Autonomous Robotics

Bayes Filtering and Smoothing

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Programming Assignments

- Due Tuesday (2/11) at 9:30am
- Any questions?
- Any comments?

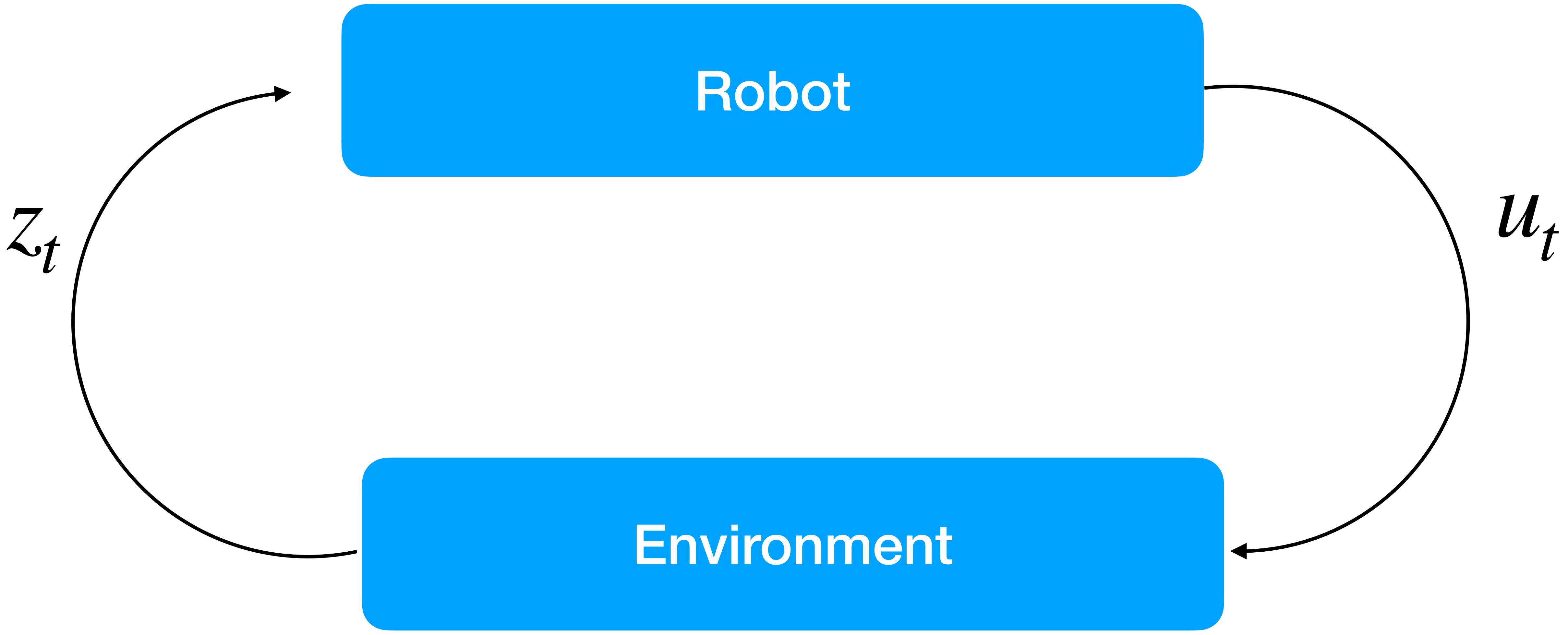
Learning Outcomes

After today's lecture, you will:

- Have reviewed the Bayes Filter from last time.
- Explain the difference between filtering and smoothing.
- Be able to work out potential exam problems related to Bayes Filter.

Probabilistic Interaction Model

$$\texttt{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$



$$x_t \sim p(x_{t-1}, u_t)$$

$$z_t \sim g(x_t)$$

$g(z \mid x)$ is the probability of z given x.

