Autonomous Robotics

Inverse Kinematics

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Announcements

Midterm next Tuesday (March 18) **from 5:45 - 7:15pm in CS 1221.**

Grading: HW 2 is underway, everything else has been graded and returned to you.

Midterm survey. Please complete ASAP!

Learning Outcomes

After today's lecture, you will:

- Be able to define the inverse kinematics problem.

Forward Kinematics

Task space:

Position of a robot's end-effector. Assume $\mathbb{R}^n$.

Joint space:

Space of possible robot configurations (e.g., angle of all joints). Assume $\mathbb{R}^m$.

Forward kinematics is the mapping from joint space to task space:

$r = f(q)$, where $r \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$.

Given a robot's joint configuration, determine where its end-effector is relative to a base frame of reference. Why useful?

DH Convention

DH convention provides a method for specifying a transform between a reference frame centered on one joint to that at another.

Let the transform at joint n be ${}^{n}_{n-1}T$.

$$
{}^{n}_{n-1}T = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & r_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & r_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & R & & t \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Transform for the entire arm is $F(q) = {}^{2}_{1}T\,{}^{3}_{2}T \cdots {}^{n}_{n-1}T$.

To rotate [x, y, z], we matrix multiply [x, y, z, 1] to obtain [x', y', z', 1].

Forward Kinematics under DH Convention

$F(q) = {}^{2}_{1}T \cdots {}^{n}_{n-1}T.$

Note that F is a function of the robot's configuration, q.

End-effector position in end-effector frame is [0, 0, 0]

Putting this together, $f(q) = F(q) \cdot [0,0,0,1].$

Differential Kinematics

- Relate velocity of end-effector to velocity of joints.
- Forward kinematics: $r = f(q)$.
- Velocity: $\dot{r} = J(q) \cdot [\dot{q}_1, \ldots \dot{q}_n]$ where $J(q)$ is the Jacobian of the robot's end-effector with respect to its configuration.

$$J = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \cdots & \frac{\partial y}{\partial q_n} \\ \frac{\partial \theta}{\partial q_1} & \frac{\partial \theta}{\partial q_2} & \cdots & \frac{\partial \theta}{\partial q_n} \end{bmatrix}$$

Differential Drive Kinematics

- Differential drive: two independently controlled wheels. Why useful?

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{\dot{\phi}_r r}{d} - \frac{\dot{\phi}_l r}{d} \end{bmatrix}$$

