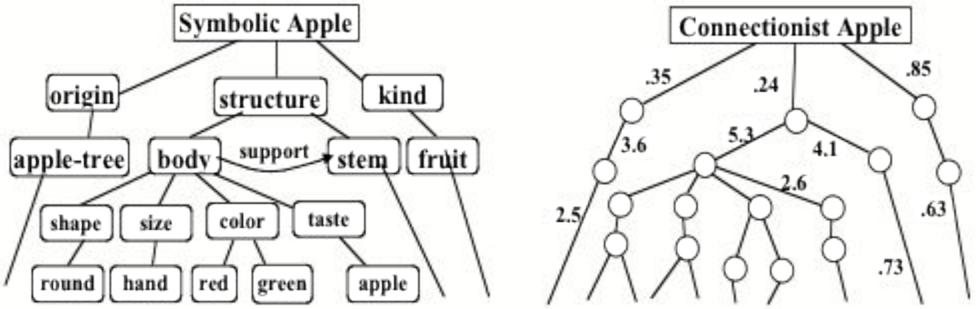
Symbolic vs Connectionist

Rival approach: **connectionist**

- Probabilistic models
- Neural networks
- **Extremely popular** last 20 years



Stanford CS231n

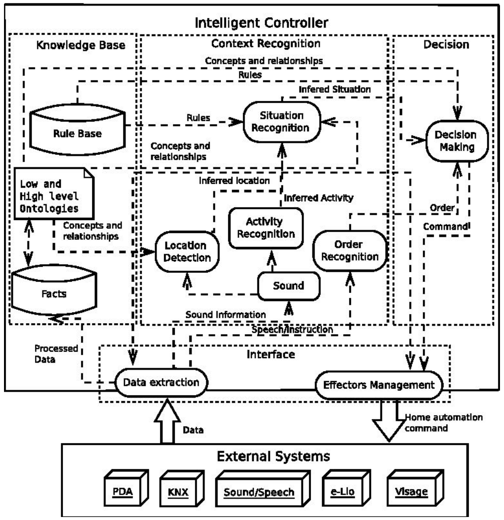


M. Minsky

Symbolic vs Connectionist

Analogy: Logic versus probability

- Which is better?
- Future: combination; best-of-both-worlds.
  - "Neurosymbolic AI"
  - **Example**: Markov Logic Networks



Outline

- Introduction to logic
  - Arguments, validity, soundness
- Propositional logic
  - Sentences, semantics, inference
- First order logic (FOL)
  - Predicates, objects, formulas, quantifiers


BEGRIFFSSCHRIFT,

EINE DER ARITHMETISCHEN NACHGEBILDETE

FORMELSPRACHE

DES REINEN DENKENS.

Basic Logic

- Arguments, premises, conclusions
  - Argument: a set of sentences (premises) + a sentence (a conclusion).
  - **Validity:** argument is valid iff it's necessary that if all premises are true, the conclusion is true.
  - **Soundness:** argument is sound iff valid & premises true.
  - **Entailment:** when argument is valid, premises entail conclusion.

Propositional Logic Basics

Logic Vocabulary:

- Sentences, symbols, connectives, parentheses
  - Symbols: P, Q, R, … (**atomic** sentences)
  - Connectives:

$\wedge$	and	[conjunction]
$\vee$	or	[disjunction]
$\Rightarrow$	implies	[implication]
$\Leftrightarrow$	is equivalent	[biconditional]
$\neg$	not	[negation]

  - Literal: P or negation ¬P

Propositional Logic Basics

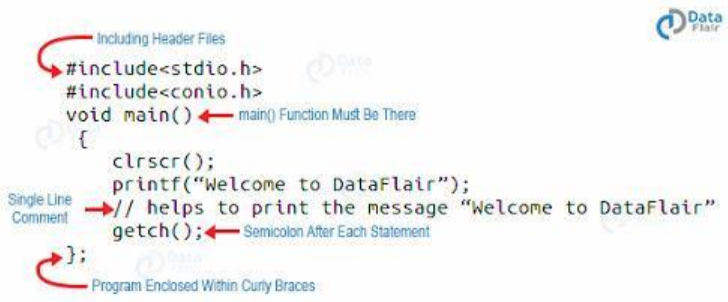
Examples:

- $(P \lor Q) \Rightarrow S$
  - "If it is cold or it is raining, then I need a jacket"
- $Q \Rightarrow P$
  - "If it is raining, then it is cold"
- $\neg R$
  - "It is not hot"

Propositional Logic Basics

Several rules in place

- Precedence: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- Use parentheses when needed
- Sentences: **well-formed** or not well-formed:
  - $P \Rightarrow Q \Rightarrow S$ **not well-formed (not associative!)**



Sentences & Semantics

- Think of symbols as defined by user
- Sentences: built up from symbols with connectives

    - **Interpretation**: assigning True / False to symbols
    - **Semantics**: interpretations for which sentence evaluates to True
    - **Model**: (of a set of sentences) interpretation for which all sentences are True

Evaluating a Sentence

- Example:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

- Note:
  – If P is false, P $\Rightarrow$ Q is true regardless of Q ("5 is even implies 6 is odd" is True!)
  – Causality unneeded: "5 is odd implies the Sun is a star" is True!)

Evaluating a Sentence: Truth Table

- **Ex**:

P	Q	R	¬P	Q∧R	¬P∨Q∧R	¬P∨Q∧R⇒Q
0	0	0	1	0	1	0
0	0	1	1	0	1	0
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	0	0	1
1	1	1	0	1	1	1

- **Satisfiable**
  - There exists some interpretation where the sentence is true.

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \wedge r$

(ii) $(\neg p \vee \neg q) \rightarrow (p \vee \neg r)$

- A. Both
- B. Neither
- C. Just (i)
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \land r$

(ii) $(\neg p \lor \neg q) \rightarrow (p \lor \neg r)$

- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \wedge r$

(ii) $(\neg p \vee \neg q) \rightarrow (p \vee \neg r)$

- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Plug interpretation into each sentence.

For (i): $(\neg p \rightarrow \neg q)$ will be false so $\neg(\neg p \rightarrow \neg q)$ will be true and r is true by assignment.

For (ii): $(\neg p \vee \neg q)$ is true and $(p \vee \neg r)$ is false which makes the implication

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- a. $A \lor (\neg A \rightarrow B)$
- b. $A \lor B$
- c. $A \lor (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A ="Aldo is Italian" and B ="Bob is English". Formalize "Aldo is Italian or if Aldo isn't Italian then Bob is English".

- **a. $A \vee (\neg A \rightarrow B)$**
- **b. $A \vee B$ (equivalent!)**
- c. $A \vee (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- **a. A $\vee$ (¬A $\longrightarrow$ B)**
- **b. A $\vee$ B (equivalent!)**
- c. A $\vee$ (A $\longrightarrow$ B)
- d. A $\longrightarrow$ B

Answer a. is the exact translation of the English sentence into a logic sentence. You can see that answer b. is also correct by writing out the truth table for all answers and seeing that a and b have the same truth tables.

Or you can use the fact that $\neg A \to B = A \vee B$ and that $A \vee A \vee B = A \vee B$ to prove equivalence.

Break & Quiz

Q 1.3: How many different assignments can there be to

$$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$$

- A. 2
- B. 2^n
- C. 2^{2n}
- D. 2n

Break & Quiz

Q 1.3: How many different assignments can there be to $(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$