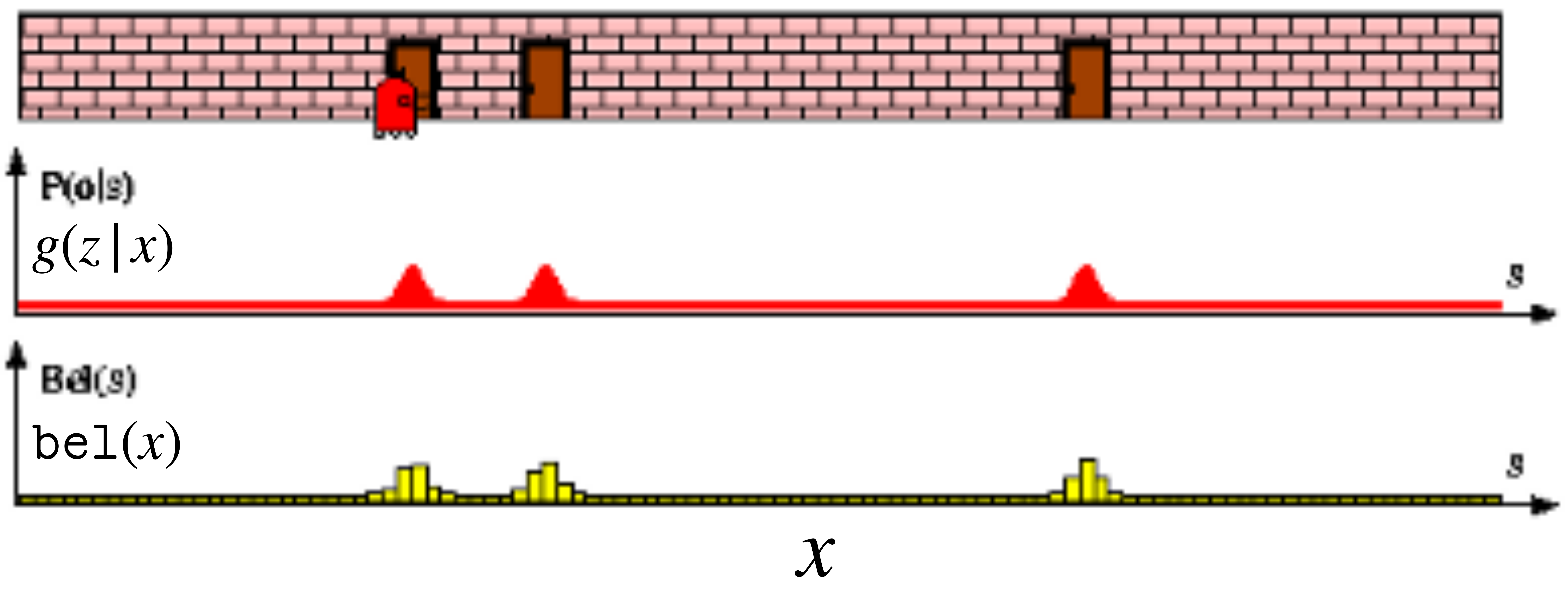
$p(x_t \mid x_{t-1}, u_t)$ is the probability of x_t given the environment is in state x_{t-1} and control u_t is taken.

Markov assumption: $p(x_t \mid x_{t-1}, u_t) = p(x_t \mid x_{0:t-1}, u_{1:t})$

State Estimation

- Define robot's belief as $\texttt{bel}(x_t)$.
- At time t, the robot has observed $z_{1:t}$ and knows it has taken actions $u_{1:t}$.
- **Goal:** Compute the posterior such that $\texttt{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$.
    - Computation should not grow with t.
    - Why necessary?
- Assumptions: know models p and g, have initial belief $\texttt{bel}(x_0)$.

Illustration



Bayes Filter

Discrete States | **Continuous States**

- **Predict:**

$$\overline{\texttt{bel}}(x_t) \leftarrow \sum_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\texttt{bel}(x_{t-1})$$

$$\overline{\texttt{bel}}(x_t) \leftarrow \int_{x_{t-1}} p(x_t \mid x_{t-1}, u_t)\texttt{bel}(x_{t-1})dx_{t-1}$$

