Autonomous Robotics

Kinematics of Wheeled Robots

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Announcements

- Homework #3 has been released. **New due date**: Thursday of next week.
- Reading assignment for next week (motion planning) due Monday at noon.
- Midterm in 2 weeks.

Learning Outcomes

After today's lecture, you will be able to:

- Define and compute differential kinematics.
- Derive the differential kinematics of mobile wheeled robots.

Forward Kinematics

Task space:

Position of a robot's end-effector. Assume $\mathbb{R}^n$.

Joint (or configuration) space:

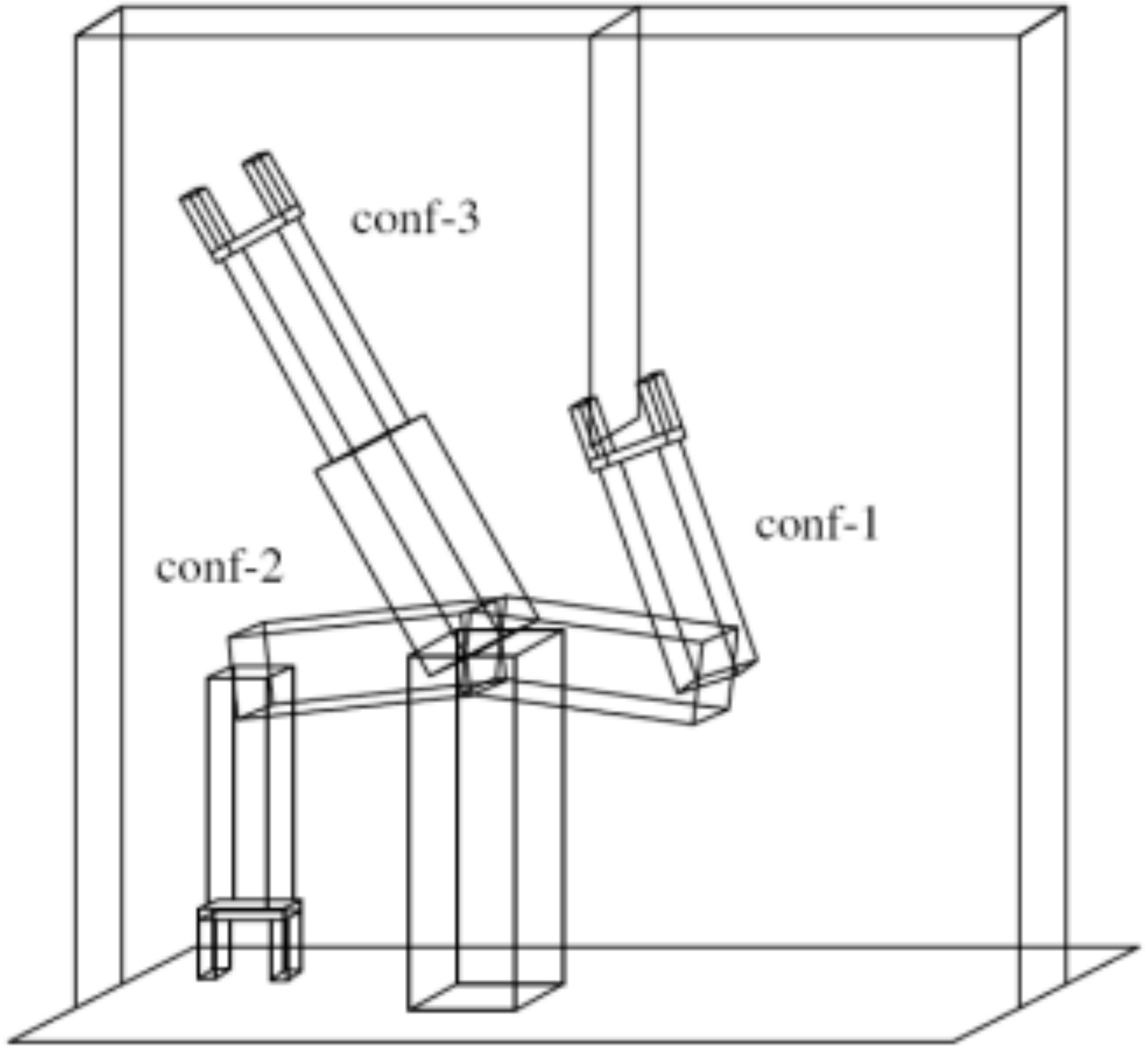
Space of possible robot configurations (e.g., angle of all joints). Assume $\mathbb{R}^m$.

