Autonomous Robotics

Forward Kinematics

Josiah Hanna
University of Wisconsin — Madison

Homework 3

Questions?

Learning Outcomes

After today's lecture, you will:

- Be able to describe the forward kinematics process.
- Understand forward kinematics calculations as repeated frame transforms.

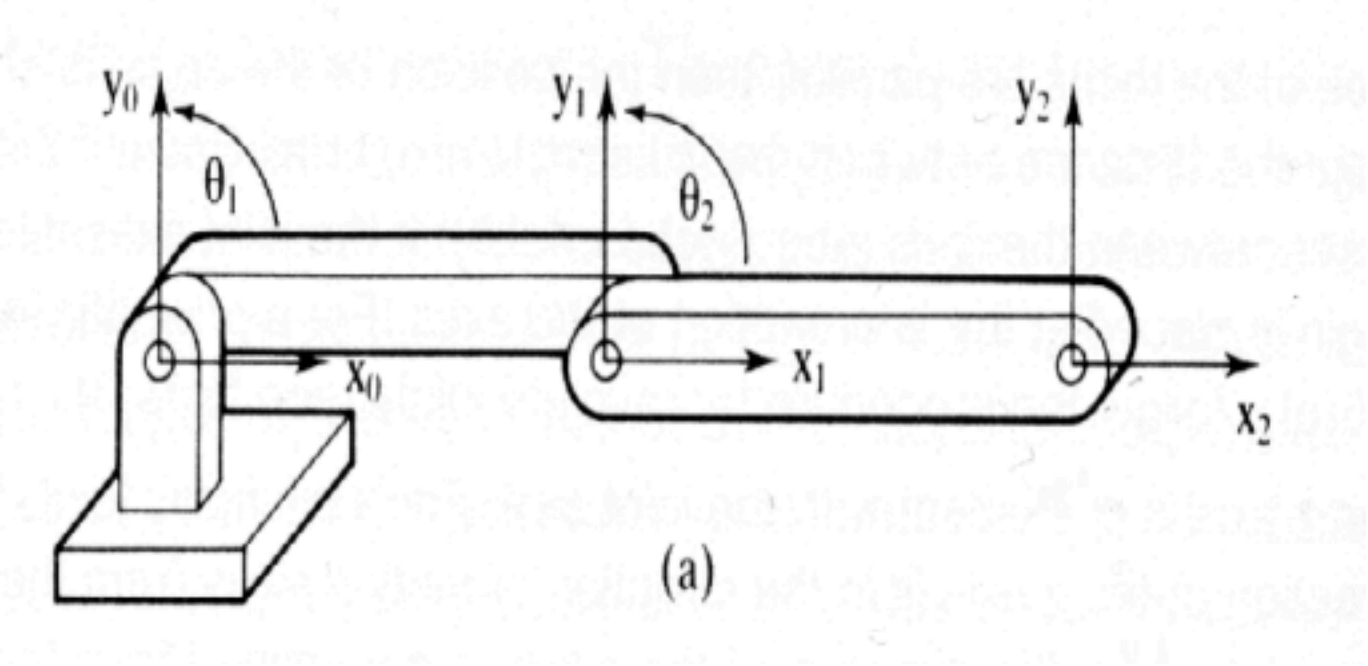
Degrees of Freedom

- The number of independent parameters that can fully define a robot's configuration.

- **Link:** Single rigid body.

