Autonomous Robotics

Rigid Body Transformations and Forward Kinematics

Josiah Hanna
University of Wisconsin — Madison

Announcements

- Homework #3 has been released. Due next week.
- Reading assignment for next week (motion planning) due Monday at noon.
- Midterm in 2 weeks.

Learning Outcomes

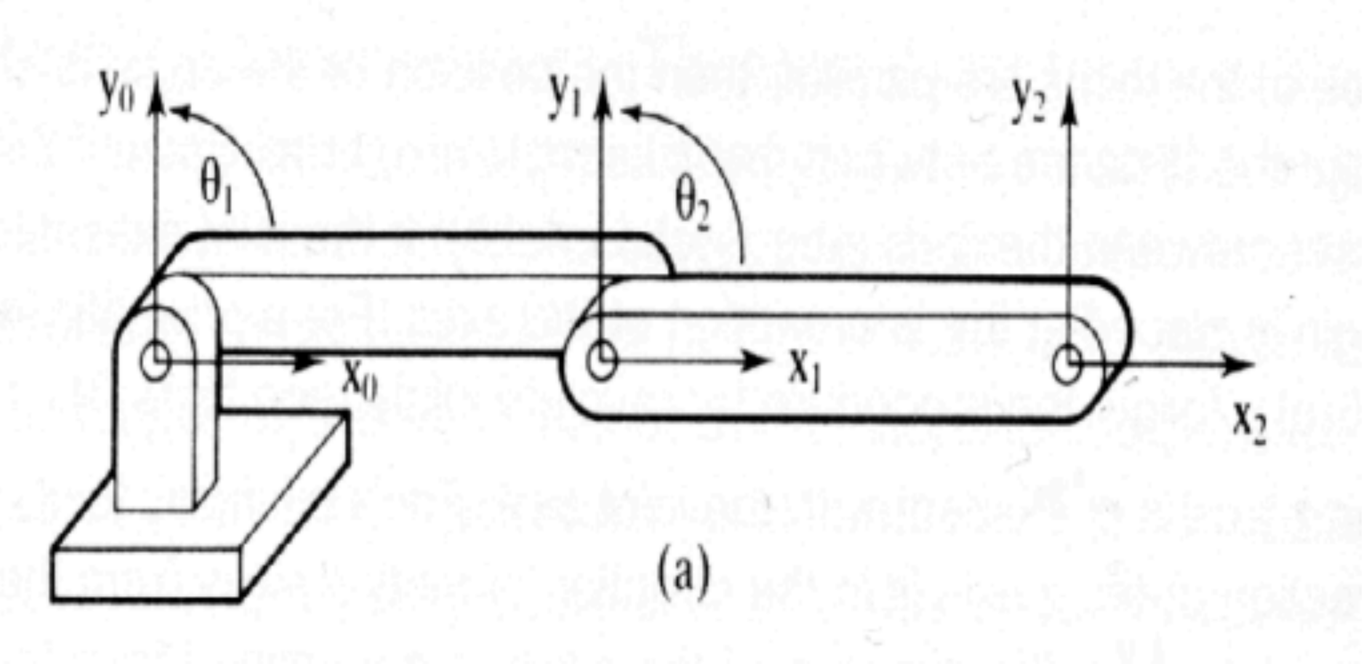
After today's lecture, you will be able to:

- Describe and compute transformations between joint reference frames.
- Describe the forward kinematics process.
- Implement forward kinematics calculations as repeated reference frame transforms.

Degrees of Freedom

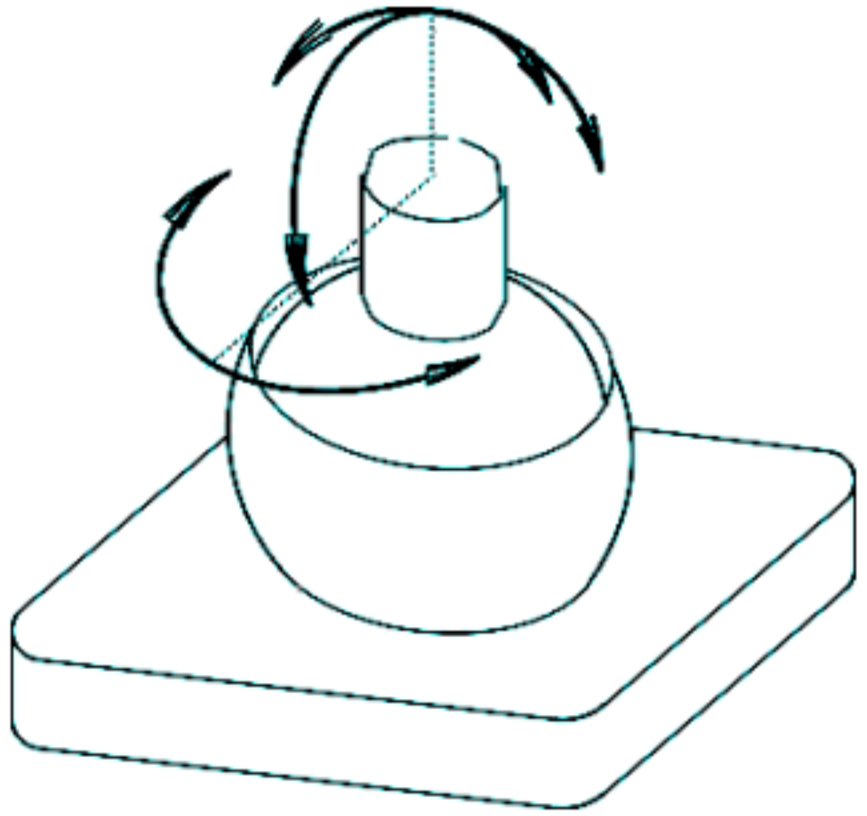
- The number of independent parameters that can fully define a robot's configuration.
- **Link:** Single rigid body.
- **Joint:** connection between links

