Advanced Topics in Reinforcement Learning

Lecture 12: Function Approximation for On-policy Prediction

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Announcements

- Homework released. Due: October 21 at 9:30AM (minute class starts)
- Read 9.7 and 16.5 for next week. Deep RL!
- Upcoming dates:
  - Literature survey due: October 30
  - Exam: November 5

Learning Outcomes

After this week, you will be able to:

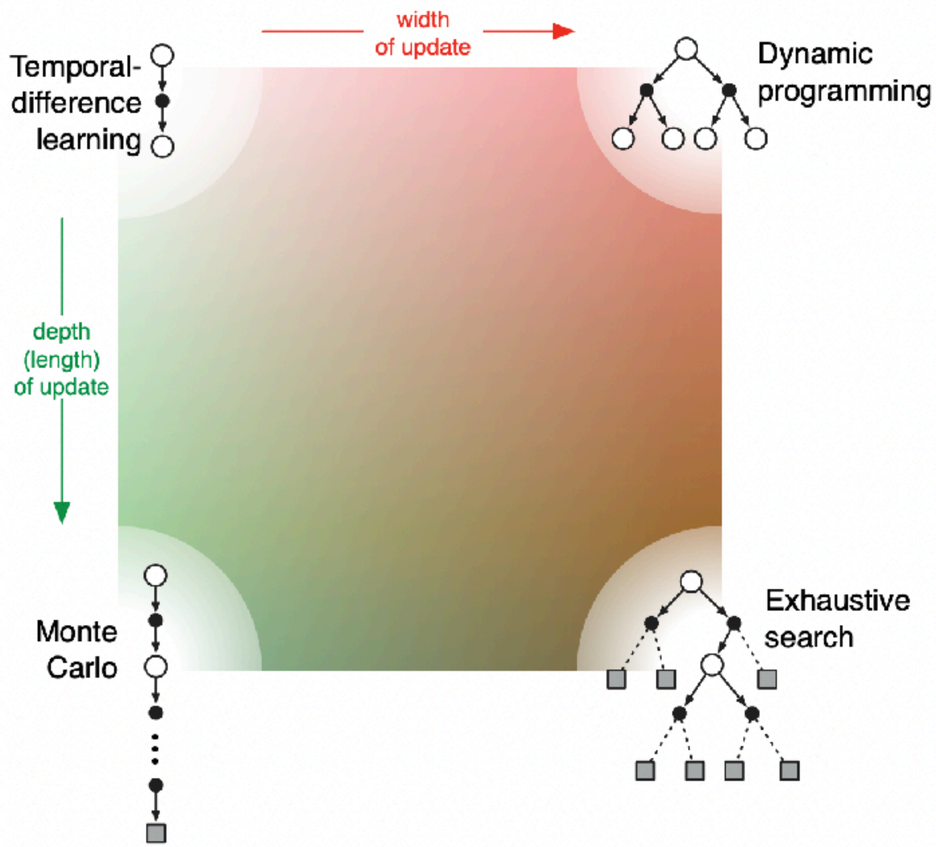
1. Generalize model-free RL algorithms from the tabular to the function approximation setting.

2. Identify challenges and opportunities with using function approximation in RL.

3. Compare and contrast convergence of different algorithms under either function approximation or off-policy learning.

Part I Summary

- Functions (policies, value functions, and models) have been represented as look-up tables.
- We have seen 4 types of algorithms:
  - Dynamic programming methods.
  - Model-free Monte Carlo methods.
  - Model-free temporal difference learning methods.
  - Model-based learning and planning methods.
- All algorithms we have seen are instances of generalized policy iteration:
  - $\pi_0 \to q_0 \to \cdots \to \pi_k \to q_k \to \pi_{k+1} \to \cdots \to q_\star \to \pi_\star$



Part I Summary

- Much intuition and understanding carries forward as we move into Part II.
  - Returns and values defined similarly.
  - On-policy and off-policy methods.
  - Exploration vs. Exploitation trade-off.
- Looking ahead:
  - The learning agent has limited capacity to model $v_\pi(s)$ for all s.
  - The learning agent may never visit the same state twice.

This Week

- Today: function approximation for on-policy prediction.
- Thursday: function approximation for off-policy prediction.