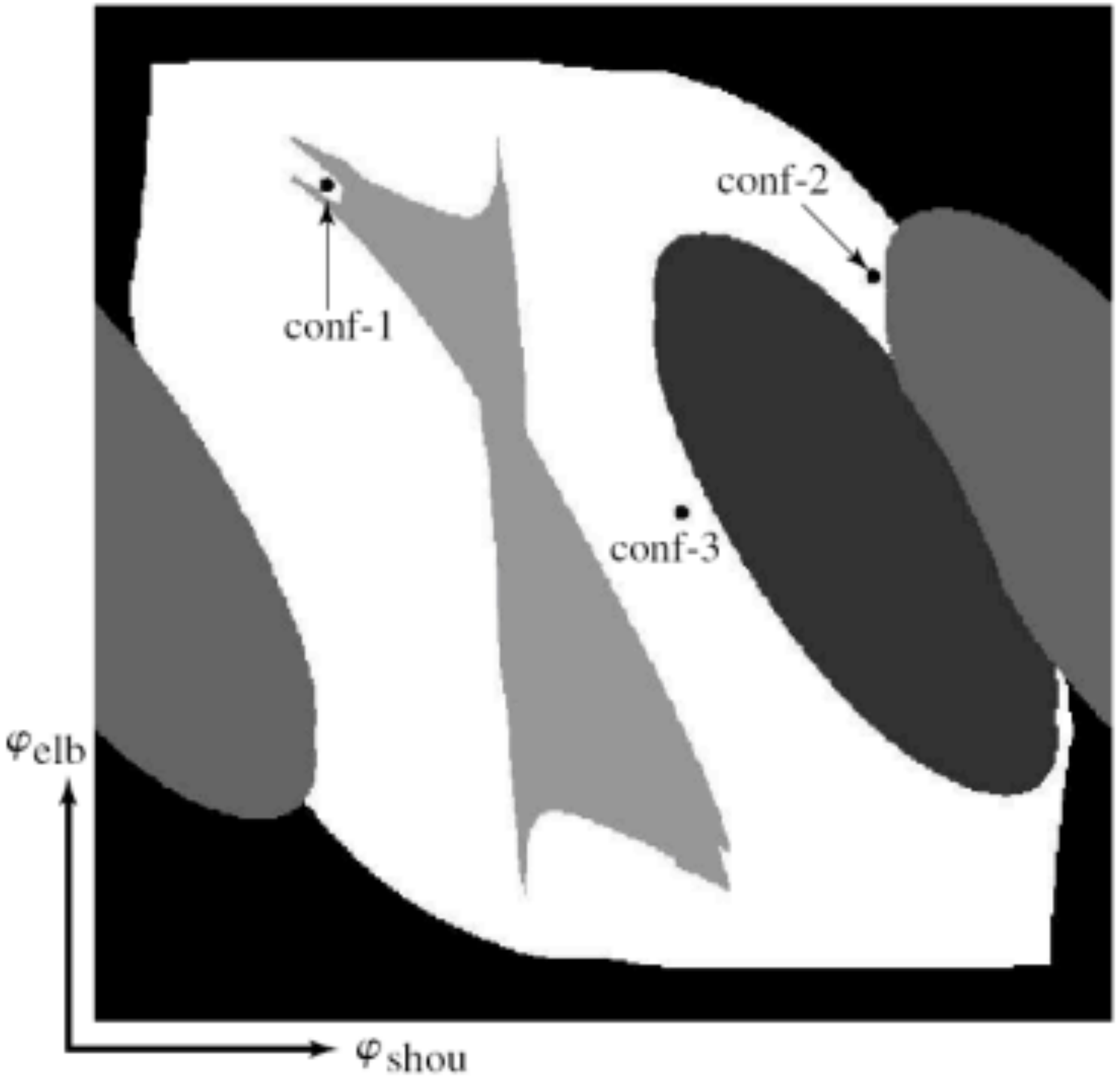
Forward kinematics is the mapping from joint space to task space:

$r = f(q)$, where $r \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$.

Given a robot's joint configuration, determine where its end-effector is relative to a base frame of reference.

Task vs Joint Space





DH Convention

DH convention provides a method for specifying a transform between a reference frame centered on one joint to that at another.

Let the transform at joint n be ${}^{n}_{n-1}T$.

$$
{}^{n}_{n-1}T = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & r_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & r_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & R & & t \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Transform for the entire arm is $F(q) = {}^{2}_{1}T\,{}^{3}_{2}T \cdots {}^{n}_{n-1}T$.