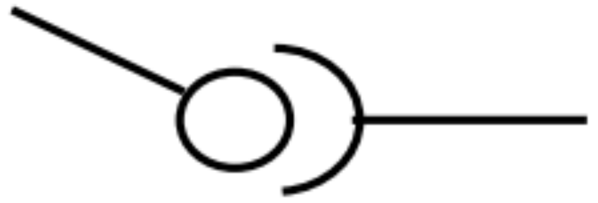
- **Joint:** connection between links

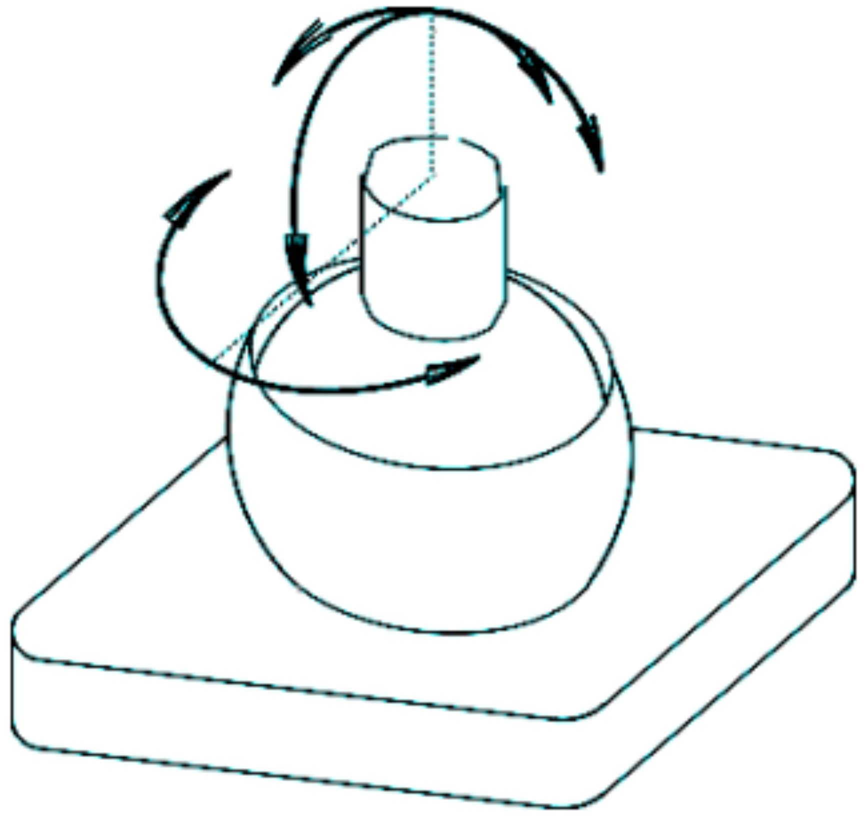
Joint Types



Revolute Joint 

1 DOF (Variable - Θ)



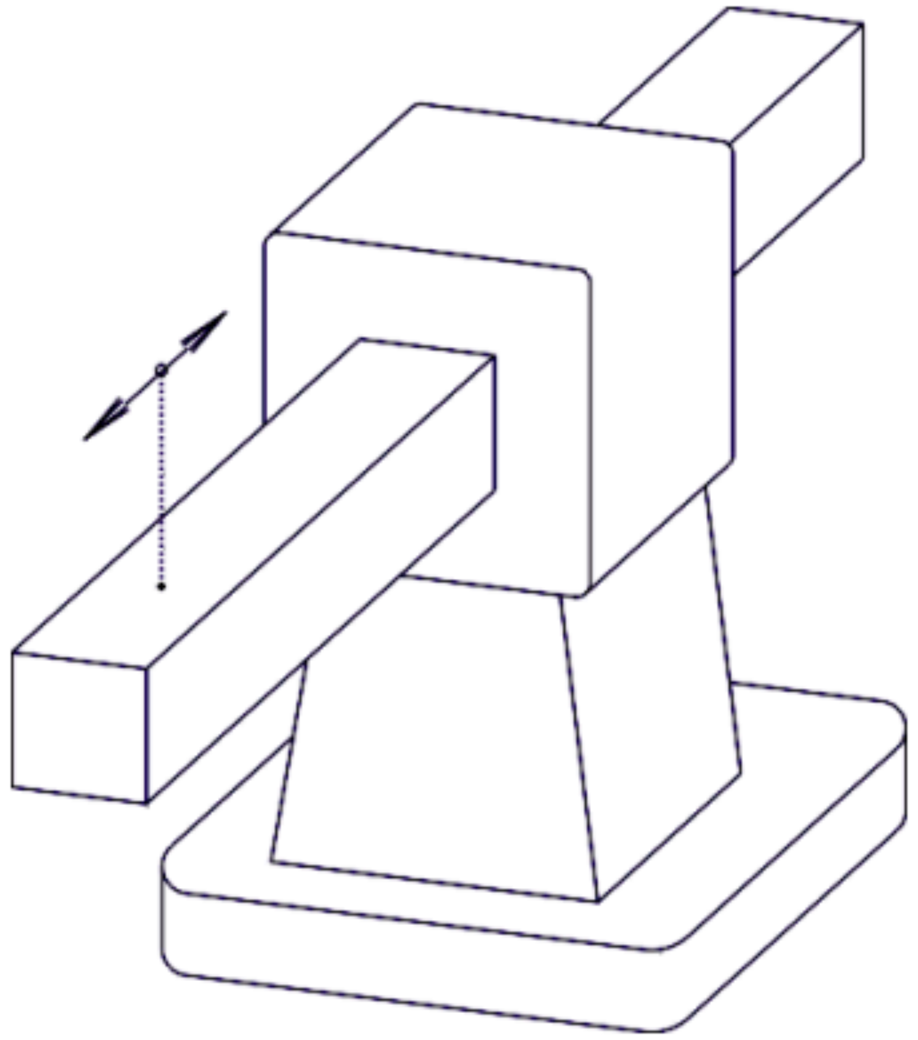


Spherical Joint

3 DOF (Variables - Θ_1, Θ_2, Θ_3)

Prismatic Joint 

1 DOF (linear) (Variables - d)



Forward Kinematics

Task space:

Position of a robot's end-effector. Assume $\mathbb{R}^n$.

Joint space:

Space of possible robot configurations (e.g., angle of all joints). Assume $\mathbb{R}^m$.

Forward kinematics is the mapping from joint space to task space:

$r = f(q)$, where $r \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$.

Given a robot's joint configuration, determine where its end-effector is relative to a base frame of reference. Why useful?

Reference Frame

Coordinate system from which rotations and translations are based.

Given a vector $x \in \mathbf{R}^d$:

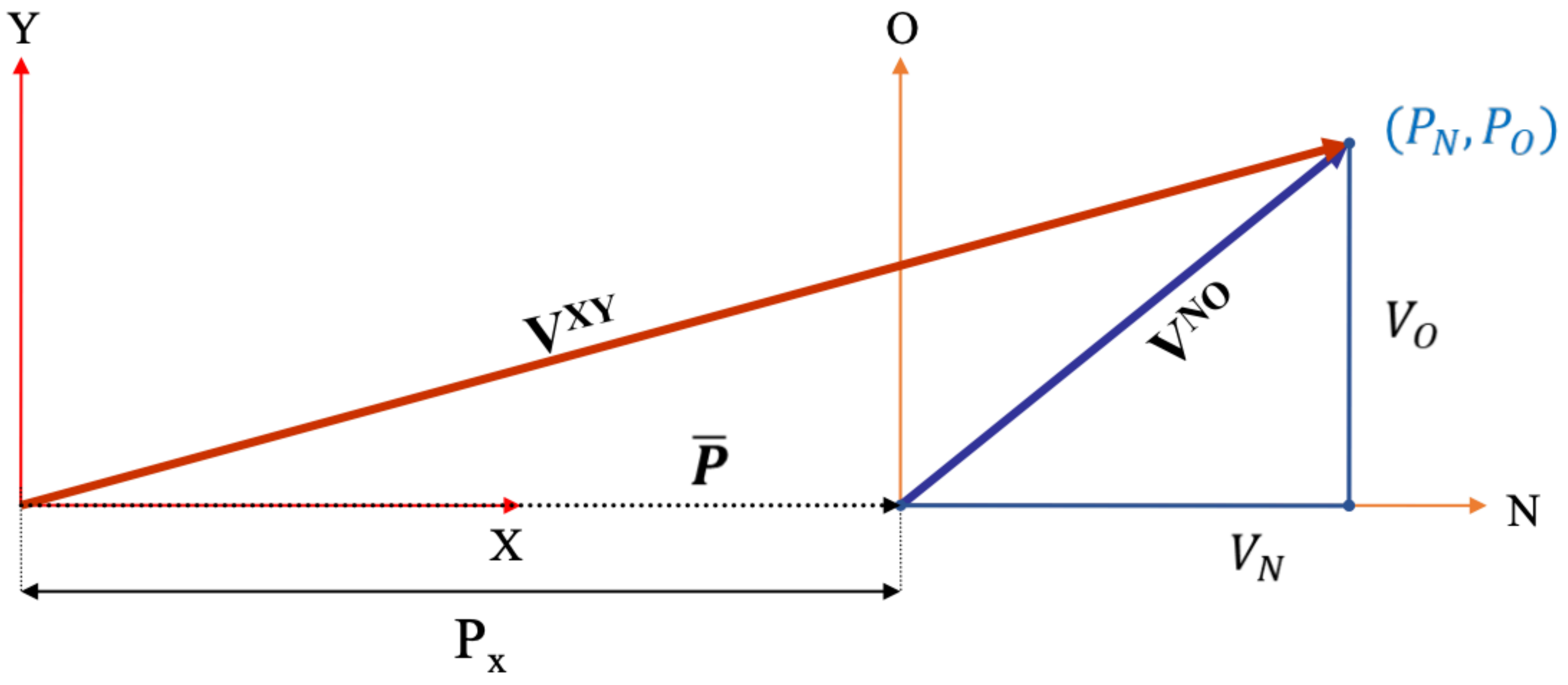
For $c \in \mathbf{R}^d$, we call $x + c$ a **translation** of x.