Function Approximation in RL

- How different from the tabular case?
  - Generalize value estimates across similar states.
- What is the benefit?
  - May only visit any given state once.
  - Too many states to store an individual value estimate for each.
- What do we lose?
  - Accurate approximation everywhere.
  - The policy improvement theorem.

See: The Big World Hypothesis and its Ramifications for Artificial Intelligence. Javed and Sutton 2024.

Function Approximation in RL

- Form of the value estimate:

$$\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$$

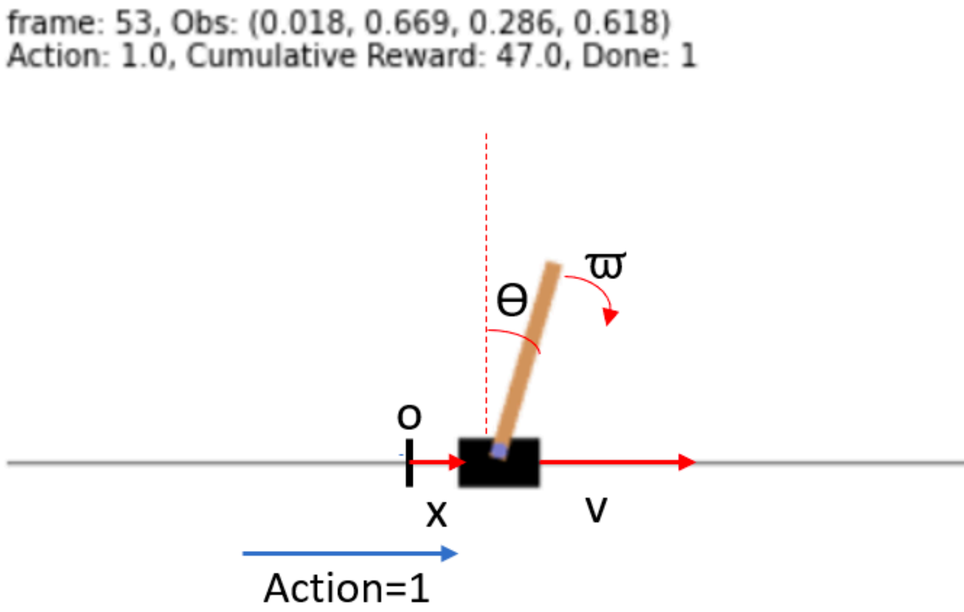
- $\mathbf{w} \in \mathbb{R}^d$ with $d \ll |\mathcal{S}|$.
- Changing $\mathbf{w}$ changes the value estimate at multiple states.
- (Tabular methods are a special case with $d = |\mathcal{S}|$).

Linear Function Approximation

- Assume value estimate is a linear function of state features.
- $$\hat{v}(s, \mathbf{w}) = \mathbf{w}^\top x(s) = \sum_{i=1}^{d} w_i x_i(s)$$
- The features, $x_i(s)$, can be non-linear functions of state variables.
  - Expressive choices for $\mathbf{x}(s)$ make linear methods more powerful than they first appear.

Linear FA Example

- What is $\mathbf{x}(s)$?
  - List of state variables: (x, v, θ, ω)
  - Any static function of the state variables.
- Suppose $\mathbf{x}(s) = (x, v, \theta, \omega)$.
  - What can you say about the value estimates as w_1 increases?



https://towardsdatascience.com/how-to-beat-the-cartpole-game-in-5-lines-5ab4e738c93f

The Prediction Objective

- As you saw in the reading, we have the following objective:

$$\overline{VE}(\mathbf{w}) = \sum_{s \in \mathcal{S}} \mu(s) \left[v_\pi(s) - \hat{v}(s, \mathbf{w}) \right]^2$$

$\mu(s)$: probability of visiting s under π

- Note: the policy is fixed because we are just considering prediction.
- Why this objective?
- Do we ever know how well we are doing?
  - μ and v_π are unknowns.

(Stochastic) Gradient Descent

- So far we have seen how to represent value estimates when $d \ll |\mathcal{S}|$ and how to evaluate different choices of $\mathbf{w}$.
- Now, how to select $\mathbf{w}$ that minimizes prediction error.
- Assuming we visit states in proportion to μ, the following update moves us towards minimal prediction error:
  - $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha(v_\pi(S_t) - \hat{v}(s, \mathbf{w}_t)) \nabla \hat{v}(S_t, \mathbf{w}_t)$
- This is the same update used for gradient-based linear regression — it's just supervised learning!