To rotate [x, y, z], we matrix multiply [x, y, z, 1] to obtain [x', y', z', 1].

Forward Kinematics under DH Convention

$F(q) = {}^{2}_{1}T \cdots {}^{n}_{n-1}T.$

Note that F is a function of the robot's configuration, q.

End-effector position in end-effector frame is [0, 0, 0]

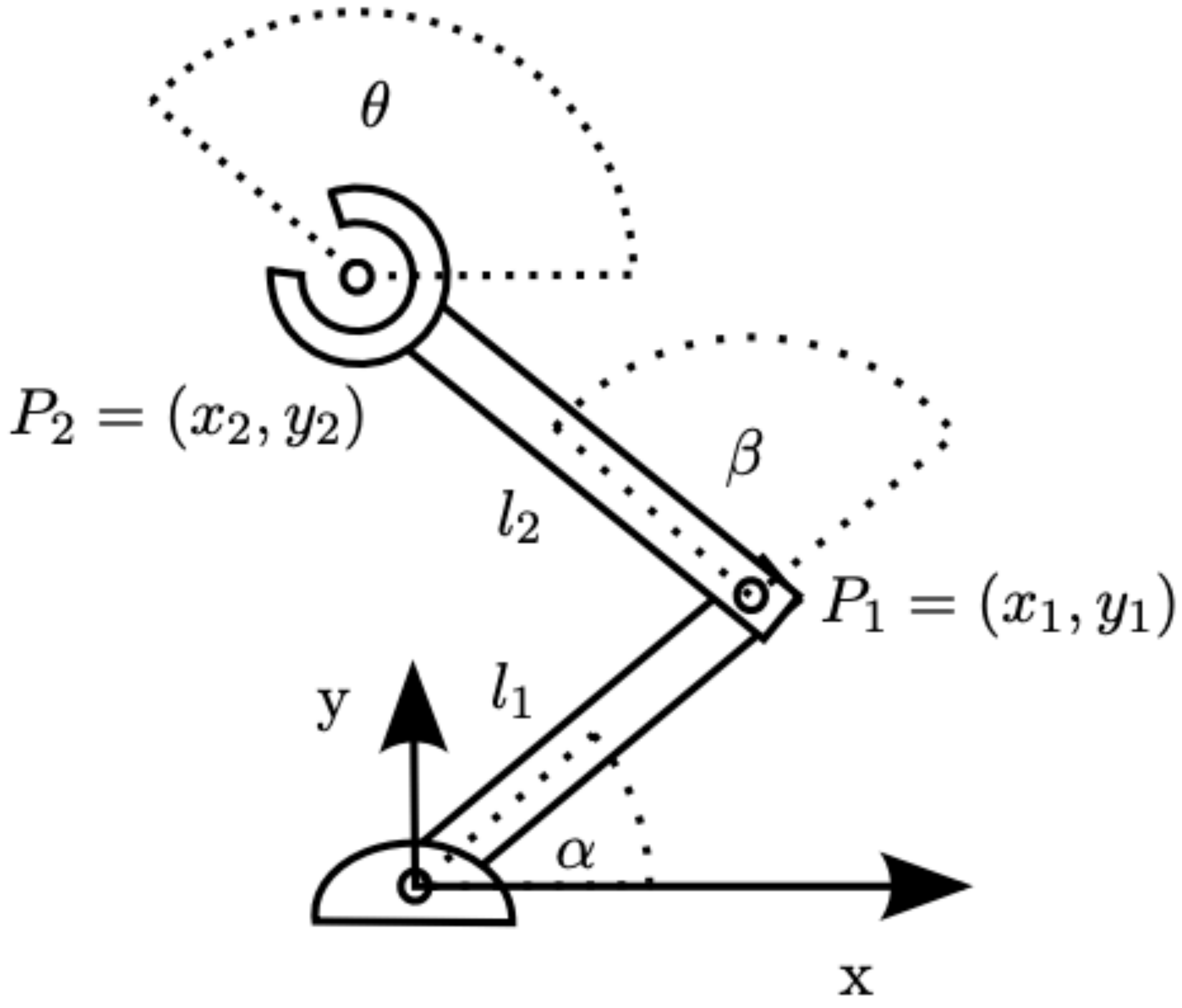
Putting this together, the forward kinematics for the end-effector is given by:
$f(q) = F(q) \cdot [0,0,0,1].$

Kinematics Practice

Consider the two-link robot arm shown here.

What are the transformation matrices for each of the 3 joints pictured here?

Hint: you can ignore the z-axis so the matrix should be 3x3.