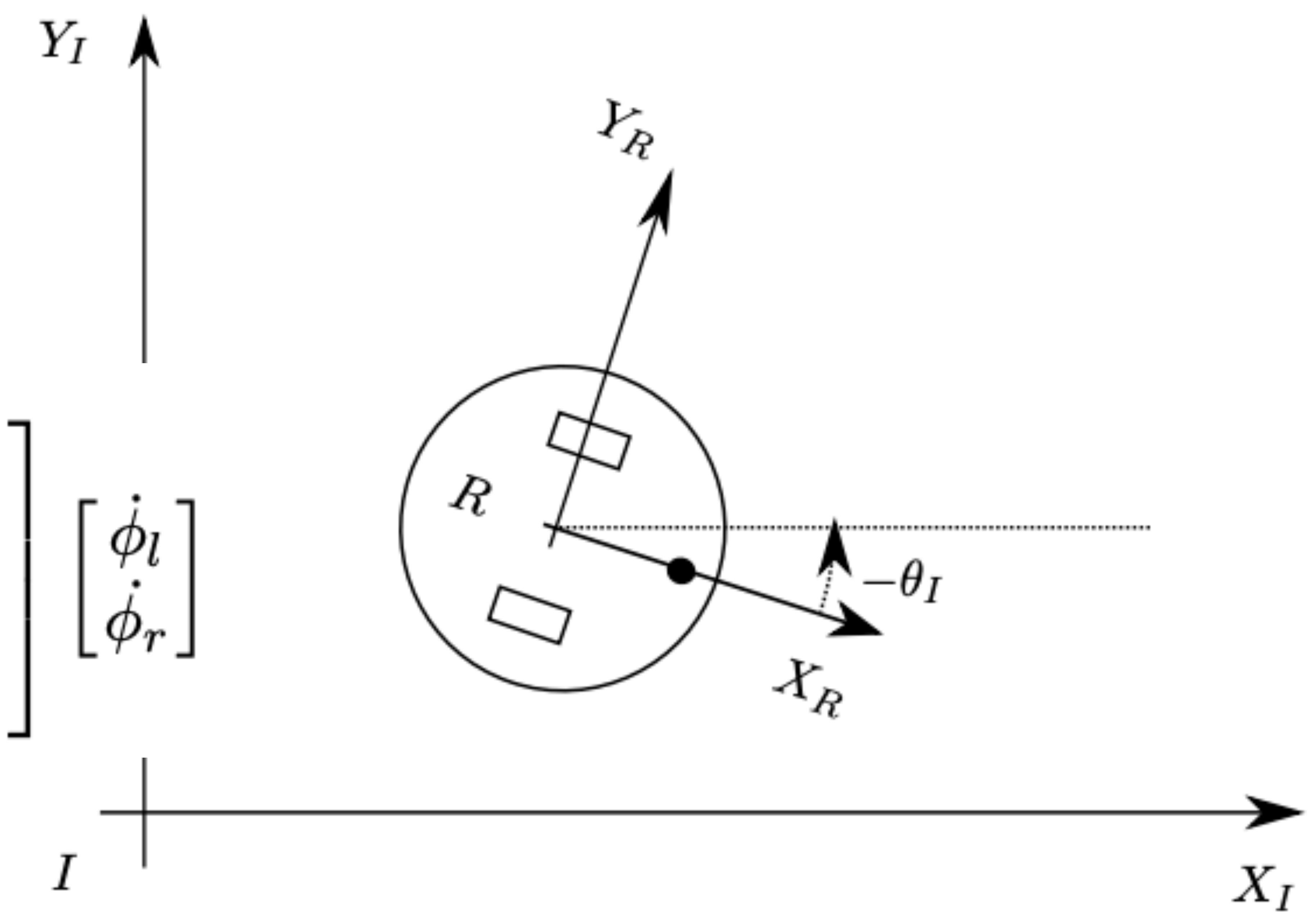
r: wheel radius

$\dot{\phi}_l$: left wheel velocity

$\dot{\phi}_r$: right wheel velocity

d: distance between wheels

$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{d} & -\frac{r}{d} \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} = \begin{bmatrix} \frac{\partial x_R}{\partial \dot{\phi}_l} & \frac{\partial x_R}{\partial \dot{\phi}_r} \\ \frac{\partial y_R}{\partial \dot{\phi}_l} & \frac{\partial y_R}{\partial \dot{\phi}_r} \\ \frac{\partial \theta}{\partial \dot{\phi}_l} & \frac{\partial \theta}{\partial \dot{\phi}_l} \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix}$$