(Stochastic) Gradient Descent

- Unlike supervised learning, we don't know the targets, $v_\pi(s)$.
- Instead, we use a noisy target, U_t:
  - $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha(U_t - \hat{v}(s, \mathbf{w}_t)) \nabla \hat{v}(S_t, \mathbf{w}_t)$
- Monte Carlo: $U_t \leftarrow G_t$
- TD(0): $U_t \leftarrow R_t + \gamma \hat{v}(s, \mathbf{w})$

Example 9.1

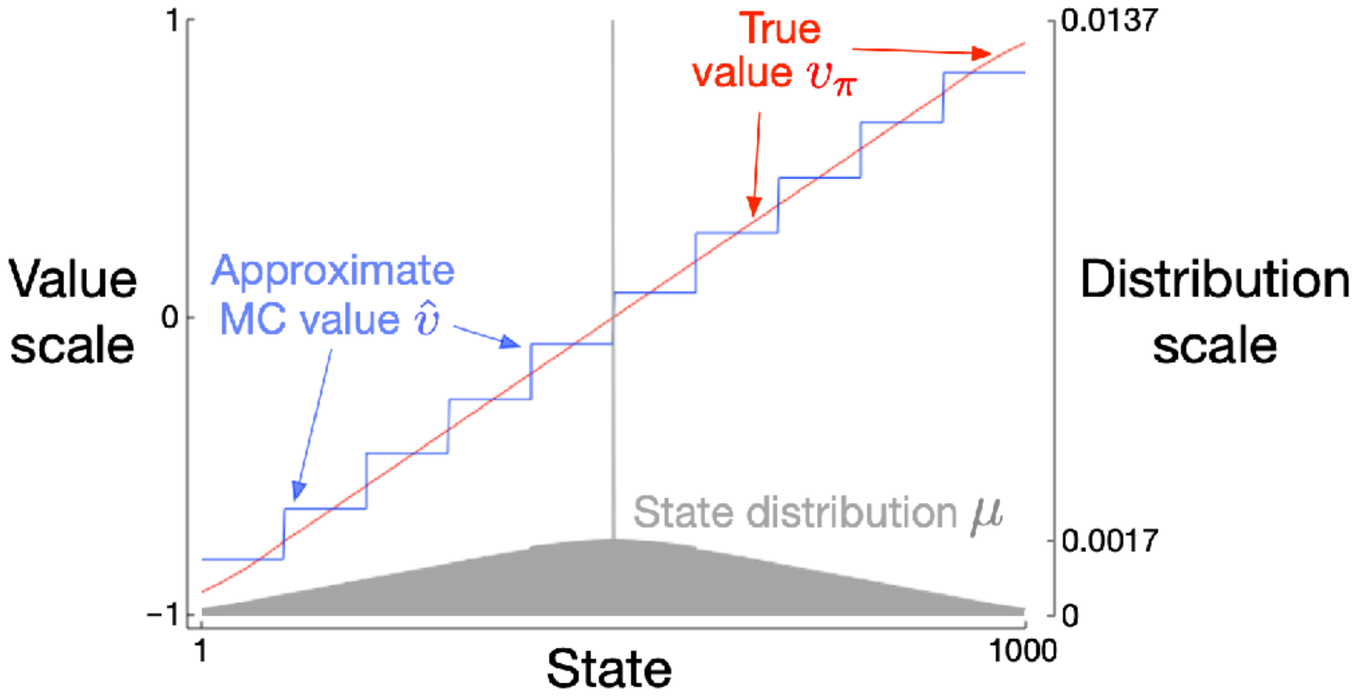


Figure 9.1: Function approximation by state aggregation on the 1000-state random walk task, using the gradient Monte Carlo algorithm (page 202).

Semi-Gradient TD(0)

- $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha(R_t + \gamma \hat{v}(S_{t+1}, \mathbf{w}_t) - \hat{v}(s, \mathbf{w}_t)) \nabla \hat{v}(S_t, \mathbf{w}_t)$
  - Why semi-gradient?
  - Why not full-gradient? (See 11.5)
- In the linear case, $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha(R_t + \gamma \hat{v}(S_{t+1}, \mathbf{w}_t) - \hat{v}(s, \mathbf{w}_t))\mathbf{x}(S_t)$
- Converges! Minimizes *the mean-squared projected Bellman error* instead of value error.