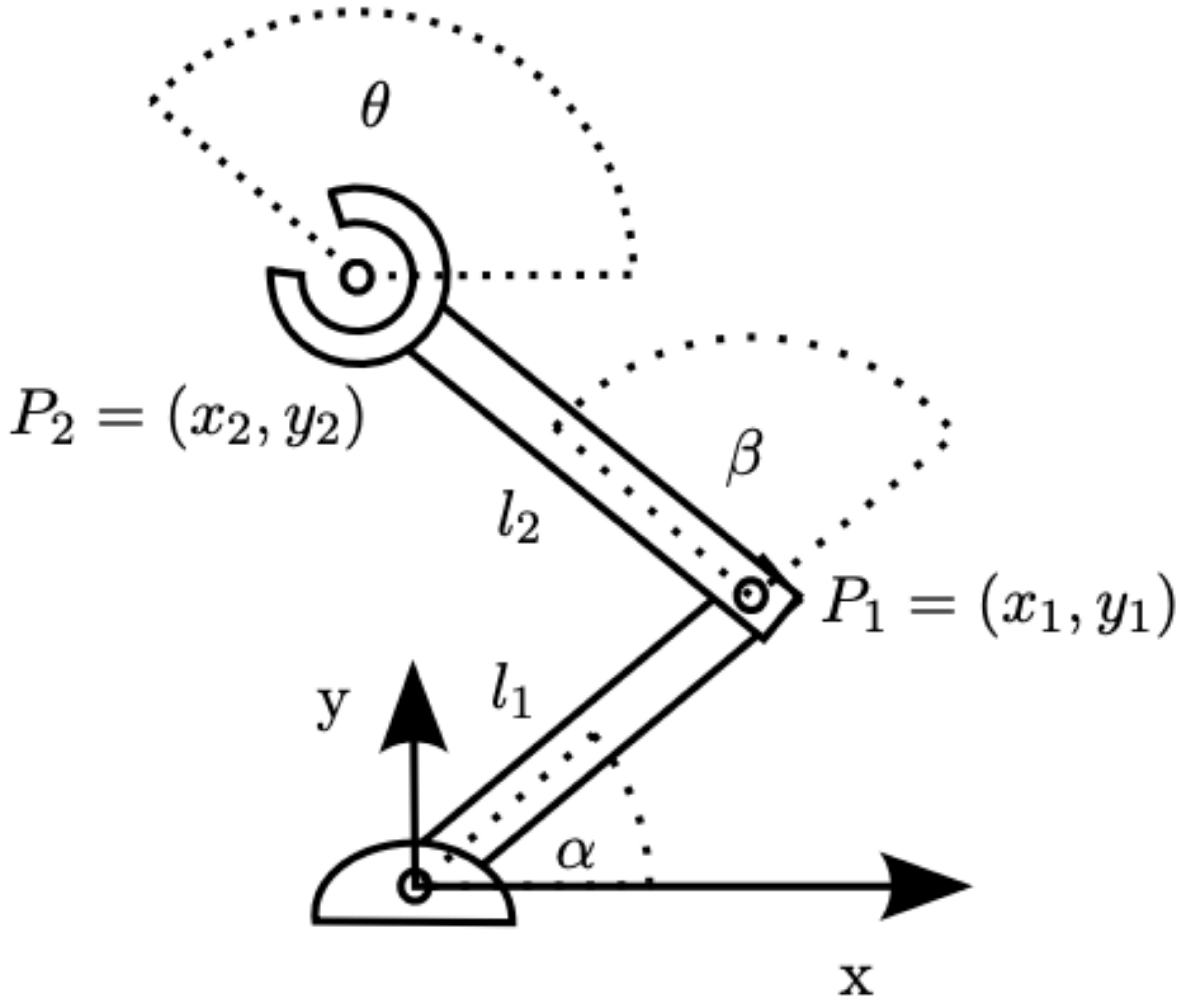
Kinematics Practice

Consider the two-link robot arm shown here.

What are the transformation matrices for each of the 3 joints pictured here?

