CS 540 Introduction to Artificial Intelligence

Statistics Review

University of Wisconsin-Madison

Spring 2023

Review: Bayesian Inference

- Conditional Prob. & Bayes:

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

- *H*: some class we'd like to infer from evidence
  - Need to plug in prior, likelihood, etc.
  - Usually do not know these probabilities. How to estimate?

Samples and Estimation

- Usually, we don't know the distribution (P)
  - Instead, we see a bunch of samples

- Typical statistics problem: **estimate parameters** from samples
  - Estimate probability *P(H)*
  - Estimate the mean $E[X]$
  - Estimate parameters $P_\theta(X)$

Samples and Estimation

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  - Estimate probability *P(H)*
  - Estimate the mean $E[X]$
  - Estimate parameters $P_\theta(X)$
- Example: Bernoulli with parameter *p (i.e., a weighted coin flip)*
  - Mean $E[X]$ is p

Examples: Sample Mean

- Bernoulli with parameter *p*
- See samples $x_1, x_2, \ldots, x_n$
  - Estimate mean with **sample mean**

$$E[X] \approx \frac{1}{n}\sum_{i=1}^{n} x_i$$

  - No different from counting heads

Break & Quiz

Q 2.1: You see samples of X given by [0,1,1,2,2,0,1,2]. Empirically estimate $E[X^2]$

A. 9/8
B. 15/8
C. 1.5
D. There aren't enough samples to estimate $E[X^2]$

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$$E[Y] \approx \frac{1}{n}\sum_i Y_i$$

$$Y = X^2$$

$$E[X^2] \approx \frac{1}{8}(0^2 + 1 + 1 + 4 + 4 + 0 + 1 + 4) = 15/8$$

A. 9/8

B. 15/8

C. 1.5

D. There aren't enough samples to estimate $E[X^2]$

Break & Quiz

Q 2.2: You are empirically estimating *P(X)* for some random variable *X* that takes on 100 values. You see 50 samples. How many of your *P(X=a)* estimates might be 0?

A. None.
B. Between 5 and 50, exclusive.
C. Between 50 and 100, inclusive.
D. Between 50 and 99, inclusive.

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If you don't see a number at all in the 50 samples then the estimated probability of that number is 0.

You can see up to 50 different values in 50 samples. On the other hand, all 50 samples might have the same value in which case 99 values were never seen.

Estimating Multinomial Parameters

- *k*-sized die (special case: *k*=2 coin)
- Face *i* has probability p_i, for *i=1..k*
- In *n* rolls, we observe face *i* showing up n_i times

$$\sum_{i=1}^{k} n_i = n$$

- Estimate ($p_{1,\ldots,}\ p_k$) from this data ($n_{1,\ldots,}\ n_k$)

Maximum Likelihood Estimate (MLE)

- The MLE of multinomial parameters $(\widehat{p_1}, \ldots, \widehat{p_k})$

$$\widehat{p_i} = \frac{n_i}{n}$$

- "frequency estimate"

Regularized Estimate

- Hyperparameter $\epsilon > 0$

$$\hat{p}_i = \frac{n_i + \epsilon}{n + k\epsilon}$$

- Avoids zero when *n* is small
- Biased, but has smaller variance
- Equivalent to a specific Maximum A Posterori (MAP) estimate, or smoothing

Estimating 1D Gaussian Parameters

- Gaussian distribution $N(\mu, \sigma^2)$
- Observe *n* data points from this distribution

$$x_1, \dots, x_n$$

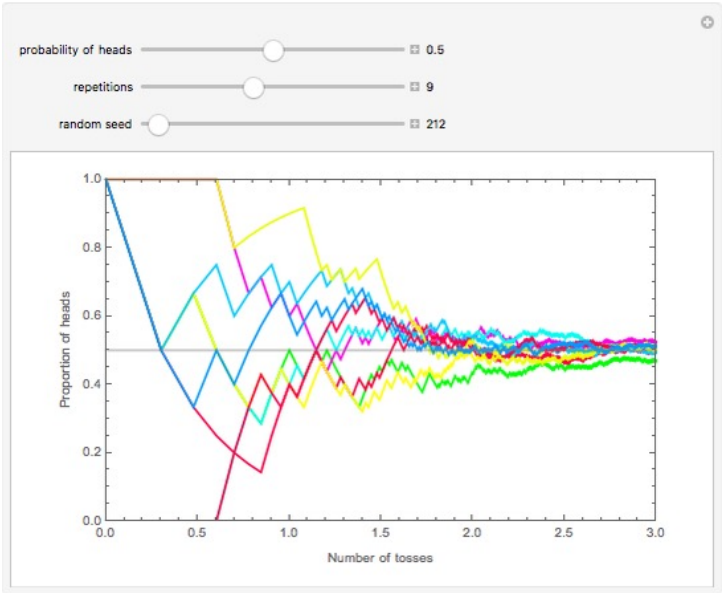
- Estimate μ, σ^2 from this data

Estimating 1D Gaussian Parameters

- Mean estimate $\hat{\mu} = \frac{x_1 + \cdots + x_n}{n}$
- Variance estimates
  - Unbiased $s^2 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2}{n-1}$
  - MLE $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2}{n}$