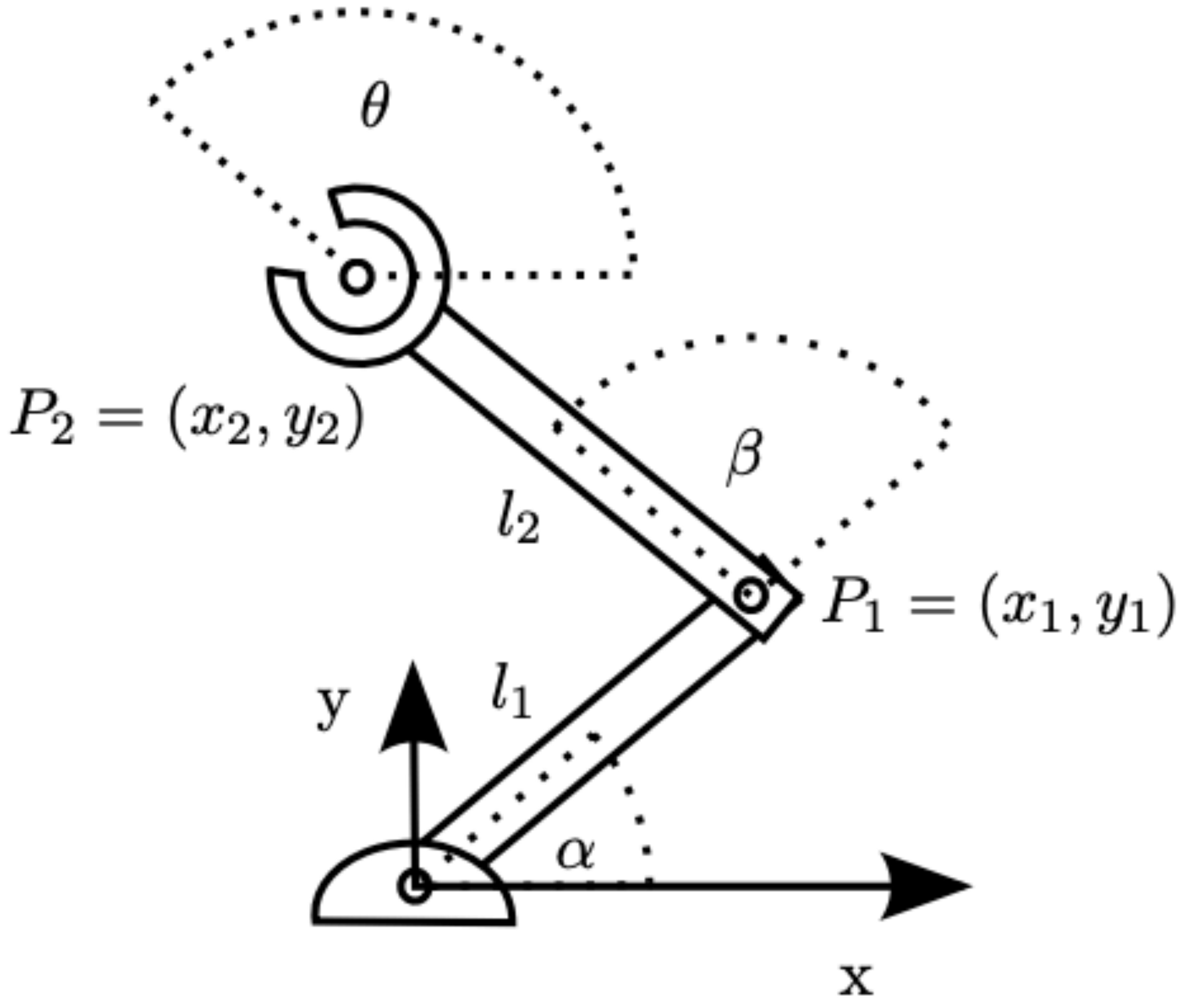


- Holonomic vs. non-holonomic: configuration determines a unique position in task space.

Kinematics Practice

Consider the two-link robot arm shown here. What is the transformation matrix for the position of the joint between the links?

Hint: you can ignore the z-axis so the matrix should be 3x3.