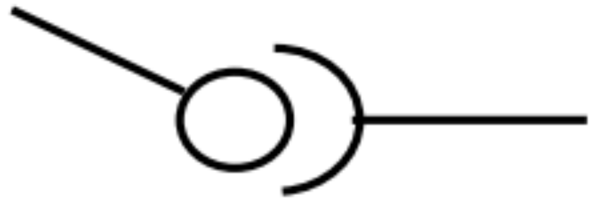
Joint Types



Revolute Joint

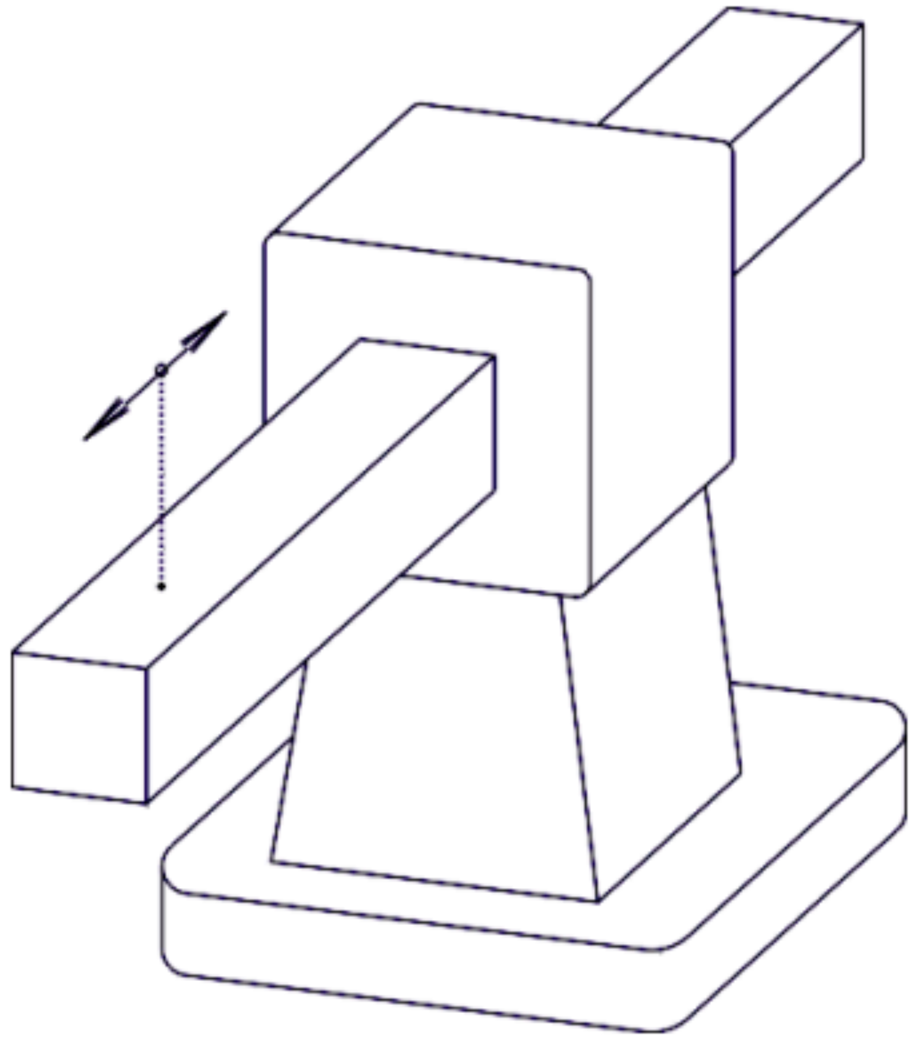
1 DOF (Variable - Θ)




Spherical Joint
3 DOF (Variables - Θ_1, Θ_2, Θ_3)

Prismatic Joint

1 DOF (linear) (Variables - d)



Forward Kinematics

Task space:

Position of a robot's end-effector. Assume $\mathbb{R}^n$.

Joint space:

Space of possible robot configurations (e.g., angle of all joints). Assume $\mathbb{R}^m$.

Forward kinematics is the mapping from joint space to task space:

$r = f(q)$, where $r \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$.

Given a robot's joint configuration, determine where its end-effector is relative to a base frame of reference. Why useful?

Reference Frames

Coordinate system from which rotations and translations are based.

Given a vector $x \in \mathbf{R}^d$:

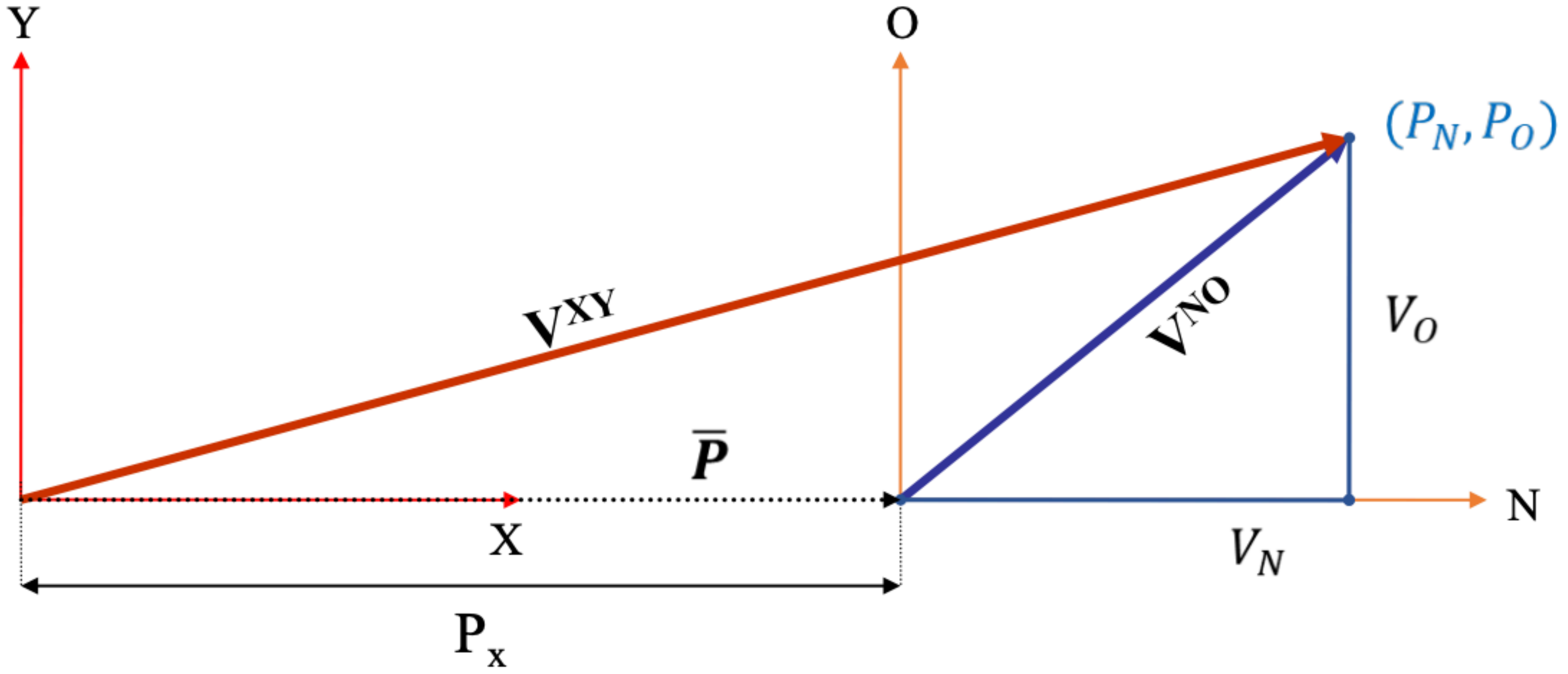
For $c \in \mathbf{R}^d$, we call $x + c$ a **translation** of x.

For $A \in \mathbf{R}^{d \times d}$, we call Ax a rotation of x.

Translating Frames

Given a vector $x \in \mathbf{R}^d$:

For $c \in \mathbf{R}^d$, we call $x + c$ a **translation** of x.