Estimation Theory

- How do we know that the sample mean is a good estimate of the true mean?
  - Law of large numbers
  - Central limit theorems
  - Concentration inequalities



Wolfram Demo

Estimation Error

- With finite samples, likely error in the estimate.
- Mean squared error
  - $MSE[\hat{\theta}] = E[(\hat{\theta} - \theta)^2]$
- Bias / Variance Decomposition
  - $MSE[\hat{\theta}] = E\left[(\hat{\theta} - E[\hat{\theta}])^2\right] + (E[\hat{\theta}] - \theta)^2$

Variance Bias

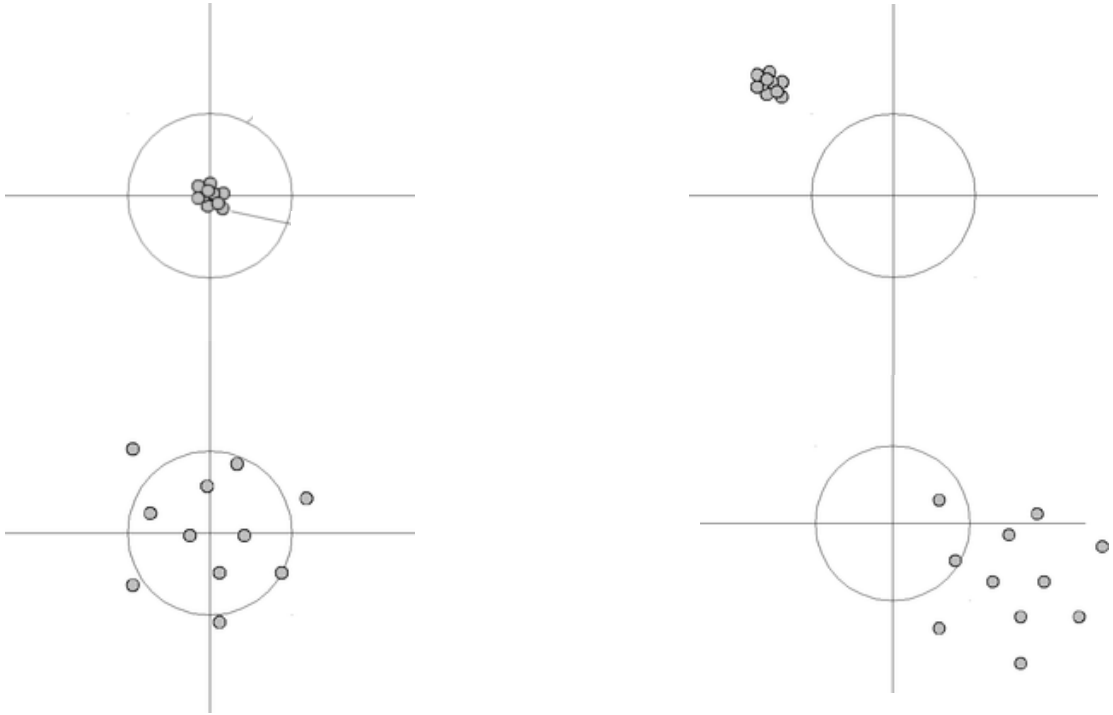
Bias / Variance

Low Bias