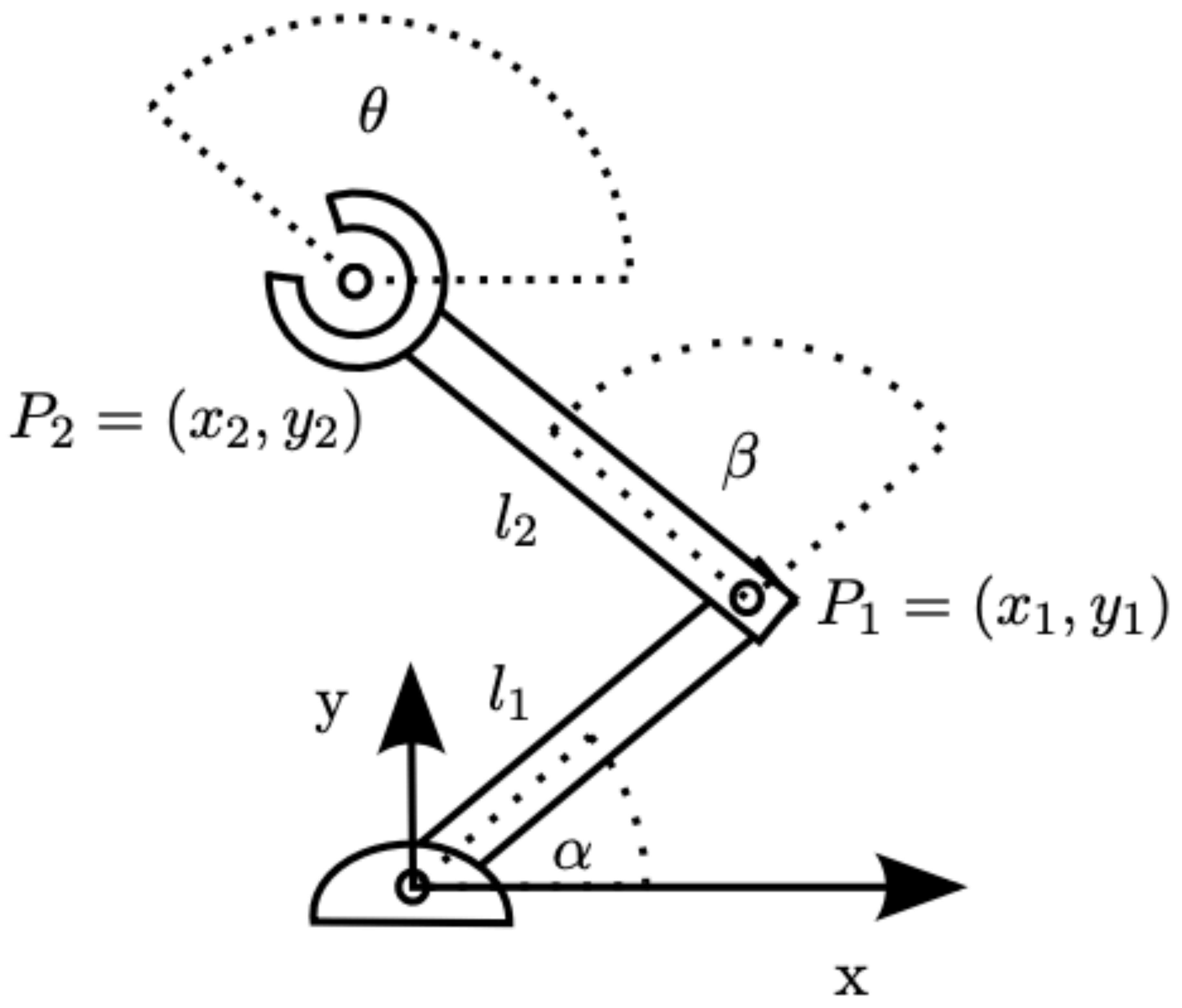


Kinematics Practice

Consider the two-link robot arm shown here. What is the transformation matrix for the position of the joint between the links?

Hint: you can ignore the z-axis so the matrix should be 3x3.

$$ {}_1^2T = \begin{bmatrix} \cos\alpha & -\sin\alpha & l_1 \cos\alpha \\ \sin\alpha & \cos\alpha & l_1 \sin\alpha \\ 0 & 0 & 1 \end{bmatrix} $$



Inverse Kinematics

Task space:

Position of a robot's end-effector. Assume $\mathbb{R}^n$.

Joint space:

Space of possible robot configurations (e.g., angle of all joints). Assume $\mathbb{R}^m$.

Inverse kinematics is the mapping from a point in task space to joint space:

$q = f^{-1}(r)$, where $r \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$.