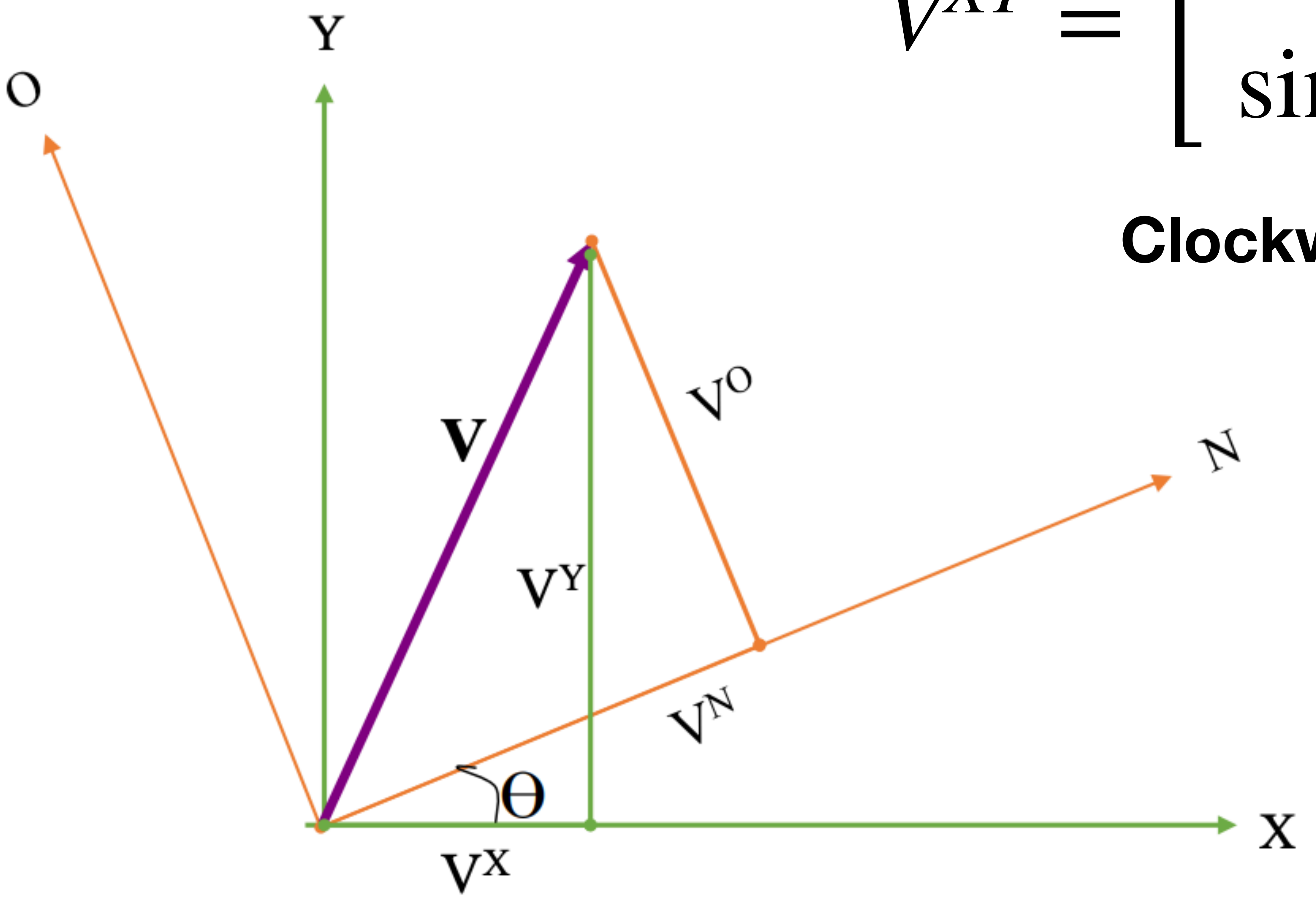
Rotating Frames

Given a vector $x \in \mathbf{R}^d$:

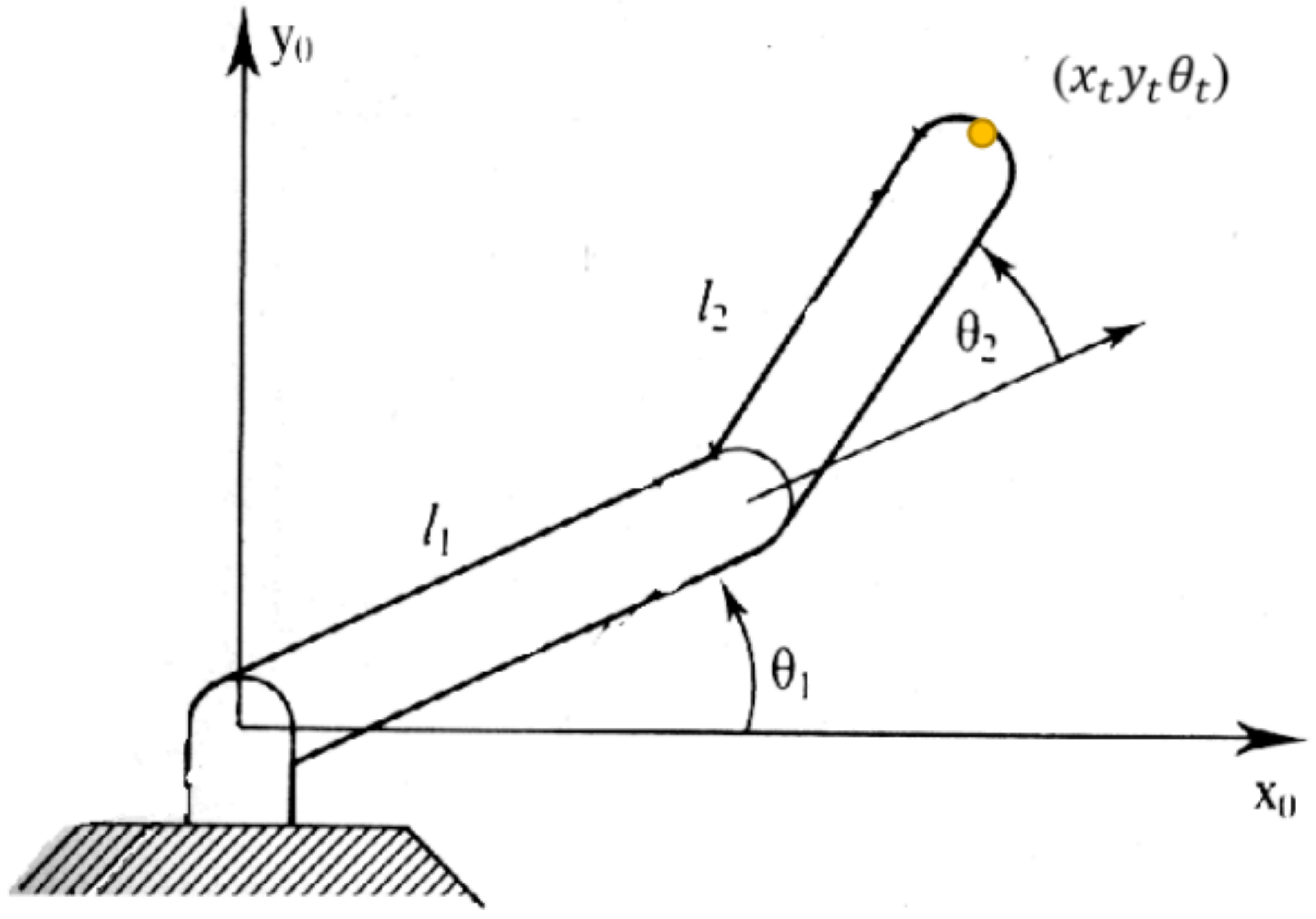
For $A \in \mathbf{R}^{d \times d}$, we call Ax a rotation of x.