Hint: you can ignore the z-axis so the matrix should be 3x3.

$$
{}^{2}_{1}T = \begin{bmatrix} \cos\alpha & -\sin\alpha & l_1 \cos\alpha \\ \sin\alpha & \cos\alpha & l_1 \sin\alpha \\ 0 & 0 & 1 \end{bmatrix}
$$



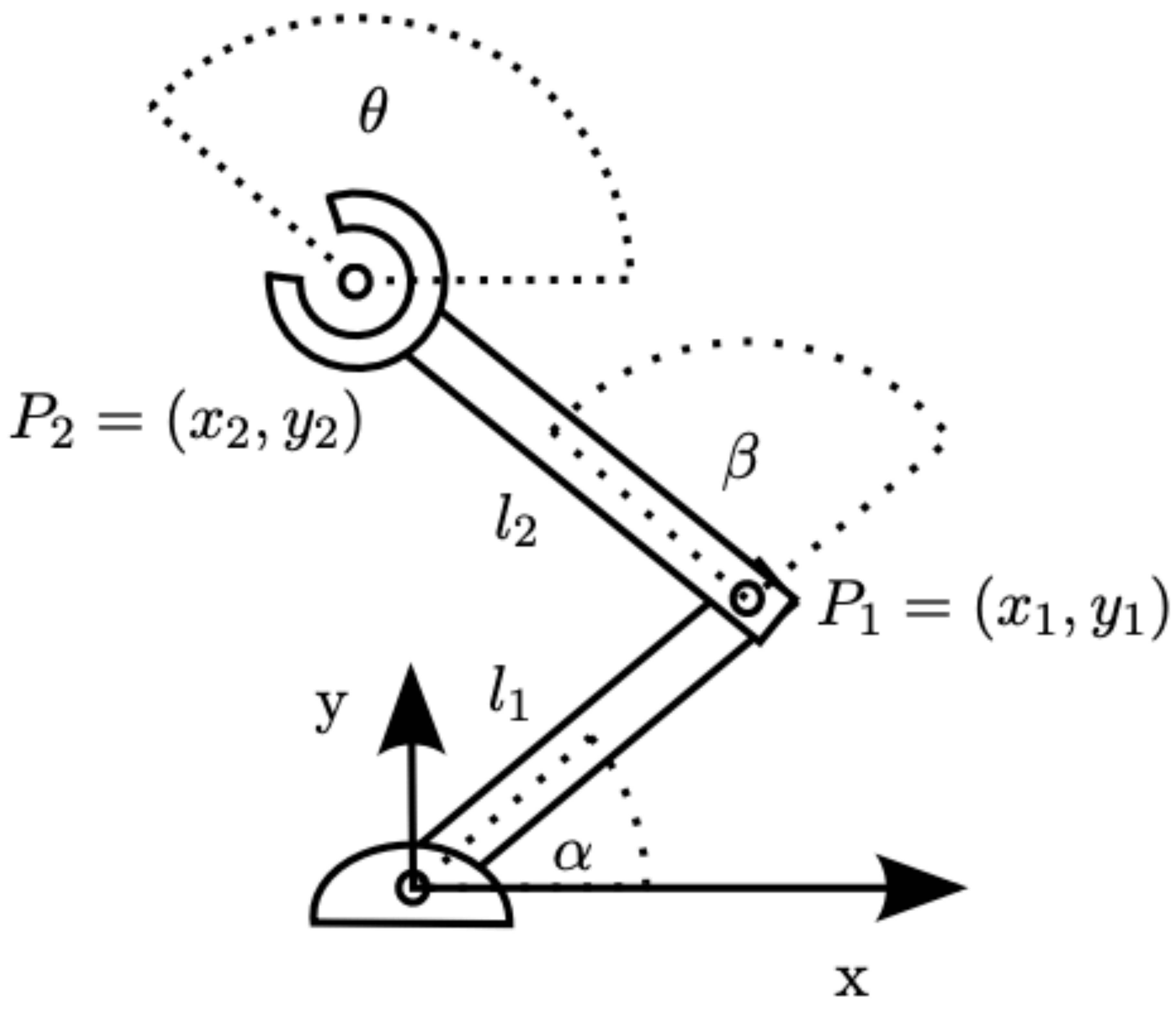
Kinematics Practice

Consider the two-link robot arm shown here.

What are the transformation matrices for each of the 3 joints pictured here?