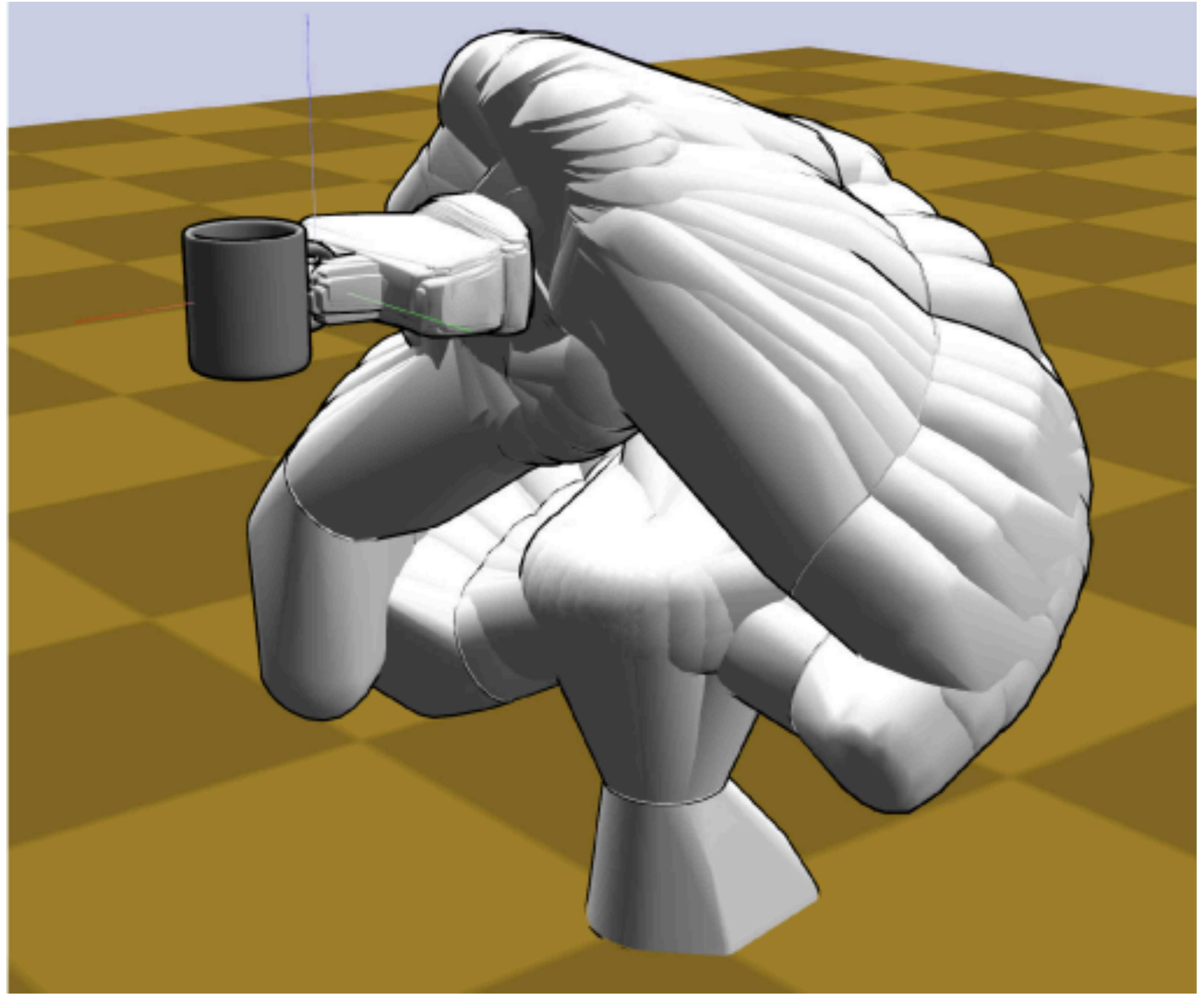
Given a desired position of the end-effector, determine the robot's joint configuration to put the end-effector there. Why useful?

Note: not all desired positions are achievable. Set of achievable positions = the robot's workspace.

Inverse Kinematics

For a target end-effector position, we may have:

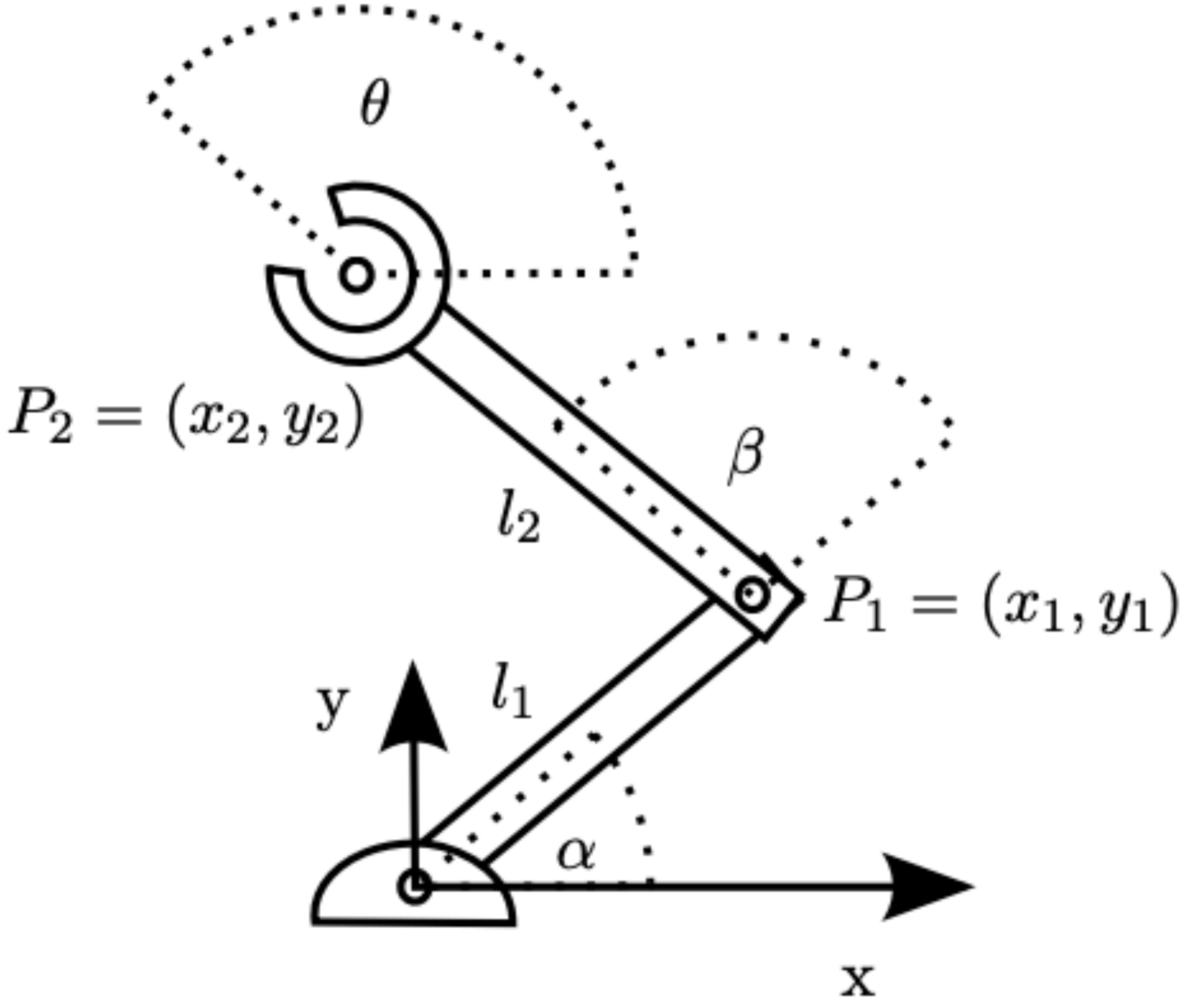
- No feasible configurations.
- A single feasible configuration.
- Many feasible configurations.



How to determine?

https://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=9793576

IK Example



Given: (x, y, θ)

Determine: α, β

$$\left.\begin{aligned} \cos\theta &= \cos(\alpha+\beta) \\ x &= l_2\cos(\alpha+\beta) + l_1\cos\alpha \\ y &= l_2\sin(\alpha+\beta) + l_1\sin\alpha \end{aligned}\right\}$$

From forward kinematics

Then solve for α, β:

$$\theta = \alpha + \beta$$

$$\cos\alpha = \frac{l_2\cos\theta - x}{l_1} \quad \sin\alpha = \frac{l_2\sin\theta - y}{l_1}$$