For $A \in \mathbf{R}^{d \times d}$, we call Ax a rotation of x.

Translating Frames

Given a vector $x \in \mathbf{R}^d$:

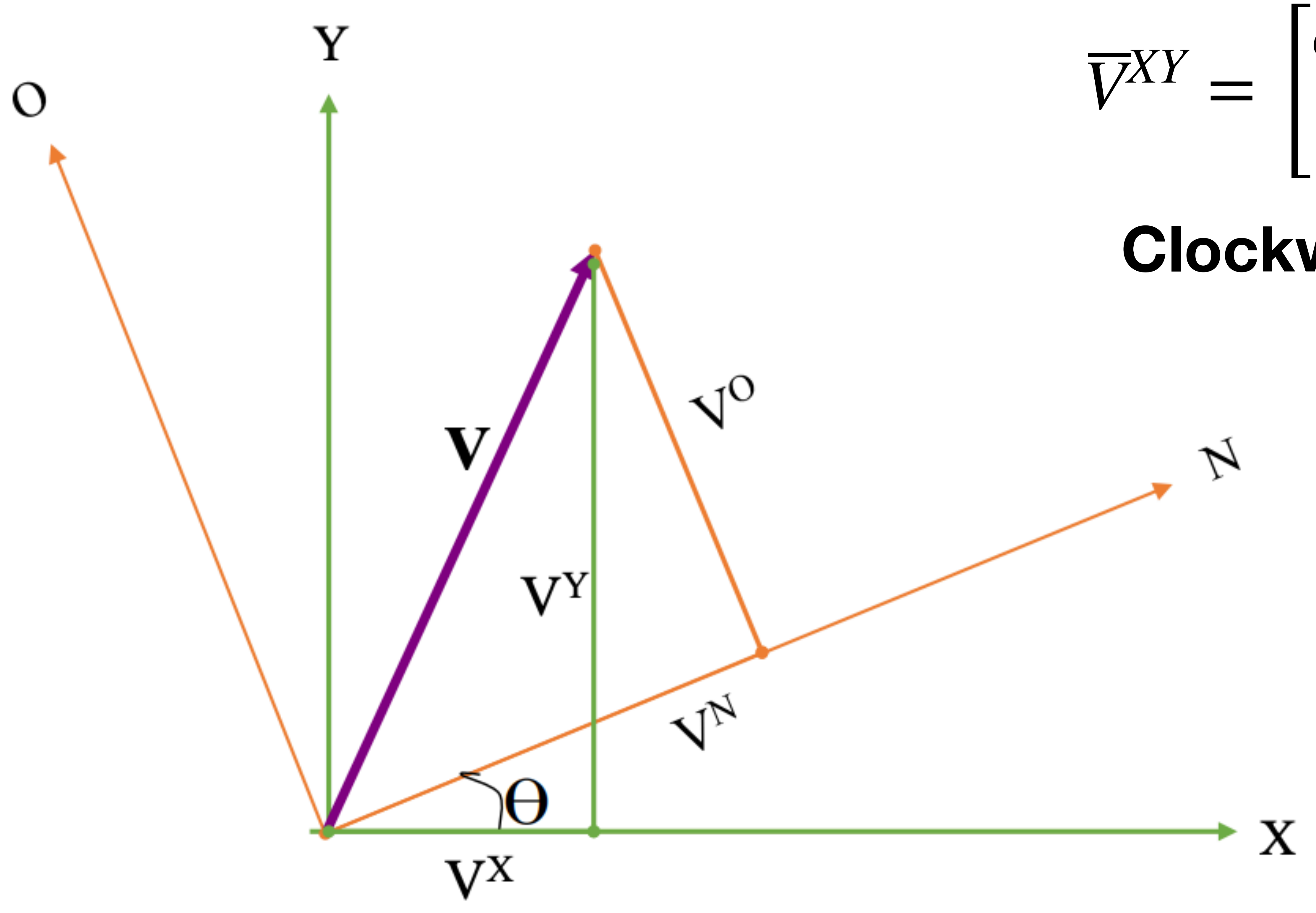
For $c \in \mathbf{R}^d$, we call $x + c$ a **translation** of x.



Rotating Frames

Given a vector $x \in \mathbf{R}^d$:

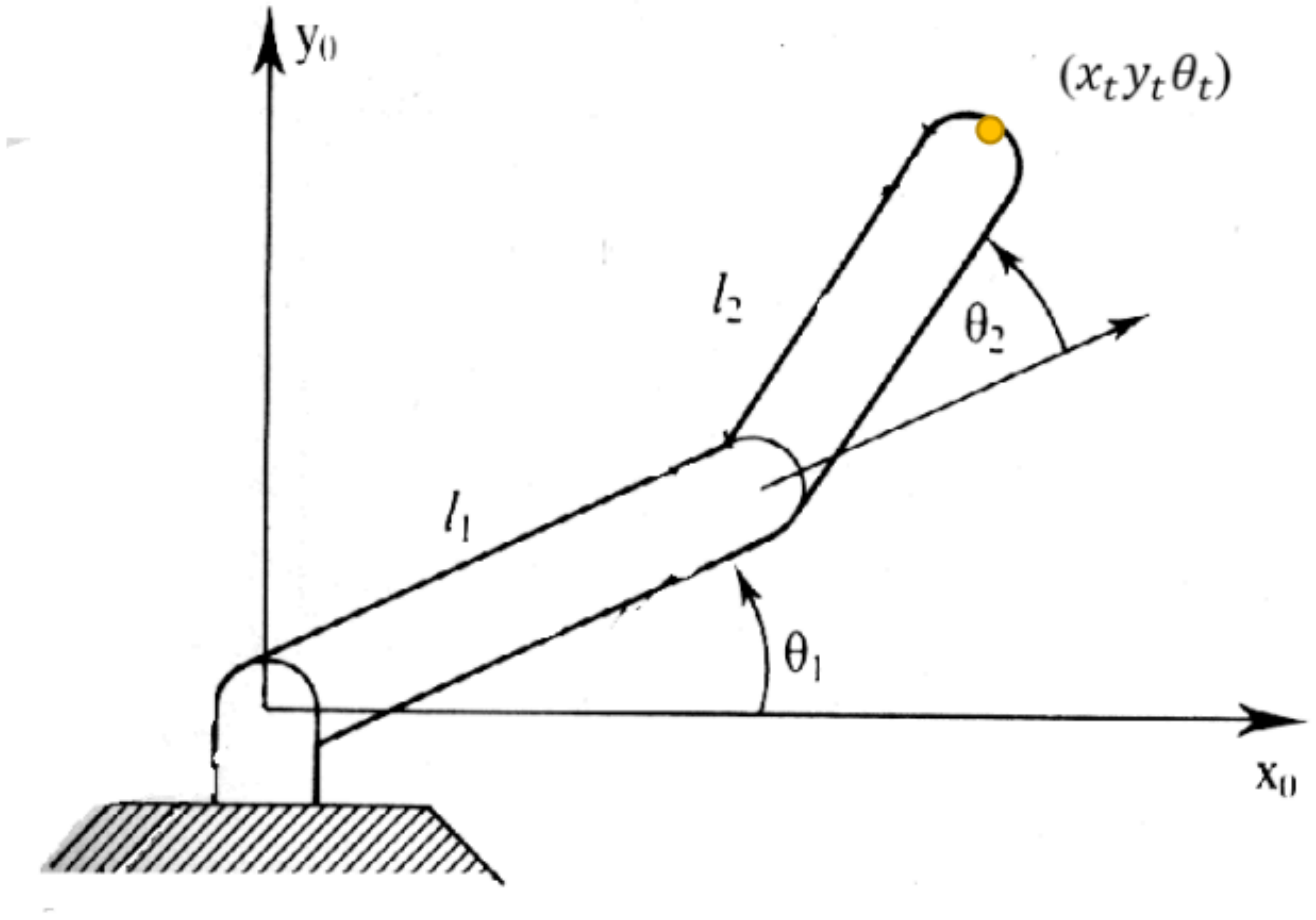
For $A \in \mathbf{R}^{d \times d}$, we call Ax a rotation of x.



$$\overline{V}^{XY} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \overline{V}^{NO}$$

Clockwise rotation matrix

Arm Kinematics

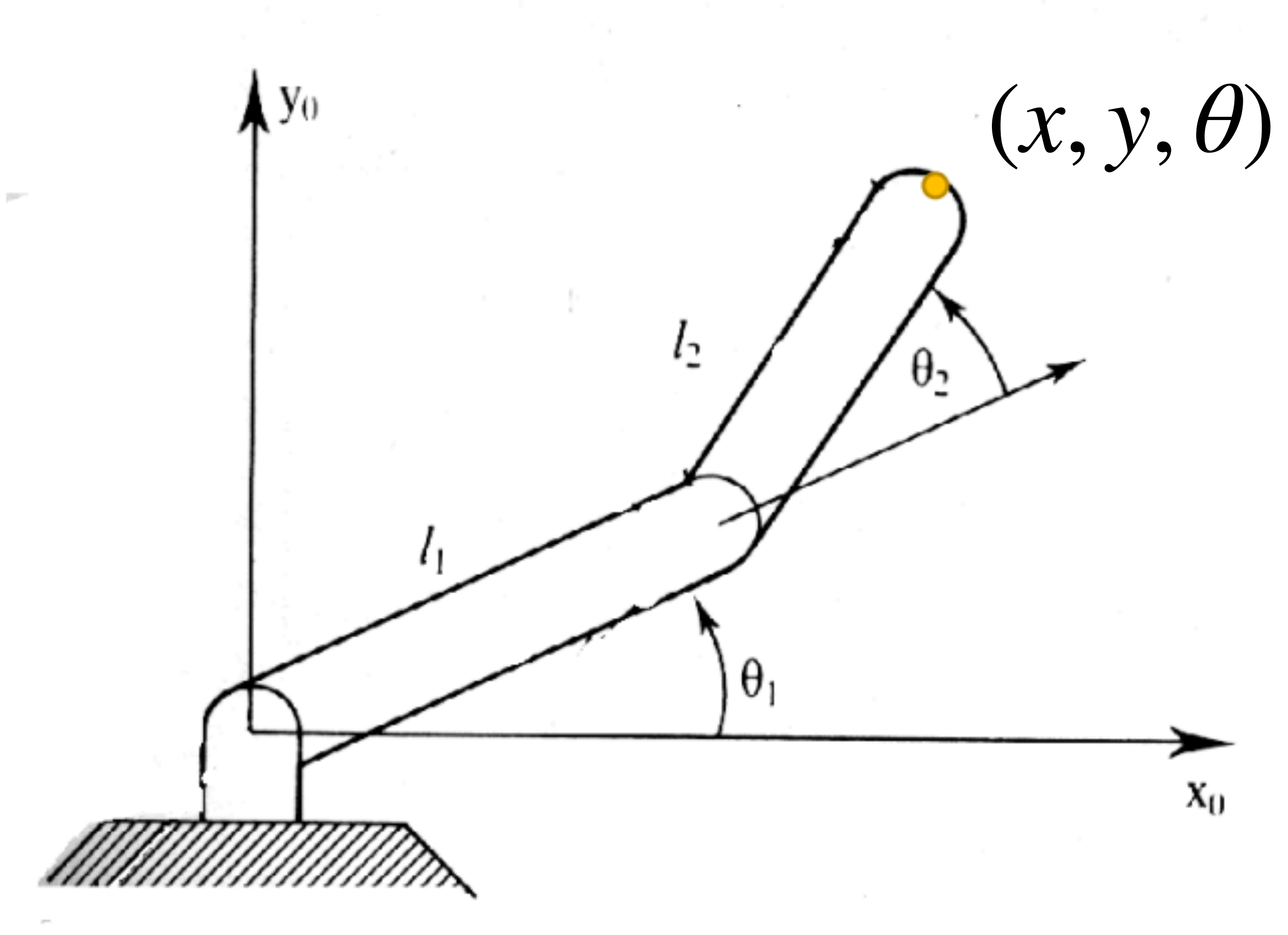


Set up:

- You have a two-link arm with base at the origin.
- The first link has length l_1 and is at angle θ_1.
- The second link has length l_2 and is at angle θ_2.