Hint: you can ignore the z-axis so the matrix should be 3x3.

$$ {}^{3}_{2}T = \begin{bmatrix} \cos\beta & -\sin\beta & l_2\cos\beta \\ \sin\beta & \cos\beta & l_2\sin\beta \\ 0 & 0 & 1 \end{bmatrix} $$



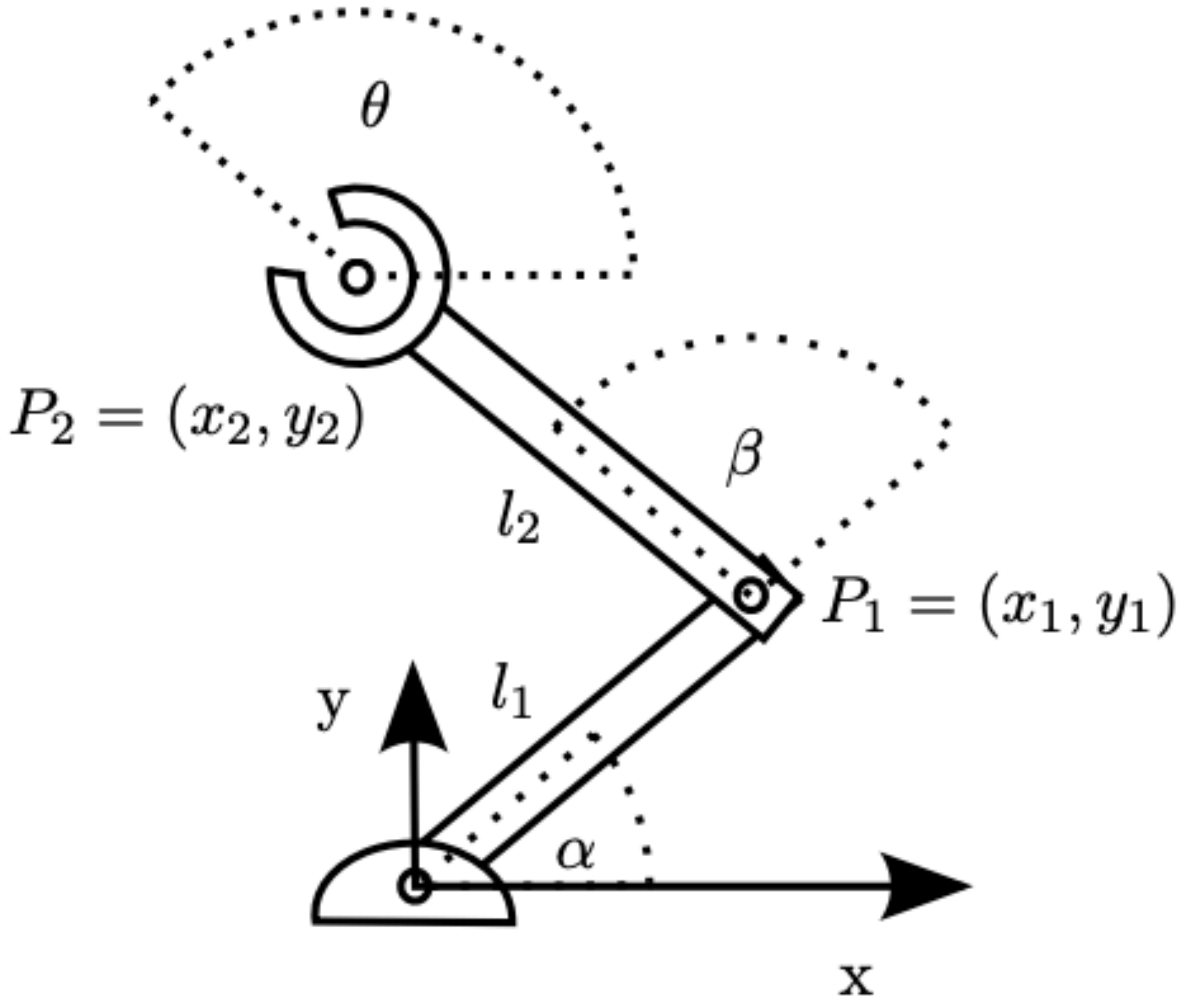
Kinematics Practice

Consider the two-link robot arm shown here.

What are the transformation matrices for each of the 3 joints pictured here?

Hint: you can ignore the z-axis so the matrix should be 3x3.

$$ {}^4_3T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Differential Kinematics