$$\overline{V}^{XY} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \overline{V}^{NO}$$

Clockwise rotation matrix



Arm Kinematics

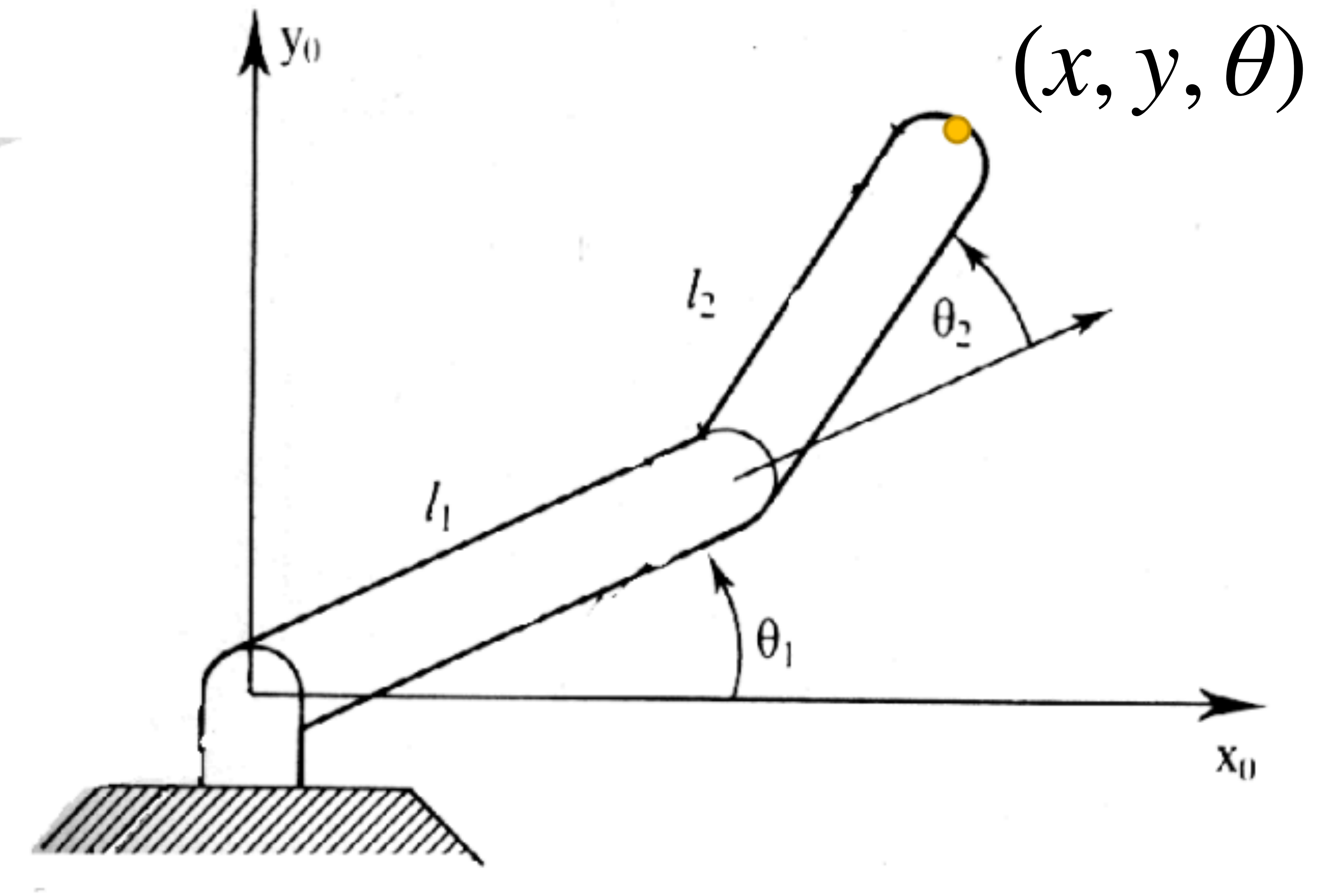


Set up:

- You have a two-link arm with base at the origin.
- The first link has length l_1 and is at angle θ_1.
- The second link has length l_2 and is at angle θ_2.

What is the position and orientation of the end-effector?