- A. 2
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Break & Quiz

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- A. 2
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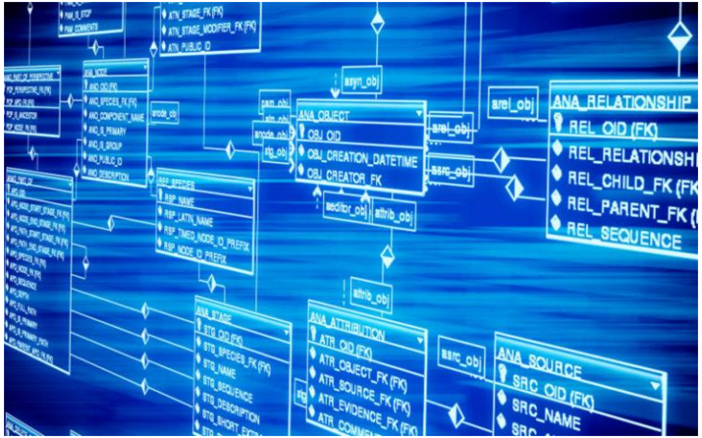
- **C. 2^{2n}**
- D. 2n

2^k assignments for k variables. We have n variables of the form x_i, y_i so k = 2n.

Knowledge Bases

- **Knowledge Base (KB)**: A set of sentences $\{A_1, A_2, \ldots A_n\}$.
  - Like a long sentence, connect with conjunction:
  - KB is $A_1 \wedge A_2 \wedge \cdots \wedge A_n$.

Model of a KB: interpretations where all sentences are True

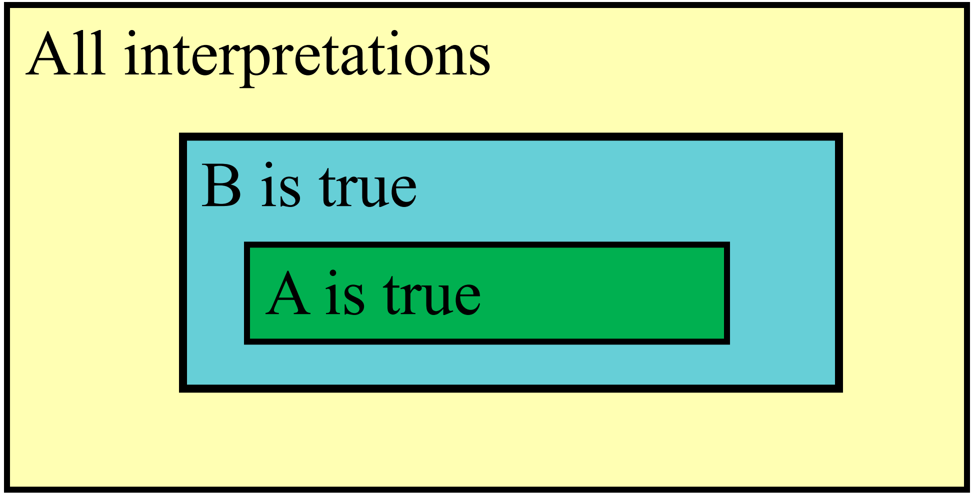


Goal: inference to discover new sentences

Entailment

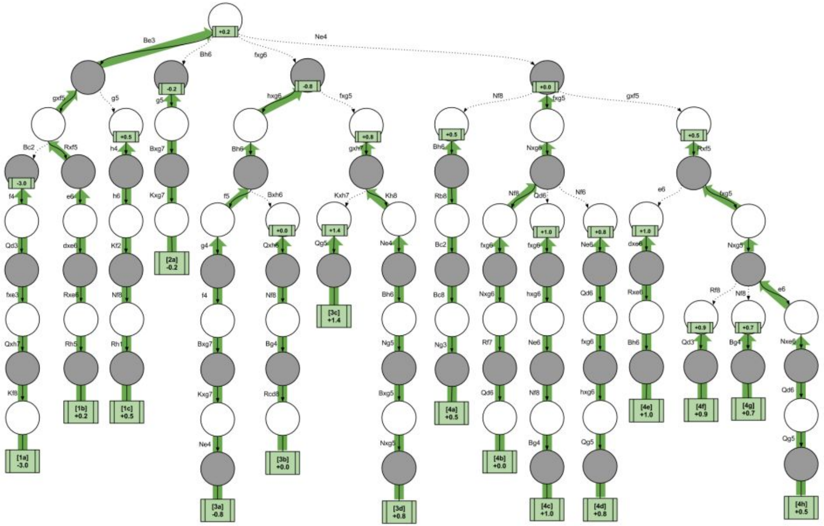
Entailment: a sentence logically follows from others