Linear Function Approximation

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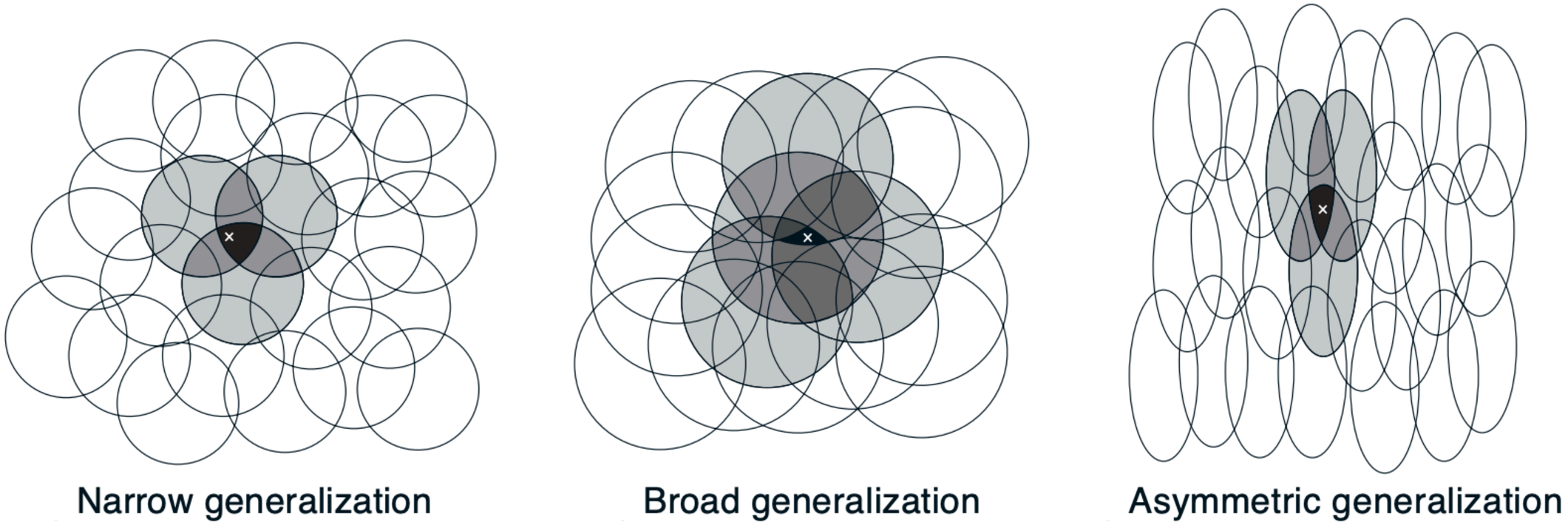
1-Hot Features / State Aggregation