What is the position and orientation of the end-effector?

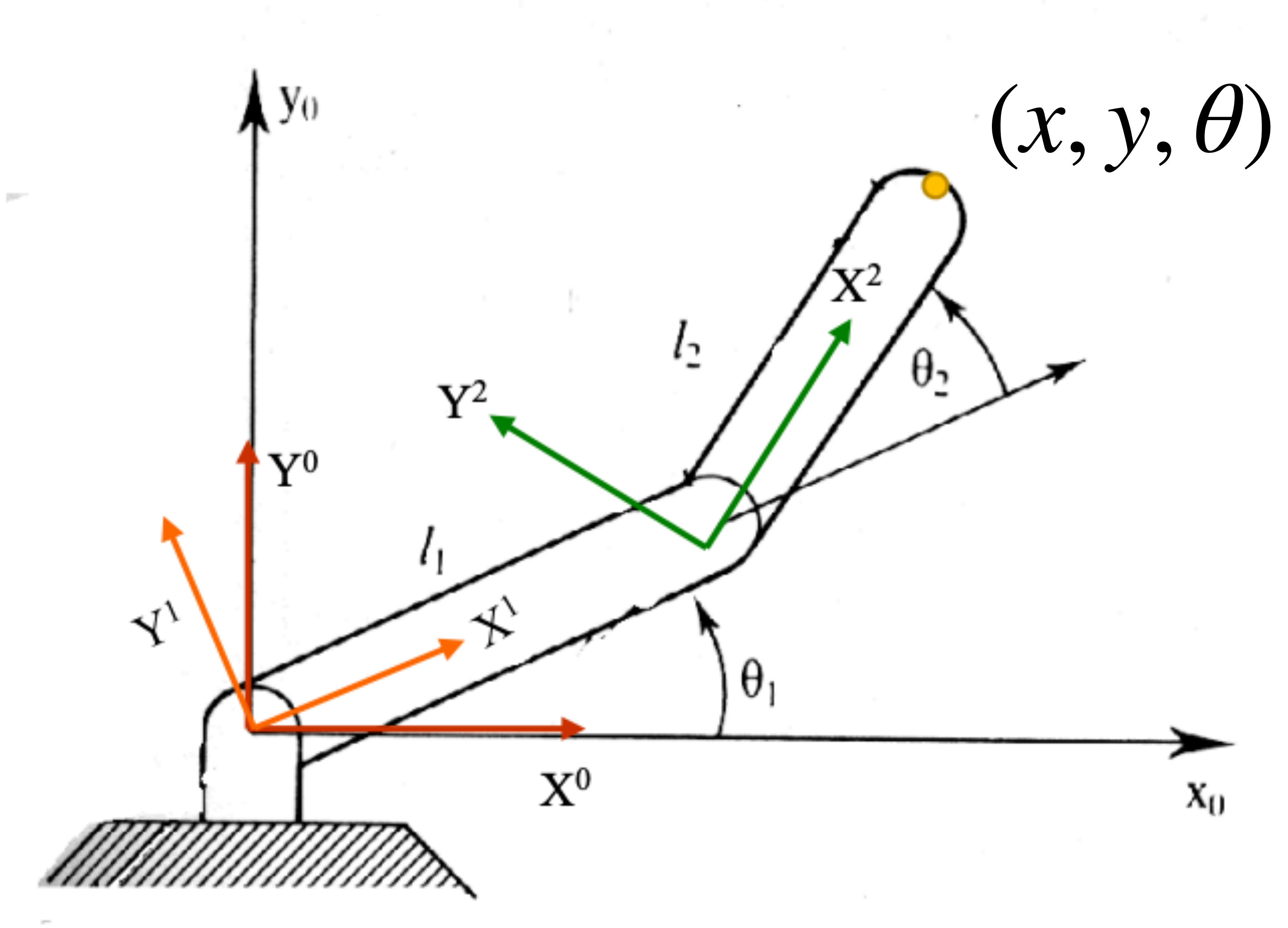
Arm Kinematics: Direct Approach



$$\theta = \theta_1 + \theta_2$$

$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

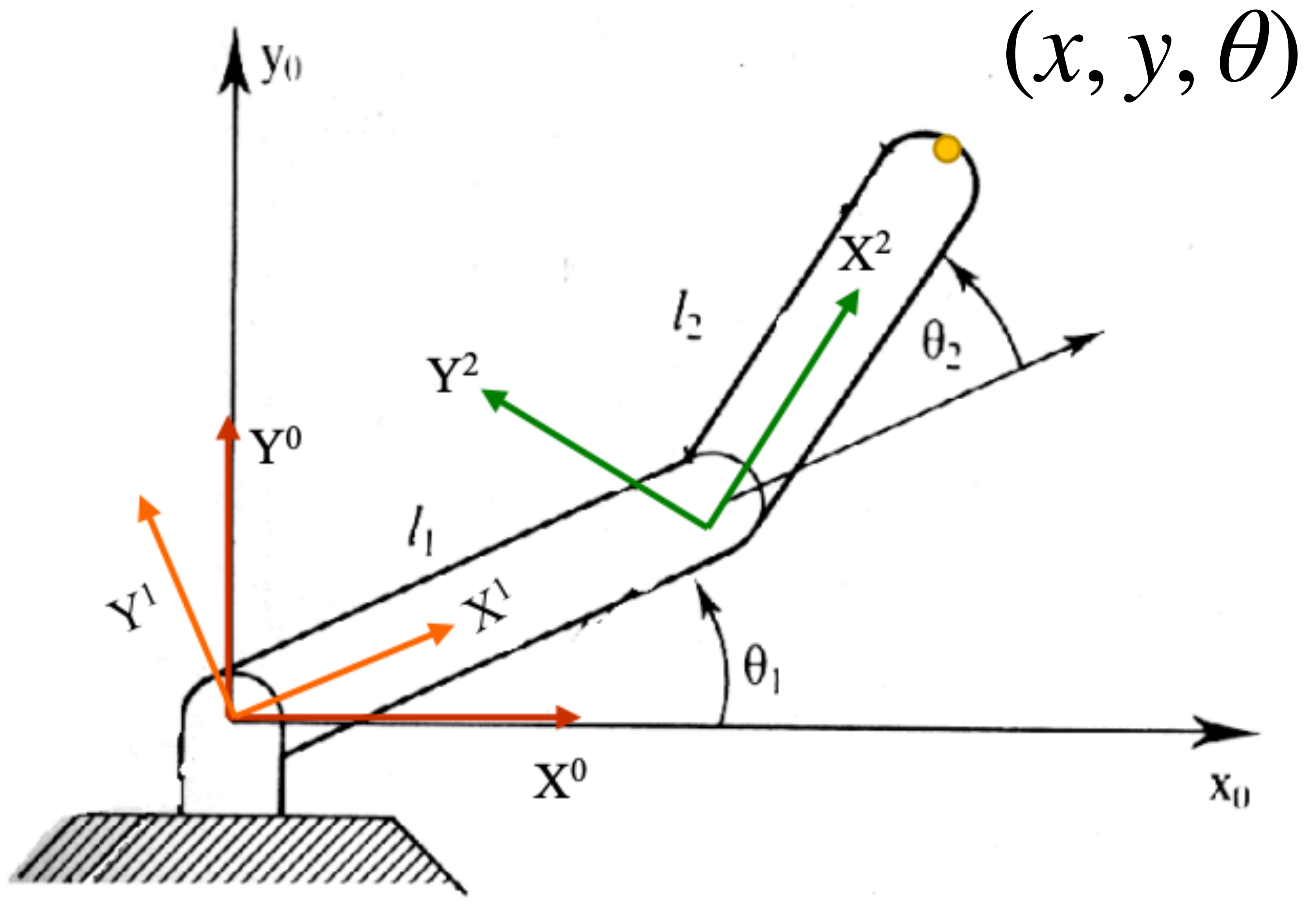
Arm Kinematics



- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$.

$$\overline{V}^{X0Y0} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \overline{V}^{X1Y1}$$

Arm Kinematics

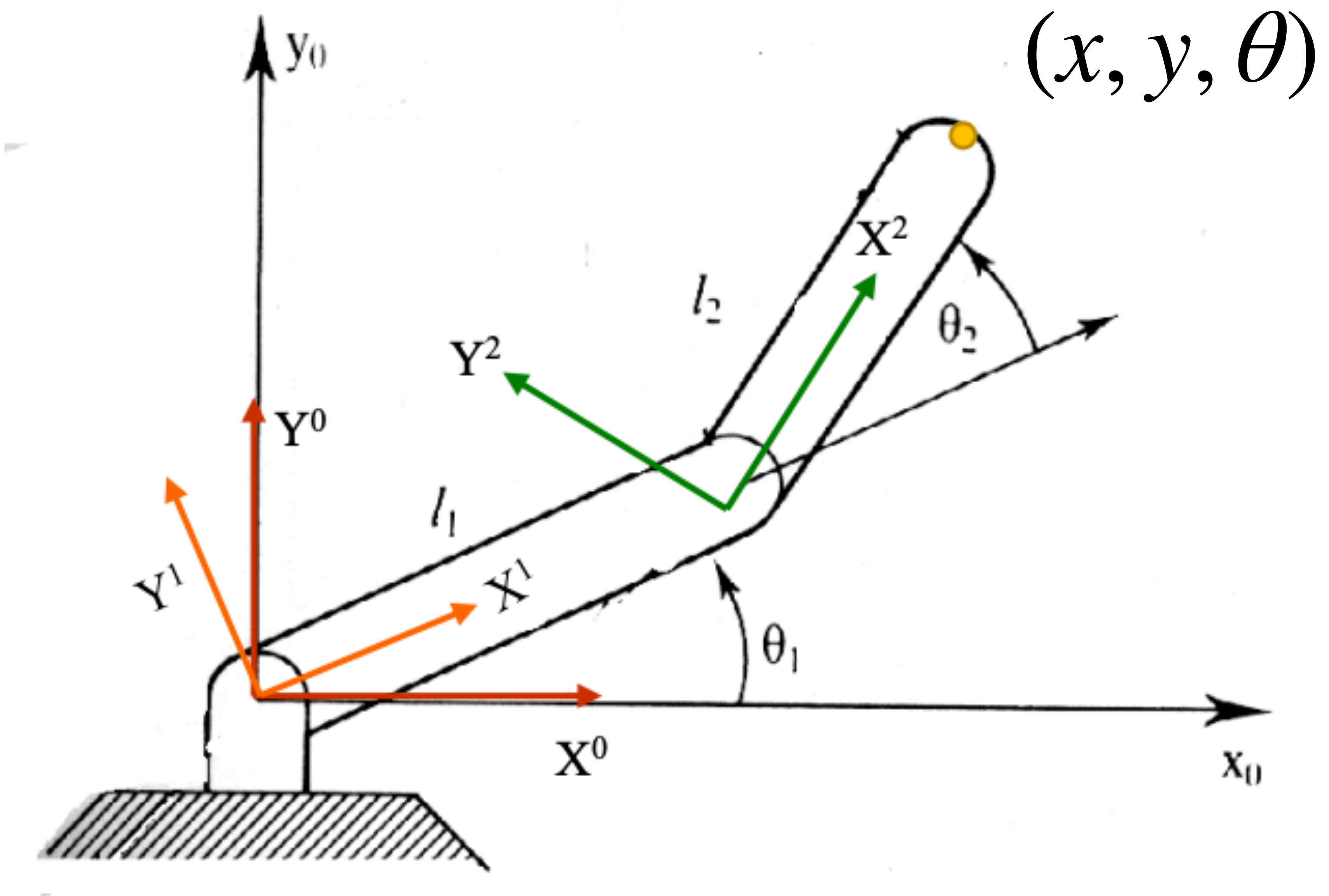


- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$.