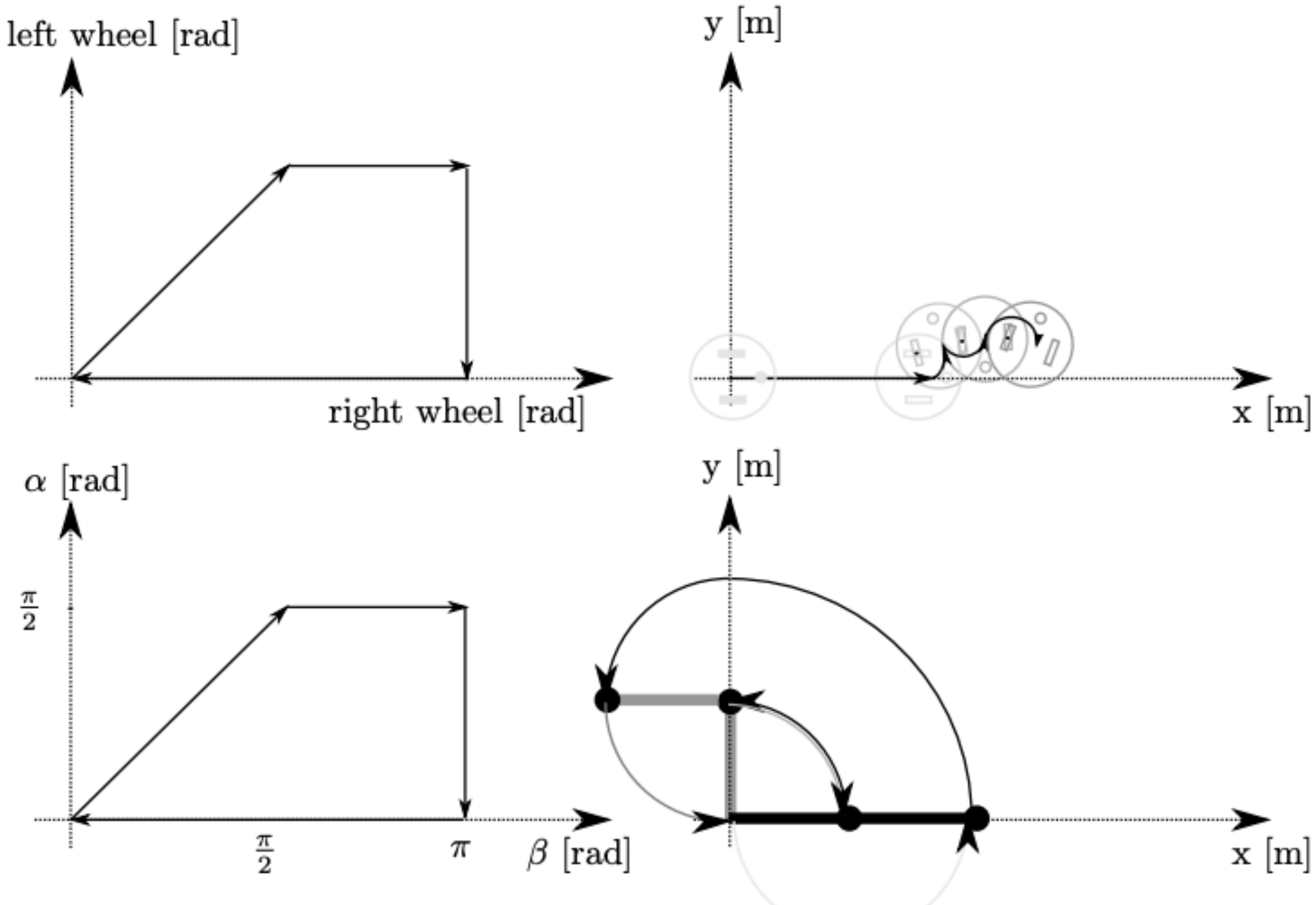
- Relate velocity of end-effector to velocity of joints.
- Forward kinematics: $r = f(q)$.
- Velocity: $\dot{r} = J(q) \cdot [\dot{q}_1, \ldots \dot{q}_n]$ where $J(q)$ is the Jacobian of the robot's end-effector with respect to its configuration.

$$J = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \cdots & \frac{\partial y}{\partial q_n} \\ \frac{\partial \theta}{\partial q_1} & \frac{\partial \theta}{\partial q_2} & \cdots & \frac{\partial \theta}{\partial q_n} \end{bmatrix}$$

Differential Kinematics

- Why useful?
  - Order of motion could matter.
  - Velocity could matter for a task.
  - In mobile robots, configuration is insufficient for determining pose in task space.