- **Correct:**

$$\texttt{bel}(x_t) \leftarrow \eta^{-1} g(z_t \mid x_t)\overline{\texttt{bel}}(x_t)$$

$$\eta = \sum_{x_t} g(z_t \mid x_t)\overline{\texttt{bel}}(x_t)$$

$$\eta = \int x_t g(z_t \mid x_t)\overline{\texttt{bel}}(x_t)dx_t$$

Alternative Bayes Filter

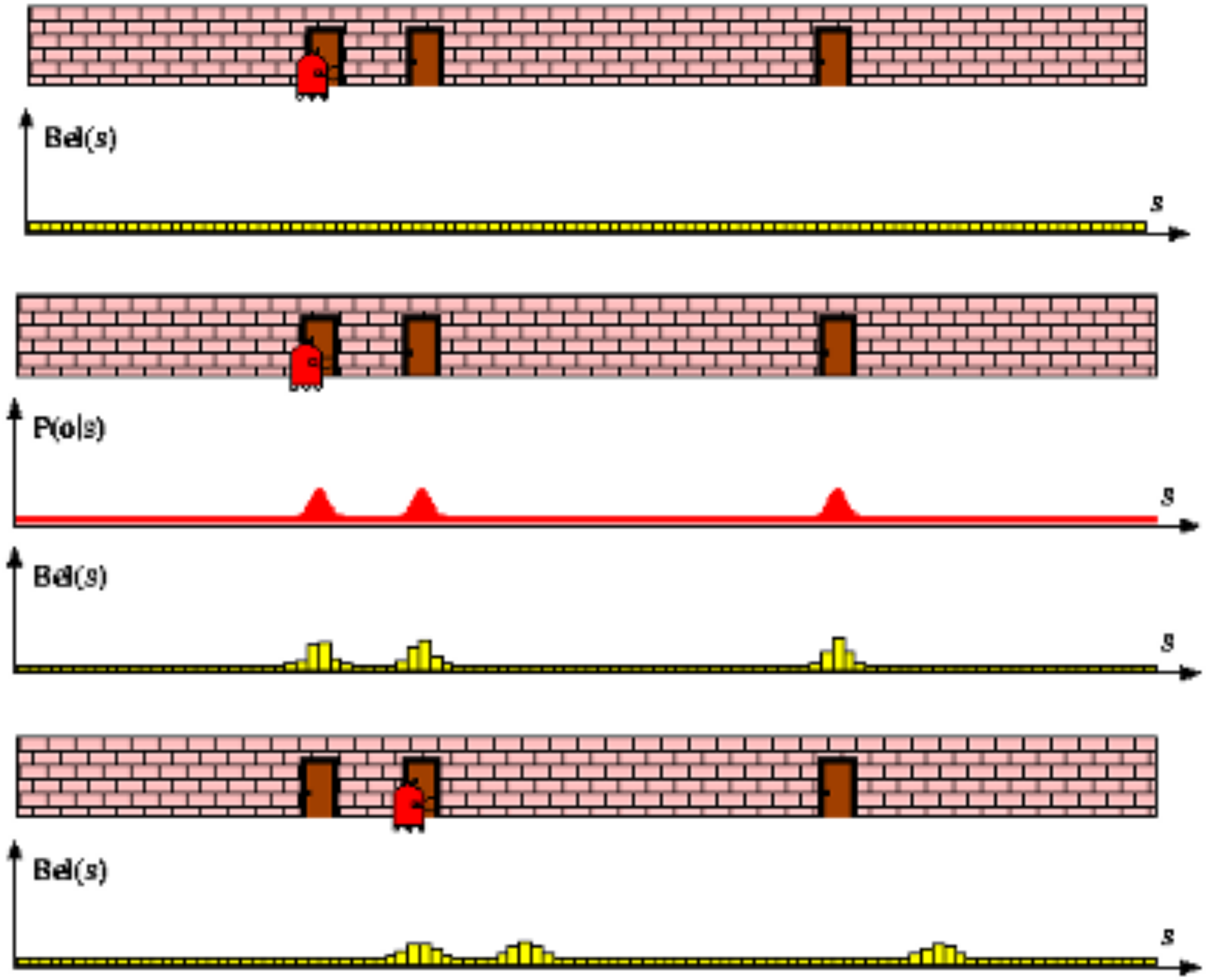
- Receive data, d_t
- If d_t is control data:

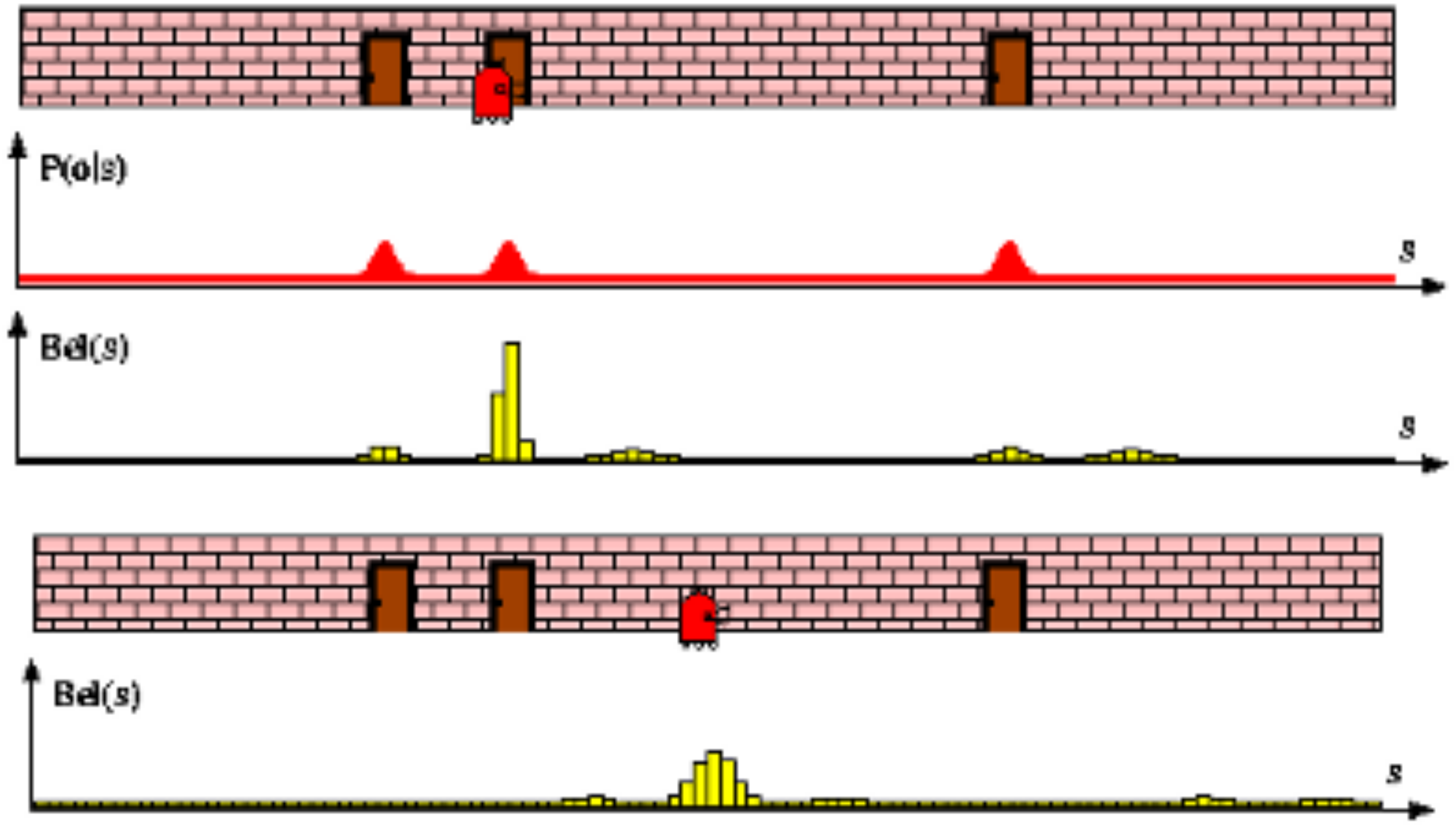
$$\texttt{bel}(x_t) \leftarrow \sum_{x_{t-1}} p(x_t \mid x_{t-1}, d_t)\texttt{bel}(x_{t-1})$$

- Else (d_t is sensor data):

$$\eta = \sum_{x_t} g(z_t \mid x_t)\texttt{bel}(x_t) \qquad \texttt{bel}(x_t) \leftarrow \eta^{-1} g(z_t \mid x_t)\texttt{bel}(x_t)$$

Bayes Filter Example





Prediction increases uncertainty; Correction step decreases uncertainty.