- Like from a KB. Write $A \vDash B$
- $A \vDash B$ iff in every interpretation where A is true, B is also true



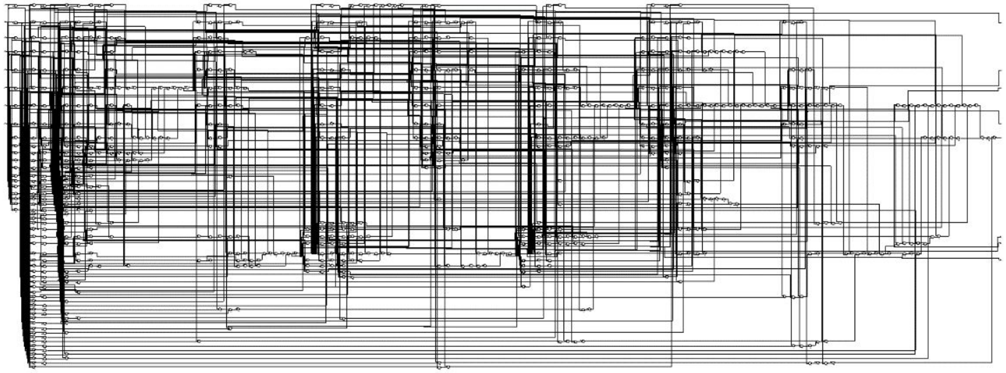
Inference

- Given a set of sentences (a KB), **logical inference** creates new sentences
  - Compare to prob. inference!
- **Challenges**:
  - Soundness
  - Completeness
  - Efficiency



Methods of Inference: **1. Enumeration**

- Enumerate all interpretations; look at the truth table
  - "Model checking"
- Downside: 2^n interpretations for n symbols



S. Leadley

Methods of Inference: **2. Using Rules**

- *Modus Ponens*: $(A \Rightarrow B, A) \models B$
- And-elimination
- Many other rules
  - Commutativity, associativity, de Morgan’s laws, distribution for conjunction/disjunction

Logical equivalences

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) && \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) && \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) && \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) && \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) && \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) && \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) && \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) && \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

You can use these equivalences to modify sentences.

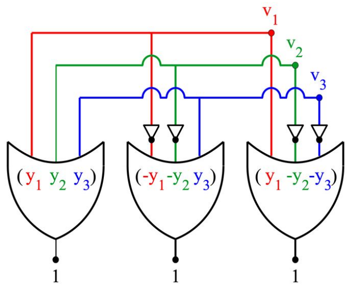
Methods of Inference: **3. Resolution**

- Convert to special form and use a single rule
- **Conjunctive Normal Form** (CNF)

$$(\neg \underbrace{A \lor B \lor C}_{\text{a clause}}) \land (\neg B \lor A) \land (\neg C \lor A)$$

Conjunction of clauses; each clause disjunction of literals

- Simple rules for converting to CNF



Conjunctive Normal Form (CNF)

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$$

- Replace all $\Leftrightarrow$ using biconditional elimination
- Replace all $\Rightarrow$ using implication elimination
- Move all negations inward using
  -double-negation elimination
  -de Morgan's rule
- Apply distributivity of $\vee$ over $\wedge$

Convert example sentence into CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$ starting sentence

$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$
biconditional elimination

$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$
implication elimination

$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$
move negations inward

$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$
distribute $\vee$ over $\wedge$

Resolution Steps

- Given KB and β (query)
- Add ¬ β to KB, show the set of satisfying interpretations is empty (implies $\neg\beta$ is False. Proof by contradiction)
- Everything needs to be in CNF
- Example KB:
  - $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
  - $\neg B_{1,1}$
- Example query: $\neg P_{1,2}$