- In the X1Y1 frame, the X2X2 frame is at position $\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$ and orientation $\begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$.

$$\overline{V}^{X0Y0} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & cos\theta_1 \end{bmatrix} \left(\begin{bmatrix} l1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \overline{V}^{X2Y2} \right)$$

Arm Kinematics



- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix} \cos(\theta_1) - \sin(\theta_1) \\ \sin(\theta_1)\ \cos(\theta_1) \end{bmatrix}$.

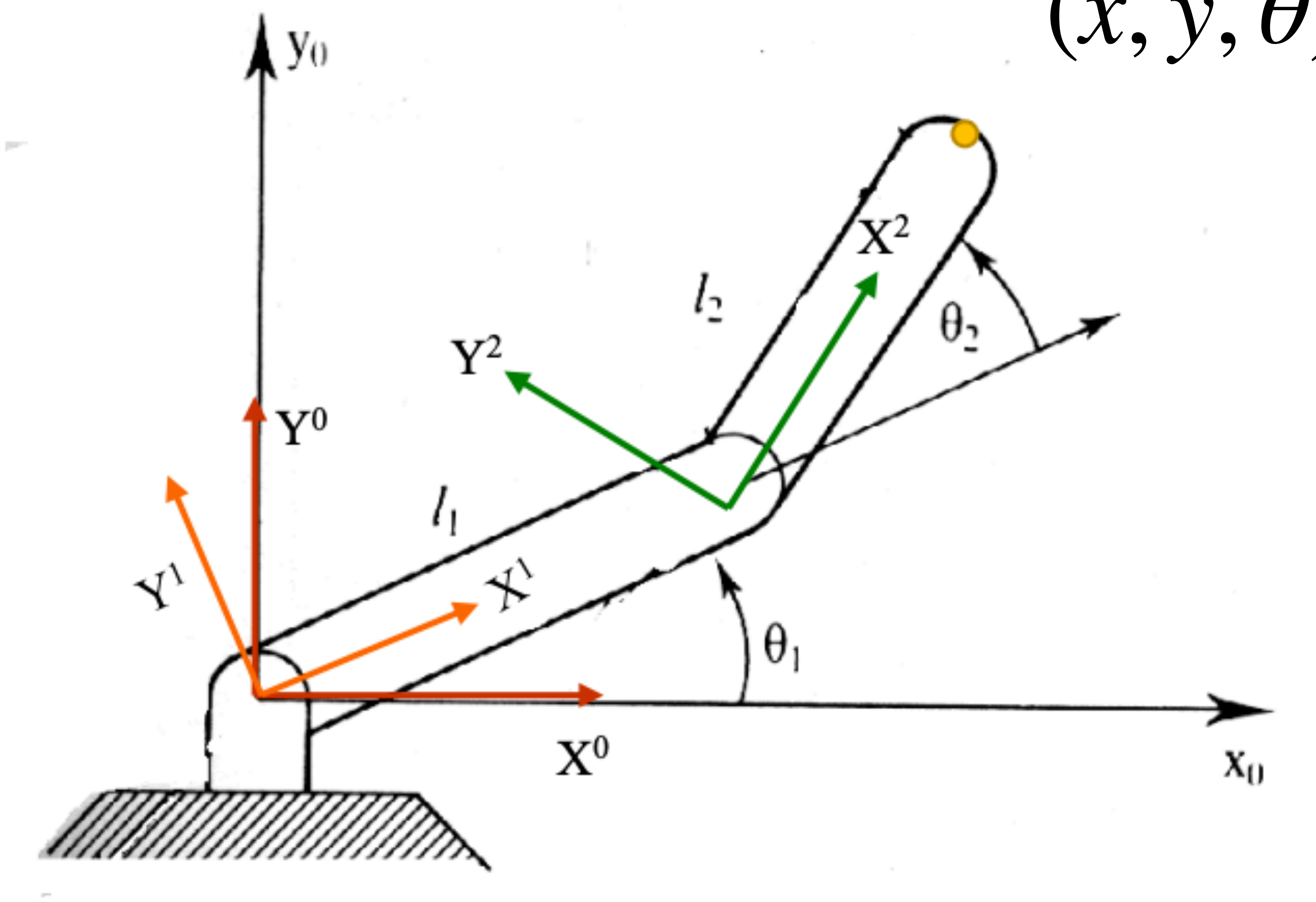
- In the X1Y1 frame, the X2X2 frame is at position $\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$ and orientation $\begin{bmatrix} \cos(\theta_2) - \sin(\theta_2) \\ \sin(\theta_2)\ \cos(\theta_2) \end{bmatrix}$.

- In the X2Y2 frame, the end effector is at position $\begin{bmatrix} l_2 \\ 0 \end{bmatrix}$.

$$\overline{V}^{X0Y0} = \begin{bmatrix} cos(\theta_1) & -\sin\theta_1 \\ \sin\theta_1 & cos\theta_1 \end{bmatrix} \left(\begin{bmatrix} l_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \end{bmatrix} \right)$$

Arm Kinematics

Can write series of rotations and translations as a single matrix



(x, y, θ)

$c_\theta = \cos\theta$

$$f(q) = \begin{bmatrix} c_{\alpha\beta} & -s_{\alpha\beta} & 0 & l_2 c_{\alpha\beta} + l_1 c_\alpha \\ s_{\alpha\beta} & c_{\alpha\beta} & 0 & l_2 s_{\alpha\beta} + l_1 s_\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$s_\theta = \sin\theta$

$$\overline{V}^{X0Y0} = f(q) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Denavit-Hartenberg (DH) Representation

- Forward kinematics is a chain of transformations applied to put the end-effector in a base frame.
- A reference frame is defined at each joint.
- DH Convention gives a basic recipe for defining the transformation from one joint to the next in terms of:
  - Length of the link between the joints.
  - The angle between the x-axis of the frames for each joint.
  - [In 3 dimensions]: the offset of the z-axis of the two joints and the angle between the z-axes.