Arm Kinematics: Direct Approach

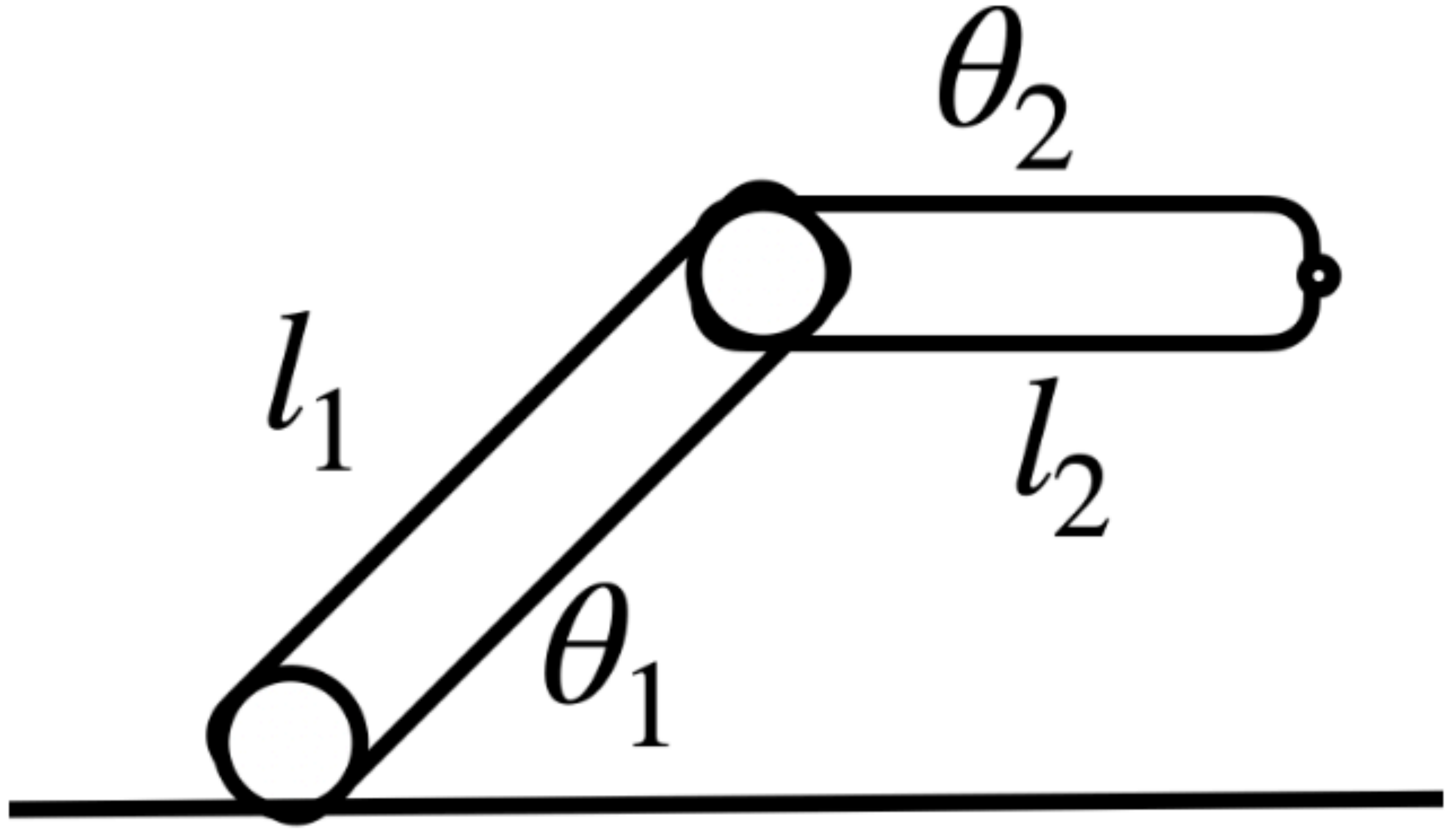


$$\theta = \theta_1 + \theta_2$$

$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

Practice

A 2-link robot arm has joint angles, θ_1, and $\theta_2$and link lengths l_1, andl_2.
Compute the end-effector's position in the robot's task space.



Practice

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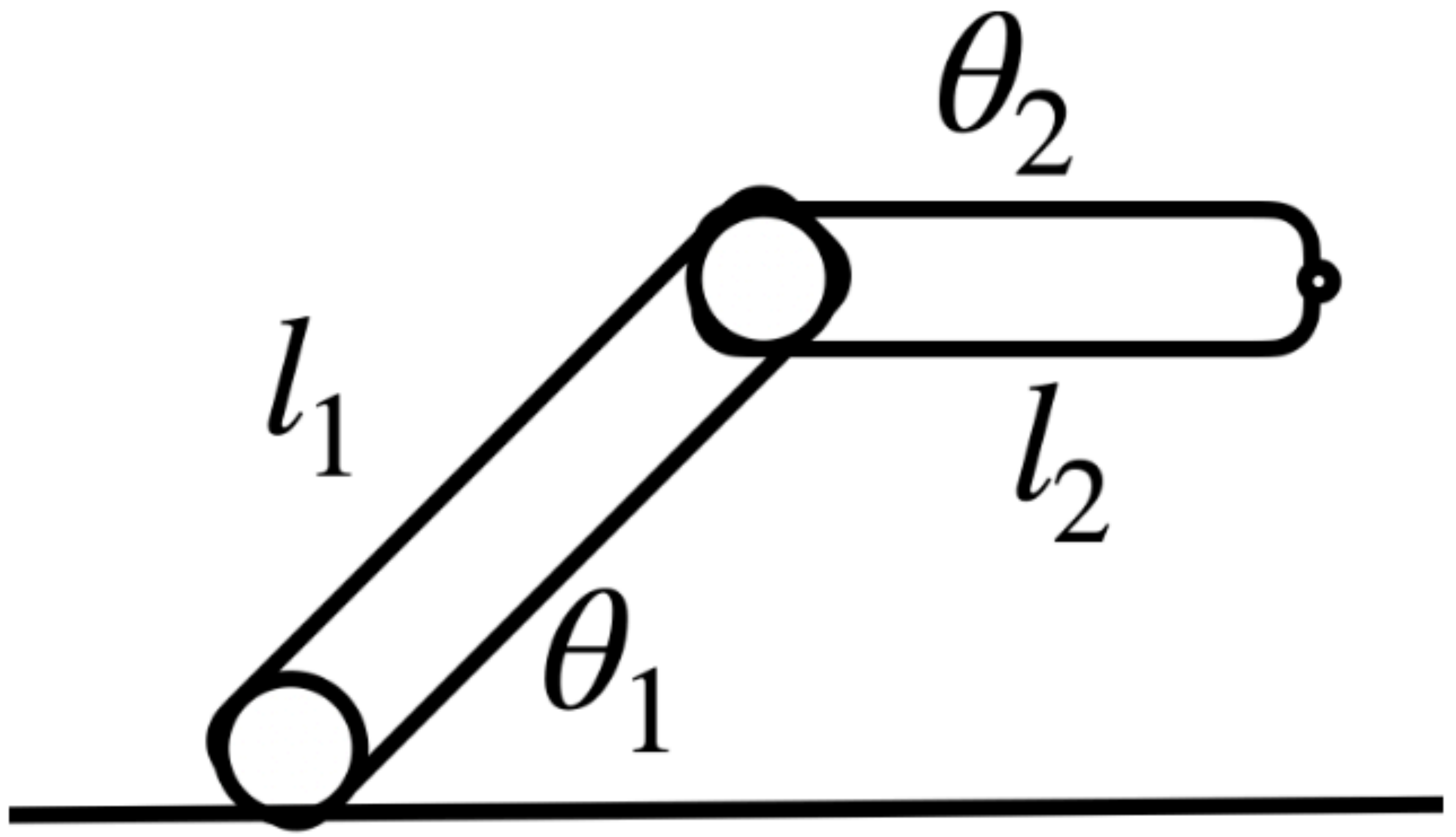
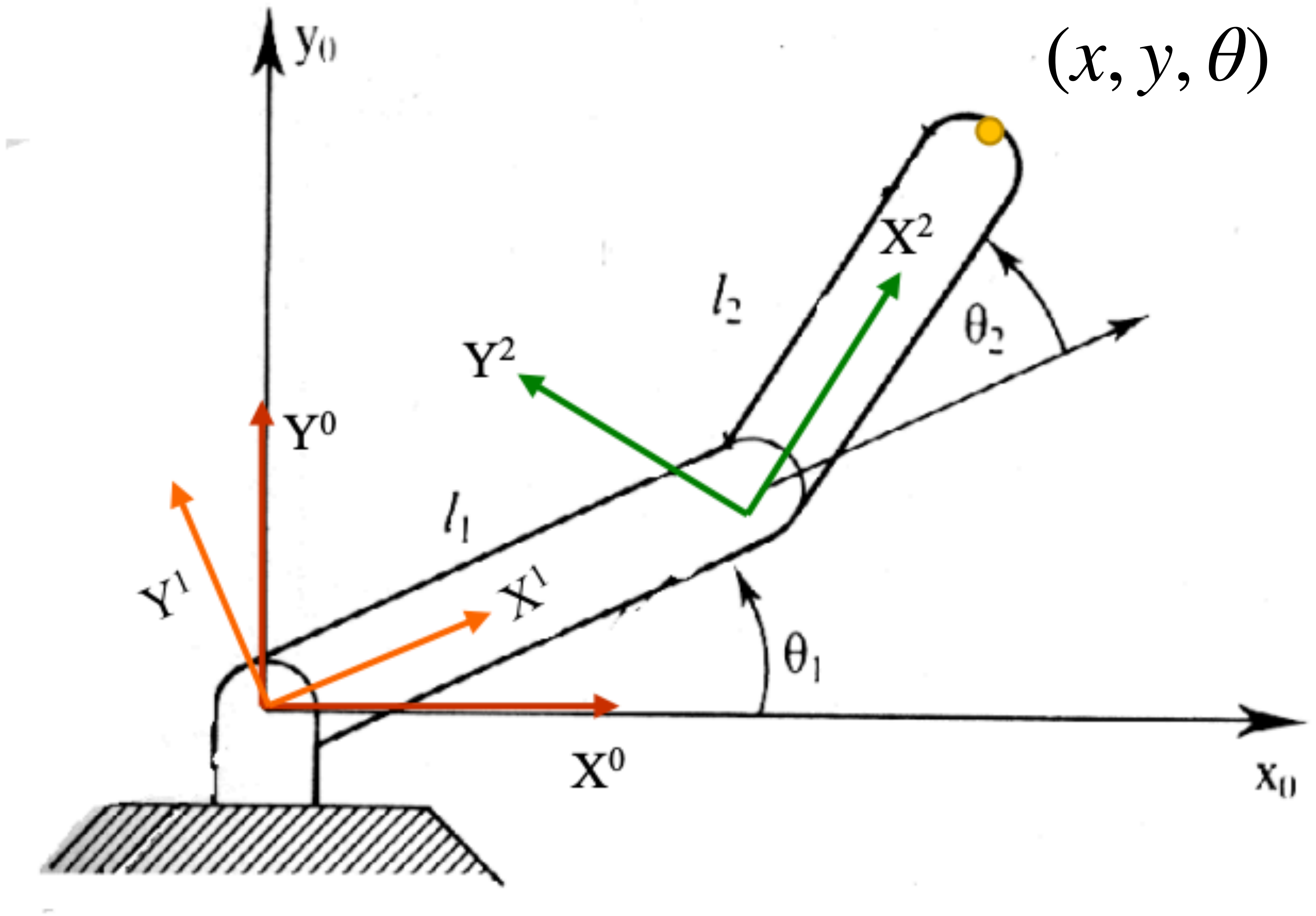