Resolution Preprocessing

- Add $\neg \beta$ to KB, convert to CNF:

  a1: $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$

  a2: $(\neg P_{1,2} \vee B_{1,1})$

  a3: $(\neg P_{2,1} \vee B_{1,1})$

  b: $\neg B_{1,1}$

  c: $P_{1,2}$ ← Query

- Want to reach goal: *empty*

Resolution

- Take any two clauses where one contains some symbol, and the other contains its complement (negative)

$$P \lor Q \lor R \qquad\qquad \neg Q \lor S \lor T$$

- Merge (resolve) them, throw away the symbol and its complement

$$P \lor R \lor S \lor T$$

- If two clauses resolve and there's no symbol left, you have reached *empty* (False). KB $|= \beta$
- If no new clauses can be added, KB does not entail β

Resolution Example

a1: $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$

a2: $(\neg P_{1,2} \vee B_{1,1})$

a3: $(\neg P_{2,1} \vee B_{1,1})$

b: $\neg B_{1,1}$

c: $P_{1,2}$

Resolution Example

a1: $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1})$

a2: $(\neg P_{1,2} \vee B_{1,1})$

a3: $(\neg P_{2,1} \vee B_{1,1})$

b: $\neg B_{1,1}$

c: $P_{1,2}$

Step 1: resolve a2, c: $B_{1,1}$

Step 2: resolve above ($B_{1,1}$) and b: *empty*

Break & Quiz

Q 2.1: Which has more rows: a truth table on *n* symbols, or a joint distribution table on *n* binary random variables?

- A. Truth table
- B. Distribution
- C. Same size
- D. It depends

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Both must specify a value for every possible set of variables. Truth tables specify a value in {0,1} and joint distribution tables specify a probability value in the range [0,1].

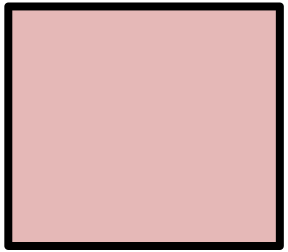
First Order Logic (FOL)

Propositional logic has some limitations

- Ex: how to say “all squares have four sides”
- No context, hard to generalize; express facts

FOL is a more expressive logic; works over

- Facts, Objects, Relations, Functions



First Order Logic Syntax

- **Term**: an object in the world
  - **Constant**: Alice, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**($term_1$, …, $term_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.
  - Constants or functions of constants.

FOL Syntax

- **Atom**: smallest T/F expression
  - **Predicate**($term_1$, …, $term_n$)
    - Teacher(Jerry, you), Bigger(sqrt(2), x)
    - Convention: read "Jerry (is)Teacher(of) you"
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - **$term_1$ = $term_2$**
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object.

FOL Syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\land$ $\lor$ $\neg$ $\Rightarrow$ $\Leftrightarrow$
    - Less(x,22) $\land$ Less(y,33)
  - Complex sentence using quantifiers $\forall$, $\exists$
- Sentences are evaluated under an interpretation
  – Which objects are referred to by constant symbols
  – Which objects are referred to by function symbols
  – What subsets defines the predicates

FOL Quantifiers

- Universal quantifier: $\forall$
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is $\Rightarrow$
  - Forms if-then rules
  - "all humans are mammals"

    $$\forall x \ \mathtt{human}(x) \Rightarrow \mathtt{mammal}(x)$$

  - Means if x is a human, then x is a mammal

FOL Quantifiers

- Existential quantifier: $\exists$
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is $\wedge$
  - "some humans are male"

    $$\exists x \ \mathbf{human(x)} \ \wedge \ \mathbf{male(x)}$$

  - Means there is an x who is a human and is a male

Break & Quiz

Q 2.1: How many entries does a truth table have for a FOL sentence with k variables where each variable can take on n values?

- A. Truth tables are not applicable to FOL.
- B. 2^k
- C. n^k
- D. It depends

Break & Quiz

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- C. $\boldsymbol{n^k}$
- D. It depends

Must have one entry for every possible assignment of values to variables. That number is (C).