Denavit-Hartenberg (DH) Representation

- Can write a single matrix to transform a point in joint n's frame into joint n-1's frame.

$$
{}^{n}_{n-1}T = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n \cos\alpha_n & \sin\theta_n \sin\alpha_n & r_n \cos\theta_n \\ \sin\theta_n & \cos\theta_n \cos\alpha_n & -\cos\theta_n \sin\alpha_n & r_n \sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & R & & t \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

- Let the point be $[x, y, z]$. Concatenate 1 to end to obtain $[x, y, z, 1]$ and then apply ${}^{n}_{n-1}T$.

$$
[x', y', z', 1] = {}^{n}_{n-1}T[x, y, z, 1]^{\top}
$$

Differential Kinematics

- Relate velocity of end-effector to velocity of joints.
- Forward kinematics: $r = f(q)$.
- Velocity: $\dot{r} = J(q) \cdot [\dot{q}_1, \ldots \dot{q}_n]$ where $J(q)$ is the Jacobian of the robot's end-effector with respect to its configuration.

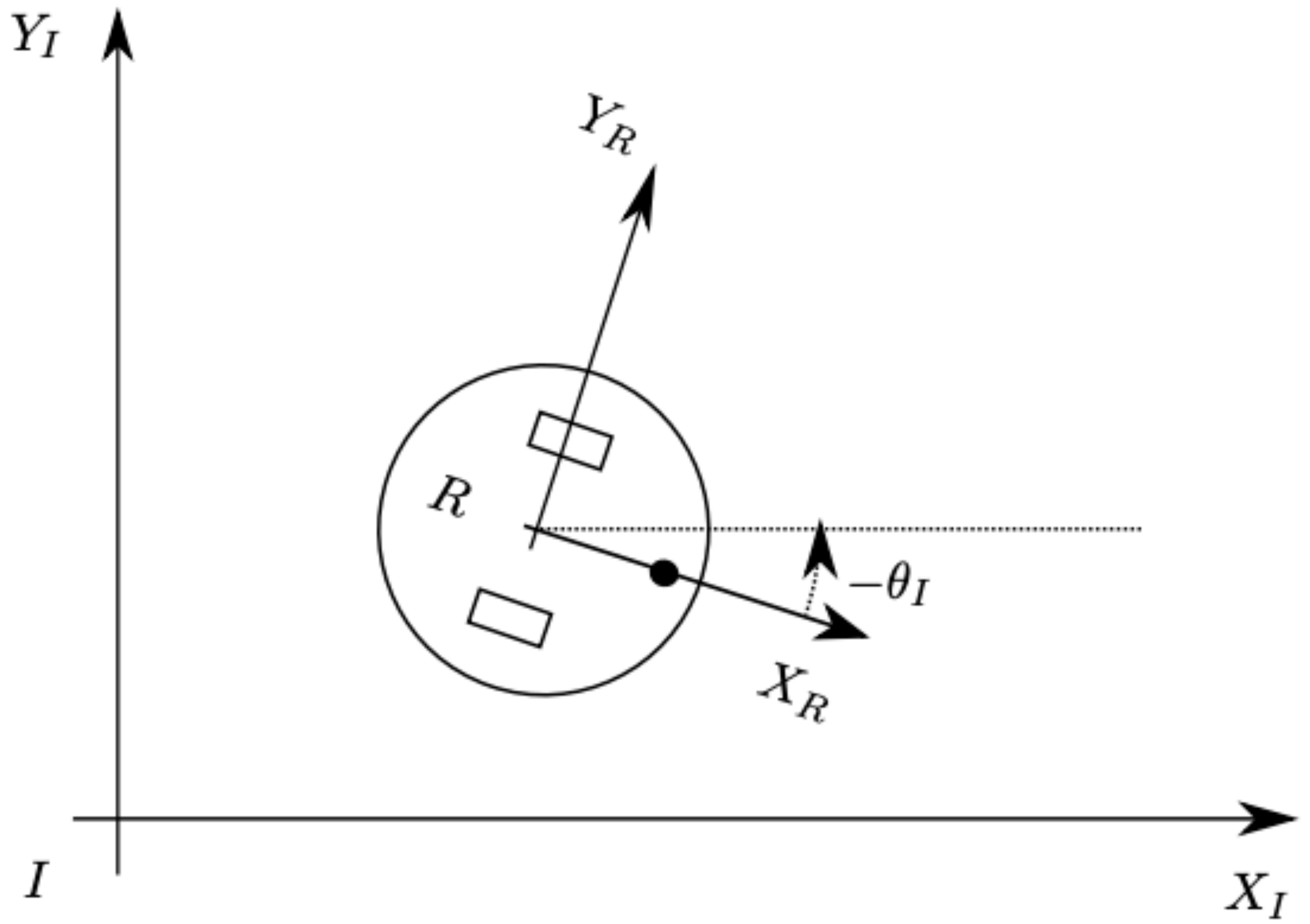
$$J = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \cdots & \frac{\partial y}{\partial q_n} \\ \frac{\partial \theta}{\partial q_1} & \frac{\partial \theta}{\partial q_2} & \cdots & \frac{\partial \theta}{\partial q_n} \end{bmatrix}$$

Differential Drive Kinematics

- Differential drive: two independently controlled wheels. Why useful?

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{\dot{\phi}_r r}{d} - \frac{\dot{\phi}_l r}{d} \end{bmatrix}$$

r: wheel radius
$\dot{\phi}_l$: left wheel velocity
$\dot{\phi}_r$: right wheel velocity
d: distance between wheels



- Holonomic vs. non-holonomic: configuration determines a unique position in task space.

Odometry

- Calculating pose of robot in inertial frame.

$$\begin{bmatrix} x_I(T) \\ y_I(T) \\ \theta(T) \end{bmatrix} = \int_0^T \begin{bmatrix} \dot{x}_I(t) \\ \dot{y}_I(t) \\ \dot{\theta}(t) \end{bmatrix} dt \approx \sum_{k=0}^{k=T} \begin{bmatrix} \Delta x_I(k) \\ \Delta y_I(k) \\ \Delta \theta(k) \end{bmatrix} \Delta t$$

- Approximation leads to errors in calculation ("drift").

Summary

- Defined basic kinematics terminology: degrees of freedom, joint, link.
- Introduced forward kinematics for a robot arm as a chain of transformations.
- Described the kinematics of a differential drive robot.

Action Items

- Planning reading for next week; send a reading response by 12 pm on Monday.
- SLAM assignment due Thursday.
- Midterm: 1 week from today