High Bias

Low Variance

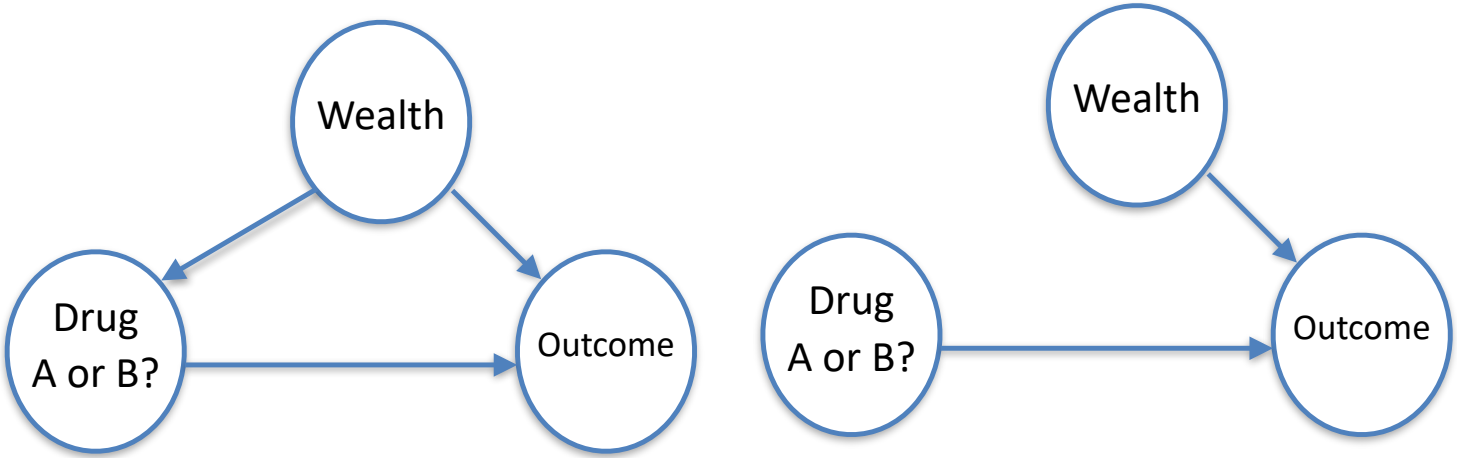
High Variance



Wikipedia: Bias-variance tradeoff

Association vs Causation

- Conditional distributions give associational relationships
  - P(Y|X) is not necessarily the causal effect of X on Y

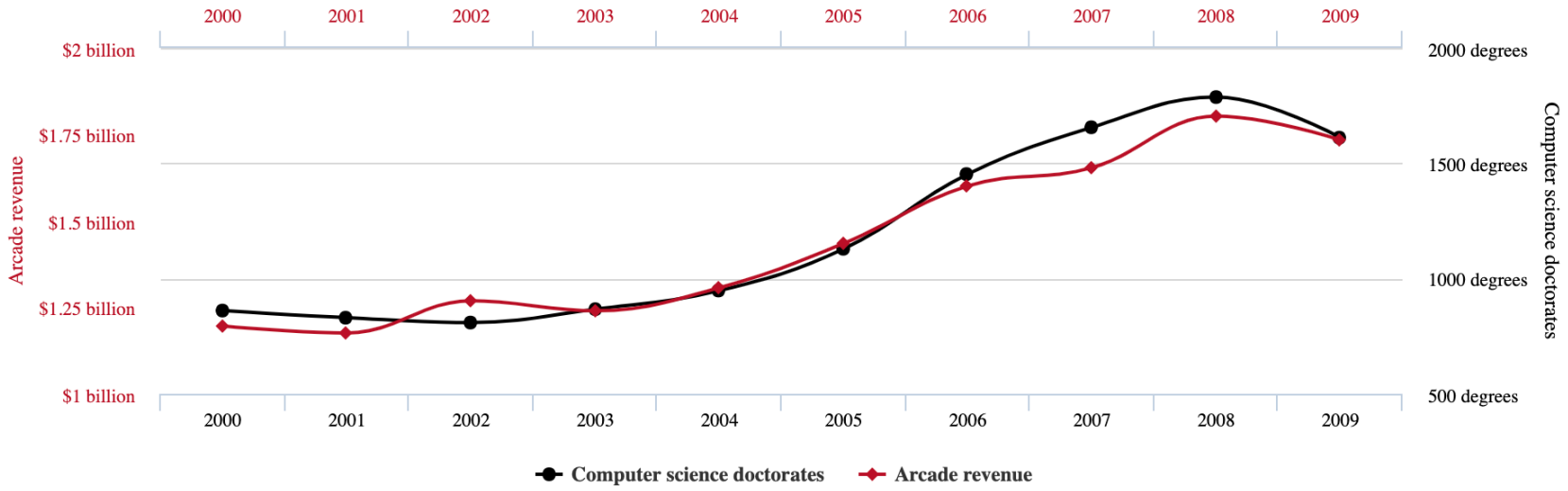


Total revenue generated by arcades

correlates with

Computer science doctorates awarded in the US

Correlation: 98.51% (r=0.985065)



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Data sources: U.S. Census Bureau and National Science Foundation