$$(x, y) = (l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2), l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2))$$

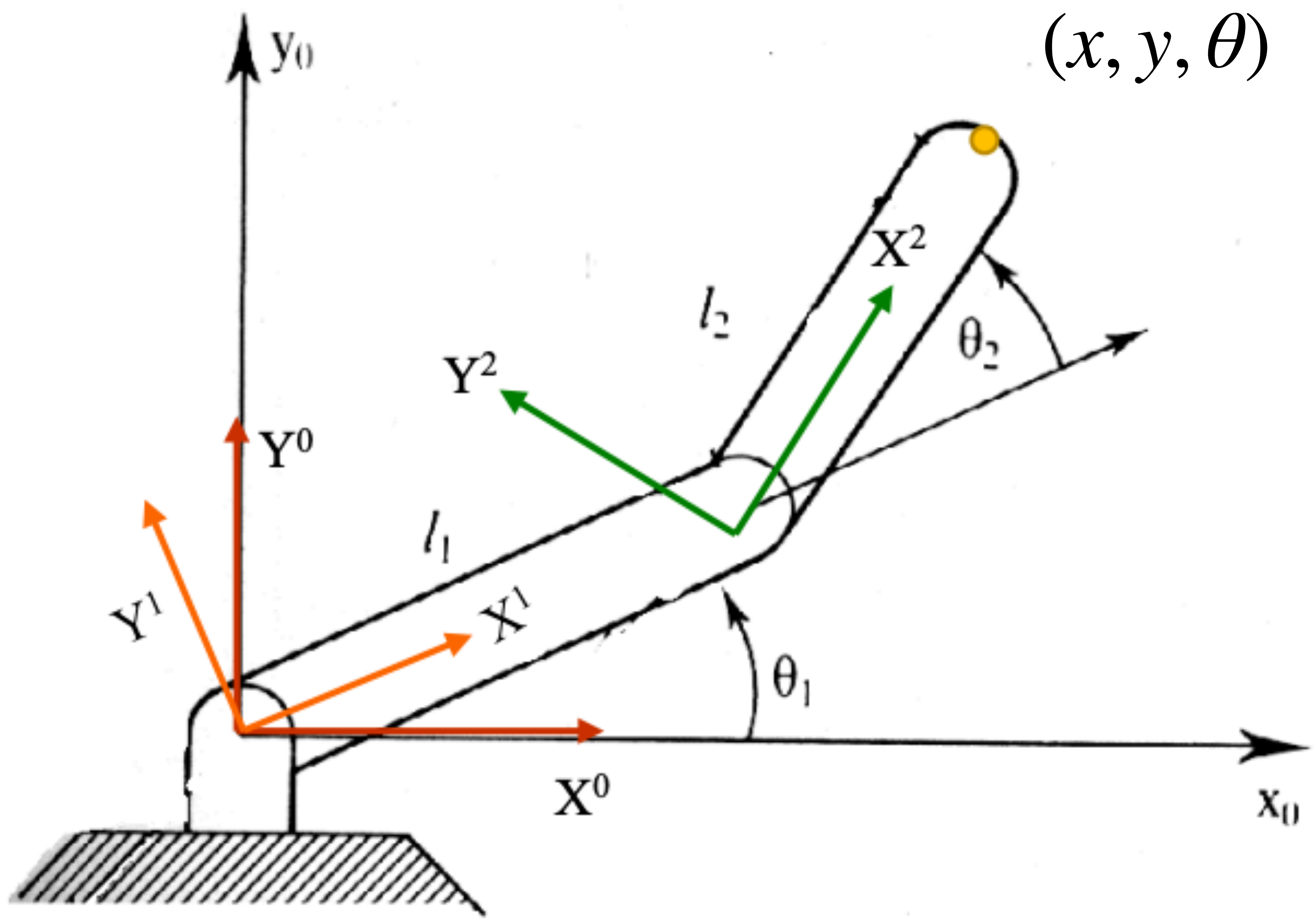
Arm Kinematics



- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix} \cos(\theta_1) - \sin(\theta_1) \\ \sin(\theta_1) \ \cos(\theta_1) \end{bmatrix}$.

$$\overline{V}^{X0Y0} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \overline{V}^{X1Y1}$$