Credit: Probabilistic Robotics

Limitations

- Intractable in general case. Why?
  - Summation/integration over state space.
  - Special cases can be tractable, e.g., Kalman Filter assumes linear Gaussian models.
  - Approximation possible, e.g., Extended KF, Particle Filter

Smoothing

- **What is it?**
  - Estimate $p(x_t \mid z_{1:T}, u_{1:T})$ for $T > t$.
  - Use new observations to refine past beliefs.
  - Compare to filtering: Expect $p(x_t \mid z_{1:T}, u_{1:T})$ to be more accurate than $p(x_t \mid z_{1:t}, u_{1:t})$.
- **Why?**
  - Future observations provide information about the past.
  - Can use inference offline for learning, e.g., map estimation or improving models.

Bayes Smoother

- Forward pass: Bayes filter over the data. Get $\texttt{bel}(x_t)$ for $t \in \{1,\ldots,T\}$.
- Backward (smoothing) pass:
  - $\texttt{bel}'(x_T) \leftarrow \texttt{bel}(x_T)$

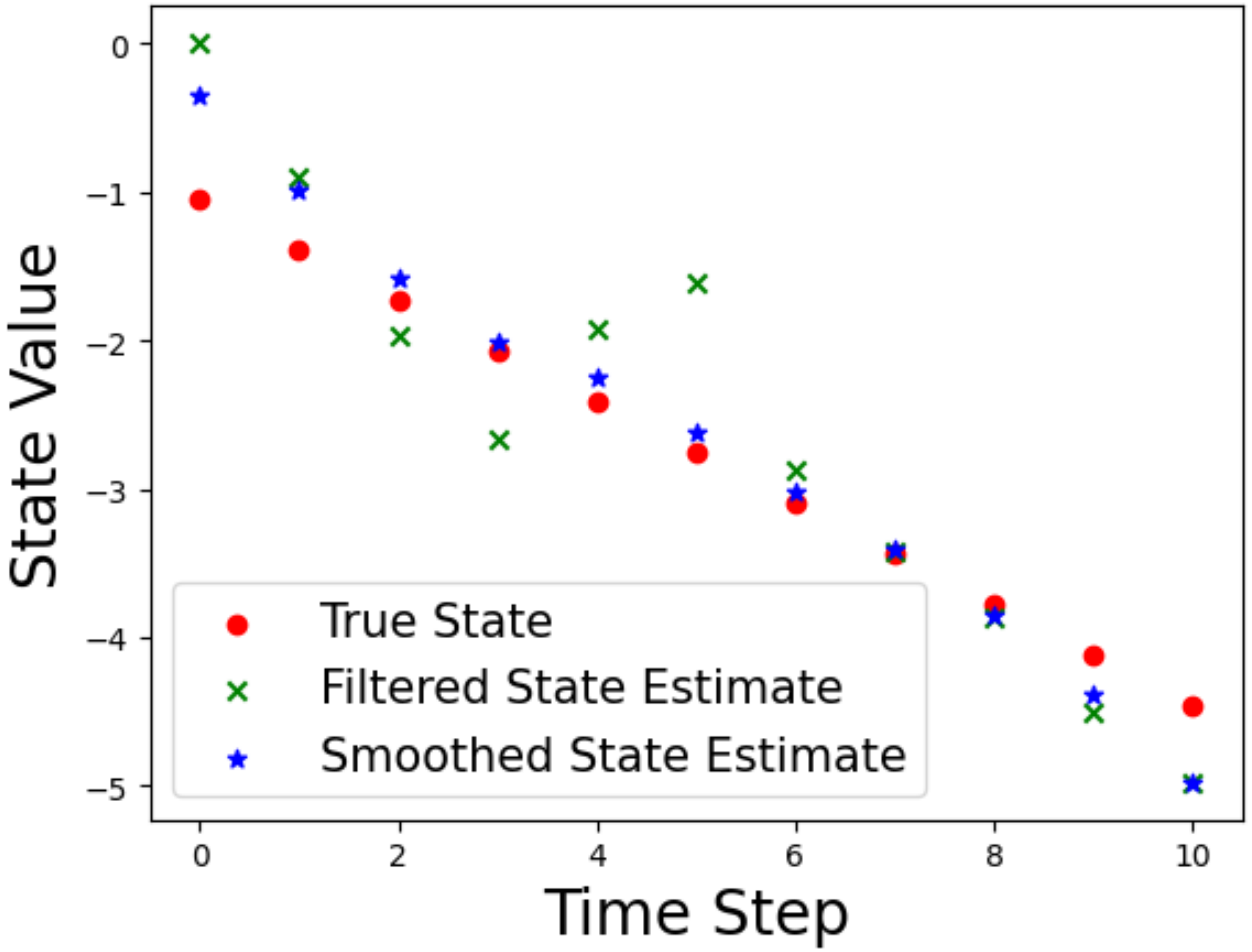
    The belief at the final time-step is just the belief from the filter.