General IK Approach

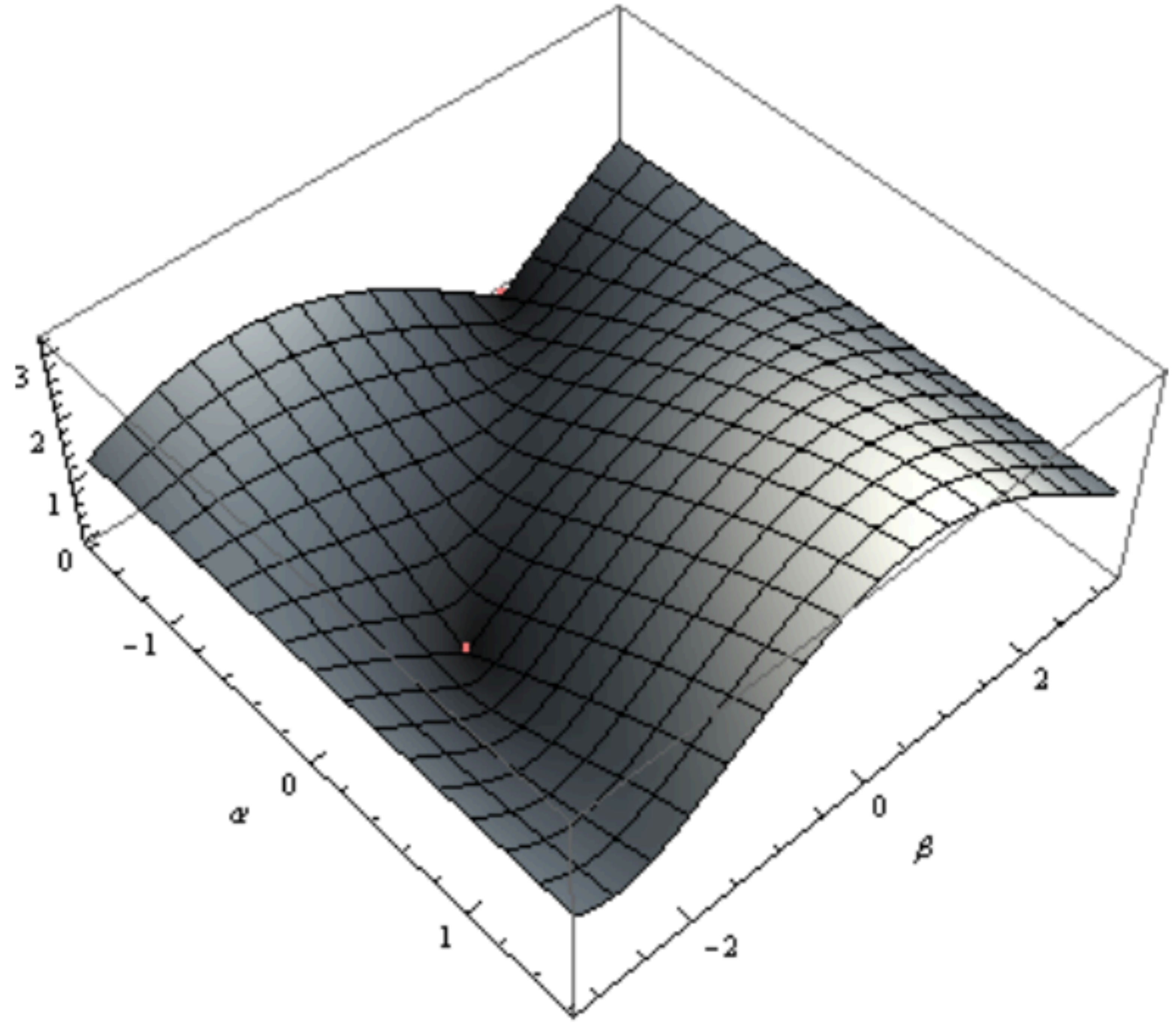
In general, it may be challenging to solve for $f^{-1}(r)$ directly.

Instead, try to find $\hat{q} \approx f^{-1}(r)$.

Specify a loss:

$$l_r(q) = ||r - F(q)||_2$$

Use optimization to find the minimizer, $q*$.



$$f_{x,y}(\alpha, \beta) = \sqrt{(s_{\alpha\beta} + s_\alpha - y)^2 + (c_{\alpha\beta} + c_\alpha - x)^2}$$

Assuming $l_1 = l_2 = 1$

Differential IK

Recall differential kinematics: $\dot{r} = J(q) \cdot \dot{q}$ where $J(q) \in \mathbb{R}^{m \times n}$ is the Jacobian of $f(q)$ and $\dot{q}$ is the time derivative of q (i.e., rate of change in q).

Differential IK: given a desired velocity in task space, determine the desired velocity that achieves it.

First idea: use $J^{-1}(q) \cdot \dot{r} = \dot{q}$. Problem?

Second idea: use the pseudo-inverse, $J^{+}(q)$. Problem?

Third idea: damp joint velocities. $\Delta q = (J^{\top} J + \lambda^2 I)^{-1} J^{+} e$