Arm Kinematics

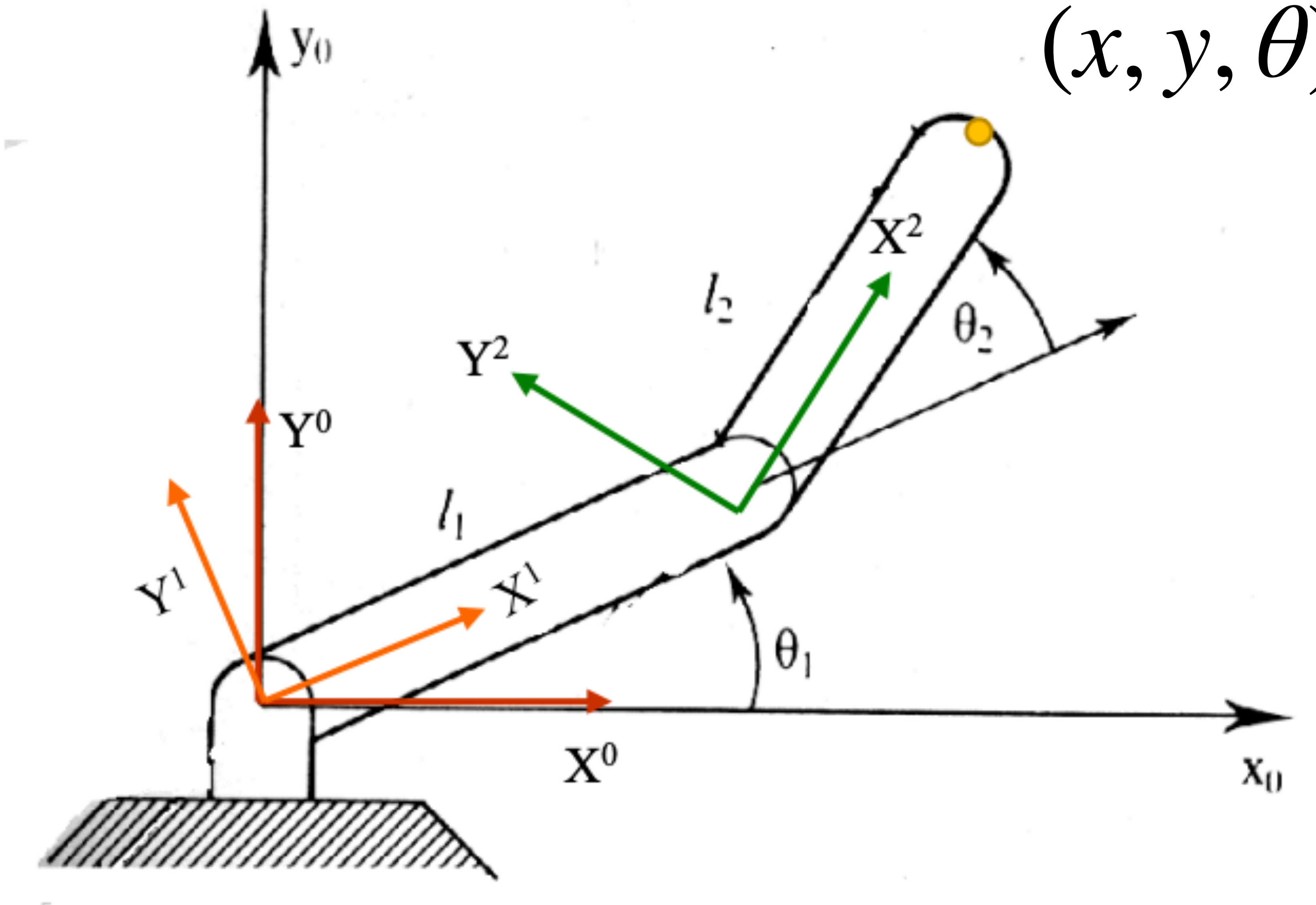


- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix} \cos(\theta_1) - \sin(\theta_1) \\ \sin(\theta_1) \ \cos(\theta_1) \end{bmatrix}$.

- In the X1Y1 frame, the X2X2 frame is at position $\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$ and orientation $\begin{bmatrix} \cos(\theta_2) - \sin(\theta_2) \\ \sin(\theta_2) \ \cos(\theta_2) \end{bmatrix}$.

$$\overline{V}^{X0Y0} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & cos\theta_1 \end{bmatrix} \left(\begin{bmatrix} l1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \overline{V}^{X2Y2} \right)$$

Arm Kinematics



(x, y, θ)

- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$.

- In the X1Y1 frame, the X2X2 frame is at position $\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$ and orientation $\begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$.

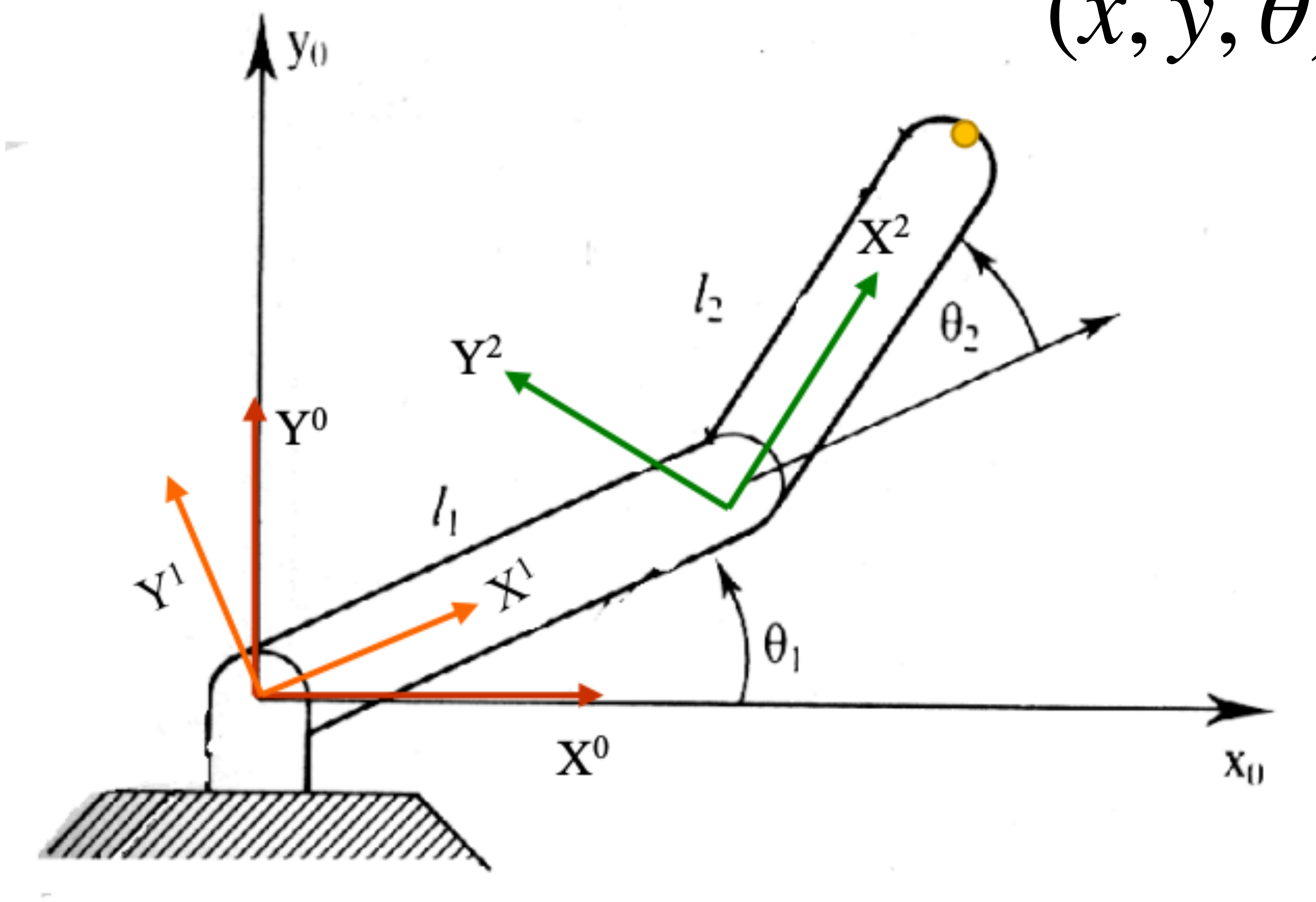
- In the X2Y2 frame, the end effector is at position $\begin{bmatrix} l_2 \\ 0 \end{bmatrix}$.