  - Work backwards in time from T to 1 using the update:

$$\texttt{bel}'(x_t) \leftarrow \texttt{bel}(x_t) \cdot \sum_{x_{t+1}} \frac{p(x_{t+1} \mid x_t, u_{t+1}) \cdot \texttt{bel}'(x_{t+1})}{\texttt{bel}(x_{t+1})}$$

Belief from filtering

Bayes Smoother



State estimates from a Kalman filter and RTS smoother — instantiations of Bayes filter and smoother.

Bayes Smoother Derivation

$$p(x_t \mid z_{1:T}, u_{1:T}) = \int_{x_{t+1}} p(x_t, x_{t+1} \mid z_{1:T}, u_{1:T}) dx_{t+1}$$

Law of total probability (Reverse marginalization)

$$= \int_{x_{t+1}} p(x_t \mid x_{t+1}, z_{1:T}, u_{1:T}) p(x_{t+1} \mid z_{1:t}, u_{1:t}) dx_{t+1}$$

Definition of conditional probability

$$= \int_{x_{t+1}} p(x_t \mid x_{t+1}, z_{1:t+1}, u_{1:t+1}) \texttt{bel}'(x_{t+1}) dx_{t+1}$$

Markov property and smoothed belief definition

$$= \int_{x_{t+1}} \frac{p(x_{t+1} \mid x_t, z_{1:t+1}, u_{1:t+1}) p(x_t \mid z_{1:t+1}, u_{1:t+1})}{p(x_{t+1} \mid z_{1:t+1}, u_{1:t+1})} \texttt{bel}'(x_{t+1}) dx_{t+1}$$

Bayes

$$= \int_{x_{t+1}} \frac{p(x_{t+1} \mid x_t, u_{t+1}) \texttt{bel}(x_t)}{\texttt{bel}(x_{t+1})} \texttt{bel}'(x_{t+1}) dx_{t+1}$$

Markov and belief definitions

Filtering Practice

Mobile robot with binary observations $z \in \{\texttt{door}, \texttt{no} - \texttt{door}\}$ and location $x \in \{1,2,3\}$.
Observation probabilities: $g(\texttt{door} \mid x = 2) = 0.8$ and otherwise $p(\texttt{door} \mid x) = 0.1$.
Initial belief: $\texttt{bel}(x_0) = 1/3$, i.e., equal probability of any start state.
Movement: the robot can move left (decrease x) or right (increase x). Actions fail with probability 0.2 and move the robot one unit in the intended direction with probability 0.8. If the robot is in state $x = 3$ and moves right then it remains in $x = 3$ with probability 1. A similar transition occurs if the robot moves left in $x = 1$.

The robot moves right and observes `door`. What is its new belief under one step of Bayes Filter?

Smoothing Practice

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The robot moves right and observes `door`. Compute a revised belief about its initial state using the Bayes smoother?

Summary

- Reviewed problem of state estimation.
- Reviewed Bayes filter as a method for state estimation.
- Discussed limitations of the Bayes filter.
- Introduced Bayes smoother.

Action Items

- Complete the first programming assignment on control.
- Read on Kalman filter for next week; send a reading response by 12 pm on Monday.