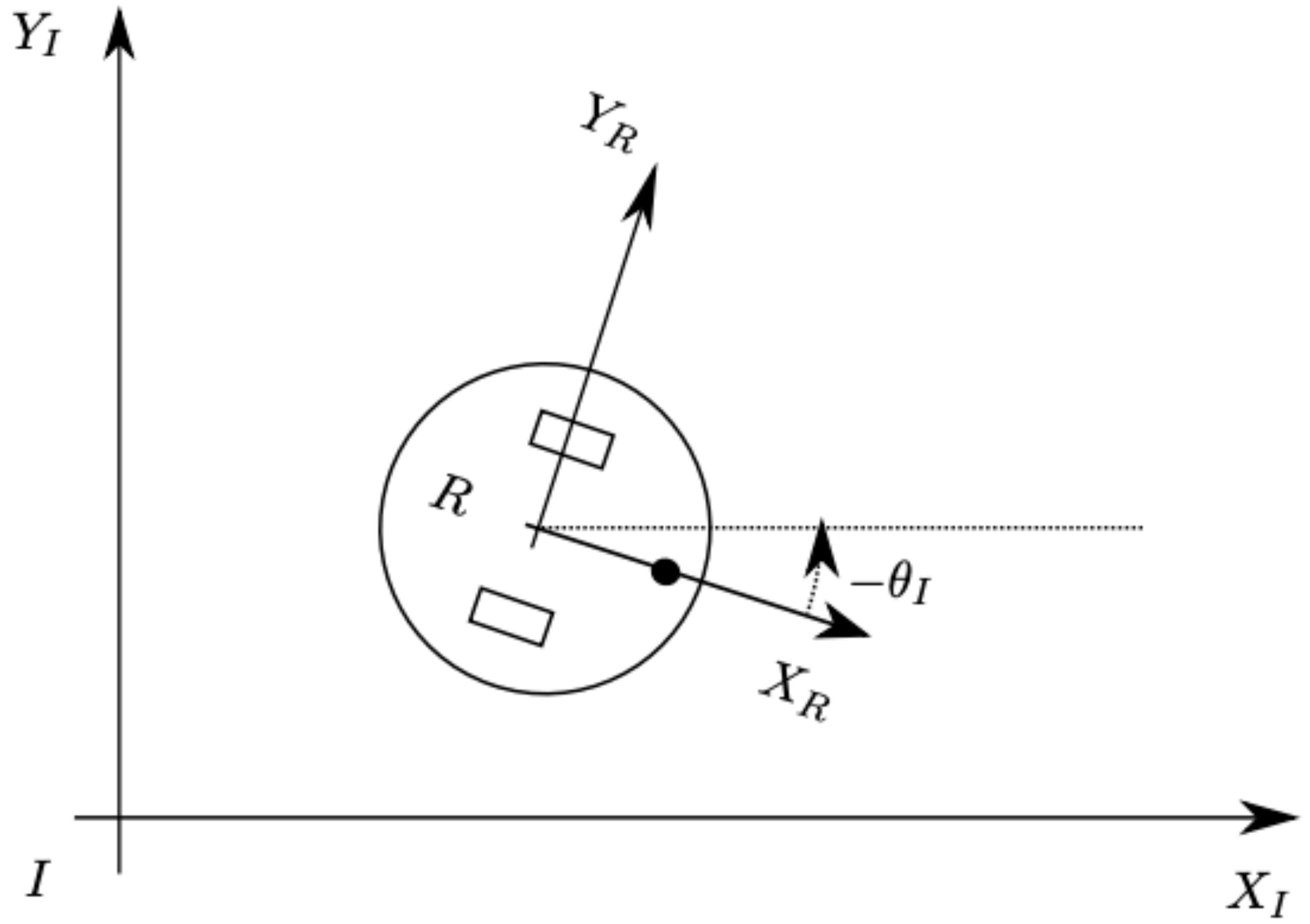
IK for Differential Drive

Given desired change in x, y, and θ, compute velocity for left and right wheel to achieve it.

Step 1: transform desired $\dot{x}, \dot{y}, \dot{\theta}$ to desired $\dot{x}_R, \dot{\theta}_R$.

Step 2: solve for $\dot{\phi}_l, \dot{\phi}_r$

$$\begin{bmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{d} & -\frac{r}{d} \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} = \begin{bmatrix} \frac{\partial x_R}{\partial \dot{\phi}_l} & \frac{\partial x_R}{\partial \dot{\phi}_r} \\ \frac{\partial y_R}{\partial \dot{\phi}_l} & \frac{\partial y_R}{\partial \dot{\phi}_r} \\ \frac{\partial \theta}{\partial \dot{\phi}_l} & \frac{\partial \theta}{\partial \dot{\phi}_l} \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix}$$



Note: ignore $\dot{y}_R$ as we assume the robot cannot move sideways.

IK in Practice

Targets not set in isolation.

Can use task knowledge to select joint configurations strategically.

Perform a *reconfiguration*

https://graphics.cs.wisc.edu/Papers/2024/WSG24/

https://graphics.cs.wisc.edu/Papers/2018/RMG18a/

Summary

- Reviewed forward kinematics.
- Introduced inverse kinematics.
- Introduced differential inverse kinematics.

Action Items

- Planning reading for next week; send a reading response by 12 pm on Monday.

- Midterm: on Tuesday!