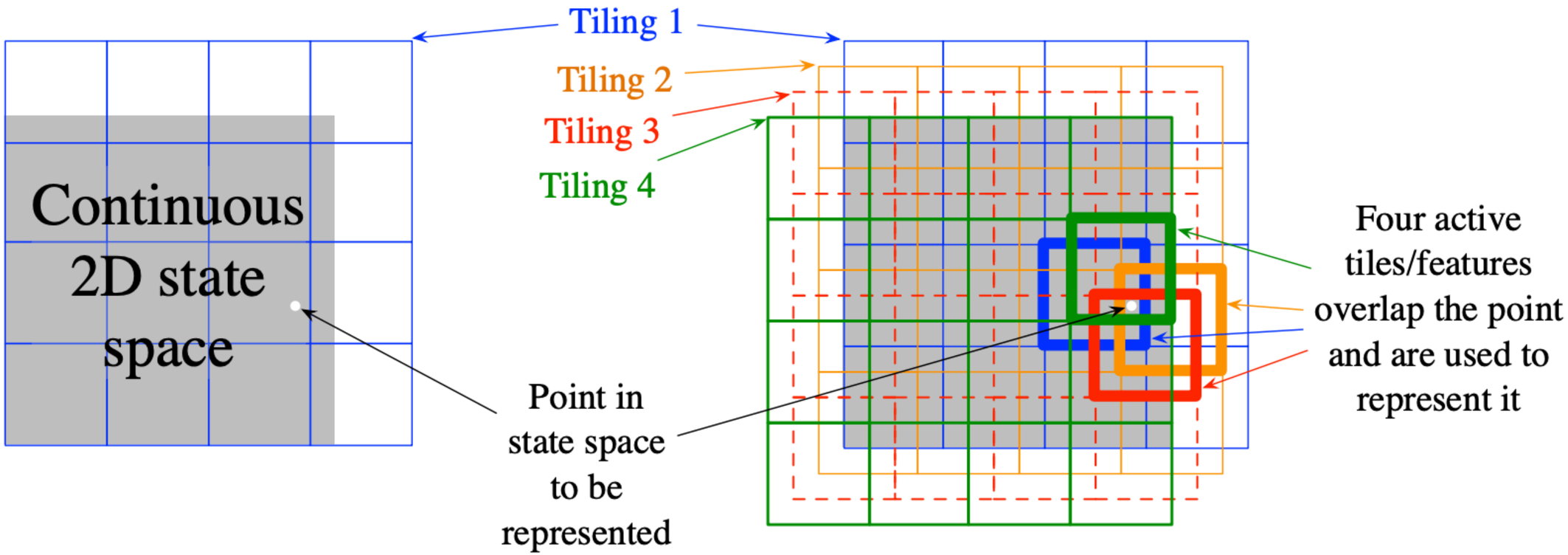
- For a finite state-space, partition state-space into d mutually exclusive groups.
- Let i be the group to which state s belongs.
- The 1-Hot feature encoding sets $x_i(s) = 1$ and $x_j(s) = 0$ for $j \neq i$.
- What does generalization look like?
- Special case is $d = |\mathcal{S}|$ in which case we recover the tabular setting.
  - Useful tip for debugging RL implementations!
  - Easily switch between easy to understand tabular experiments and more complex function approximation within same implementation.

Coarse Coding

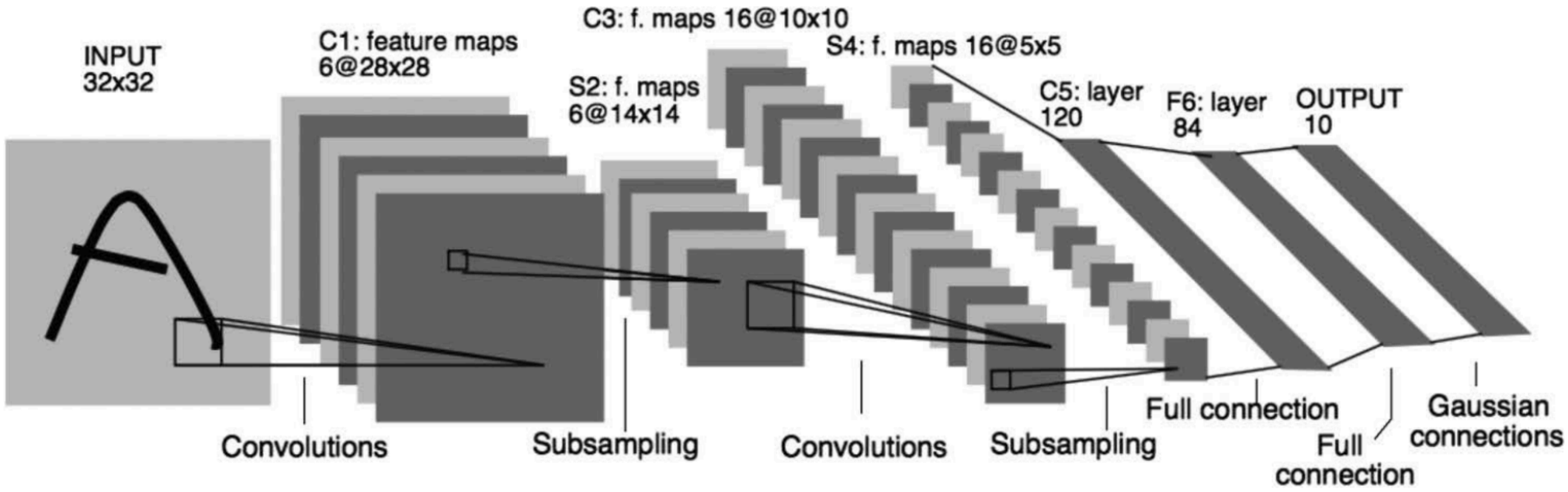


Tile Coding

- Intuitively, multiple state aggregation mappings at the same time.