$$\overline{V}^{X0Y0} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & cos\theta_1 \end{bmatrix} \left(\begin{bmatrix} l_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \end{bmatrix} \right)$$

Arm Kinematics

Can write series of rotations and translations as a single matrix

(x, y, θ)



$c_{\alpha\beta} = \cos(\alpha + \beta)$

$$f(q) = \begin{bmatrix} c_{\alpha\beta} & -s_{\alpha\beta} & 0 & l_2 c_{\alpha\beta} + l_1 c_\alpha \\ s_{\alpha\beta} & c_{\alpha\beta} & 0 & l_2 s_{\alpha\beta} + l_1 s_\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

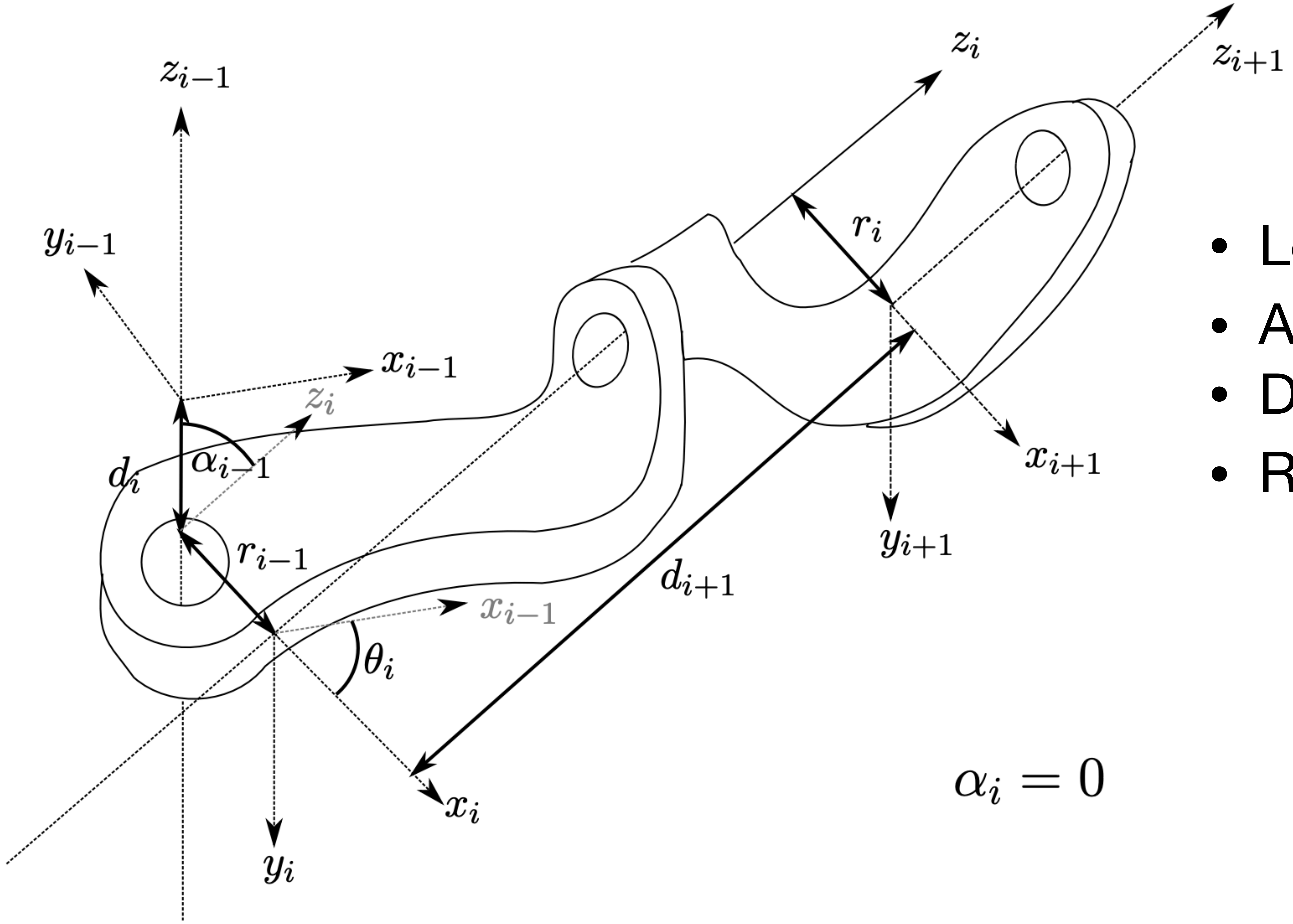
$s_{\alpha\beta} = \sin(\alpha + \beta)$

$$\overline{V}^{X0Y0} = f(q) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Denavit-Hartenberg (DH) Representation

- Forward kinematics is a chain of transformations applied to put the end-effector in a base frame.
- A reference frame is defined at each joint.
- DH Convention gives a basic recipe for defining the transformation from one joint to the next in terms of:
    - Length of the link between the joints.
    - The angle between the x-axis of the frames for each joint.
    - [In 3 dimensions]: the offset of the z-axis of the two joints and the angle between the z-axes.

Denavit-Hartenberg (DH) Representation



- Length r between z-axes of the joints.
- Angle α between z-axes of the joints.
- Distance d between z-axes of the joints.
- Rotation around common axis, θ

Credit: Modern Robotics

Denavit-Hartenberg (DH) Representation

- Can write a single matrix to transform a point in joint n's frame into joint n-1's frame.

$$
{}^{n}_{n-1}T = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & r_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & r_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & R & & t \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

- Let the point be $[x, y, z]$. Concatenate 1 to end to obtain $[x, y, z, 1]$ and then apply ${}^{n}_{n-1}T$.

$$
[x', y', z', 1] = {}^{n}_{n-1}T[x, y, z, 1]^{\top}
$$

DH Convention

DH convention provides a method for specifying a transform between a reference frame centered on one joint to that at another.

Let the transform at joint n be ${}^{n}_{n-1}T$.

$$
{}^{n}_{n-1}T = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & r_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & r_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & R & & t \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Transform for the entire arm is $F(q) = {}^{2}_{1}T\,{}^{3}_{2}T \cdots {}^{n}_{n-1}T$.

To rotate [x, y, z], we matrix multiply [x, y, z, 1] to obtain [x’, y’, z’, 1].

Forward Kinematics under DH Convention

$F(q) = {}^{2}_{1}T \cdots {}^{n}_{n-1}T.$

Note that F is a function of the robot's configuration, q.

End-effector position in end-effector frame is [0, 0, 0]

Putting this together, $f(q) = F(q) \cdot [0,0,0,1].$

Summary

- Defined basic kinematics terminology: degrees of freedom, joint, link.
- Introduced forward kinematics for a robot arm as a chain of transformations.
- Described the kinematics of a differential drive robot.

Action Items

- Planning reading for next week; send a reading response by 12 pm on Monday.
- SLAM assignment due next Thursday.
- Midterm: 2 weeks from today. Start studying!