Neural Networks



Step-size Selection

- The step-size is an important parameter in any SGD algorithm.
- Book gives rule of thumb:

$$\alpha = (\tau E[x^\top x])^{-1}$$

- Why does this make sense?
- Not often used in practice.

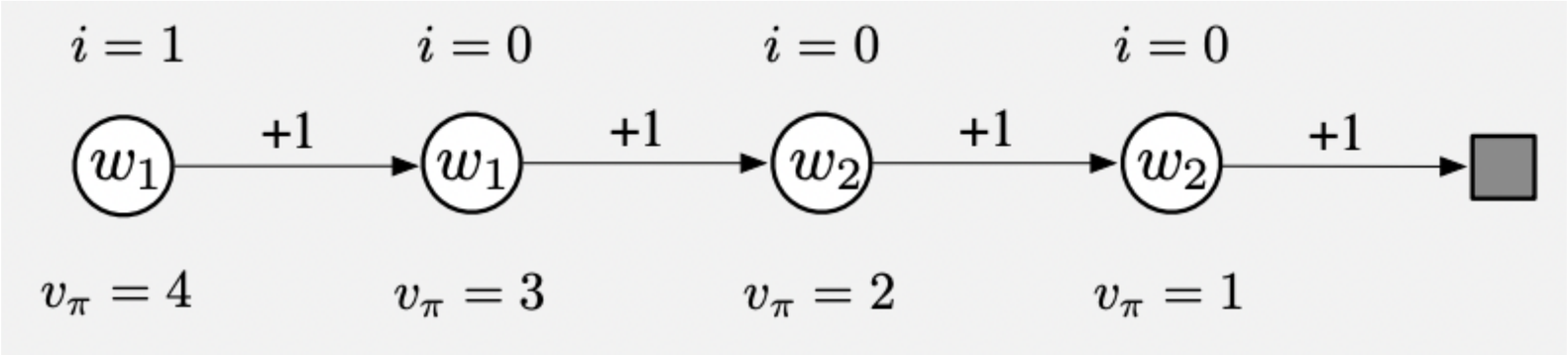
LSTD(0)

- Convergence analysis shows that Linear TD(0) converges to $\mathbf{w}_{\texttt{TD}} = \mathbf{A}^{-1}\mathbf{b}$.
  - $A = \mathbf{E}[\mathbf{x}_t(\mathbf{x}_t - \gamma \mathbf{x}_{t+1})^\top]$ and $\mathbf{b} = \mathbf{E}[R_{t+1}\mathbf{x}_t]$.
- LSTD(0) estimates A and b and then directly computes the fixed point.
  - (+) More data efficient than semi-gradient linear TD(0)
  - (-) More computation (after optimizations $O(d^2)$ vs $O(d)$ for TD(0))
- Harder to extend to deep reinforcement learning.

Interest and Emphasis

- So far, assumed we are updating states equally (same learning rate) but according to the on-policy state distribution, μ.
- We may wish to emphasize some states more.
- State interest, I_t, represents how much we care about accurate estimation in state S_t.
- Emphasis is a learned multiplier on the learning rate.
  - $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha M_t [R_t - \hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w})] \nabla \hat{v}(S_t, \mathbf{w})$
  - $M_t \leftarrow I_t + \gamma M_{t-1}$

Interest and Emphasis



- Interest is (1, 0, 1, 0)
- Semi-gradient 2-step TD converges to weight vector (3.5, 1.5)
- Emphatic 2-step TD converges to weight vector (4, 2)

On-Policy Control

- As usual, for control we will estimate action-values, $\hat{q}(s, a, \mathbf{w})$.
- For linear function approximation, features are now a function of (s,a) pairs, $\mathbf{x}(s, a)$.
- Function approximation often inherently means that making $\hat{q}(s, a, \mathbf{w})$ more accurate at one state will make it less accurate at another state.
- Now making π greedy w.r.t. $\hat{q}(s, a, \mathbf{w})$ is no longer guaranteed to improve π — no more policy improvement theorem.

Summary

- Function approximation allows us to represent state values when there are too many states for a look-up table.
- Approximation allows generalization but forces us to choose which states to approximate best.
- Linear function approximation is well understood theoretically and can be powerful with the right set of non-linear features.

Action Items

- Complete homework.
- Begin literature review.
- Begin reading Chapter 9.7 and 16.5.