Holonomic vs. Non-Holonomic



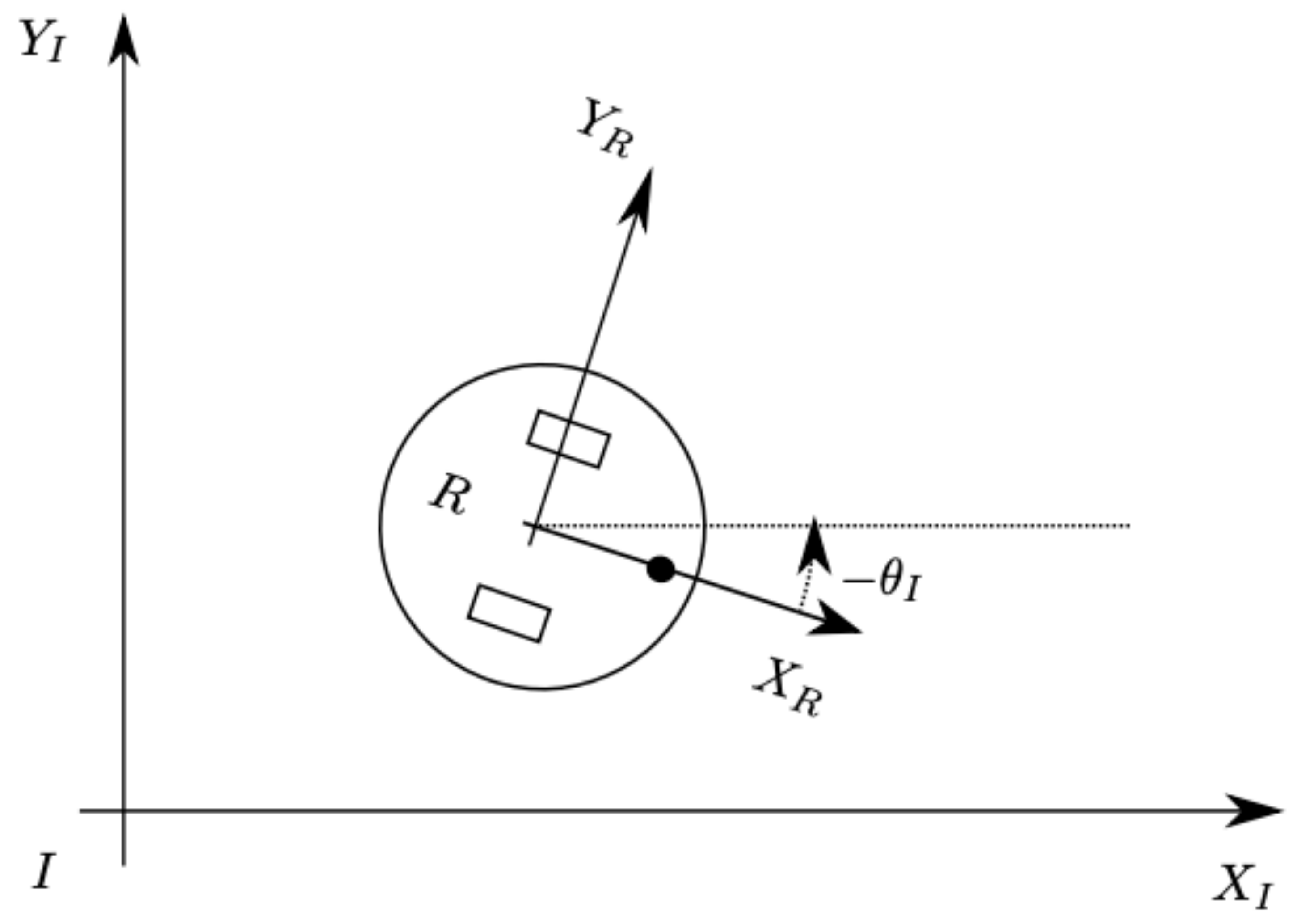
- Holonomic vs. non-holonomic: configuration determines a unique position in task space.

From robot to inertial frames

- Robot's position can be expressed in one of two frames: inertial (i.e., global) frame or robot (i.e., local) frame.
- Robot is always at the origin in its own frame.
- Goal: understand how velocities in robot's frame map to velocities in inertial frame.

$$\dot{x}_I = \dot{x}_R \cos\theta - \dot{y}_R \sin\theta$$
$$\dot{y}_I = \dot{x}_R \sin\theta + \dot{y}_R \cos\theta$$

$$\dot{\theta}_I = \dot{\theta}_R$$



From wheel velocities to robot velocities

r: wheel radius

$\dot{\phi}_l$: left wheel velocity

$\dot{\phi}_r$: right wheel velocity

d: distance between wheels

- If a wheel of radius r has velocity $\dot{\phi}$, how far will it move?
    - $r\dot{\phi}$
- Forward motion of robot midpoint is mean movement of both wheels:
    - $\dot{x}_R = \frac{r}{2}(\dot{\phi}_l + \dot{\phi}_r)$
    - $\dot{y}_R = 0$. Why?
    - $\dot{\theta} = \frac{r(\dot{\phi}_r - \dot{\phi}_l)}{d}$

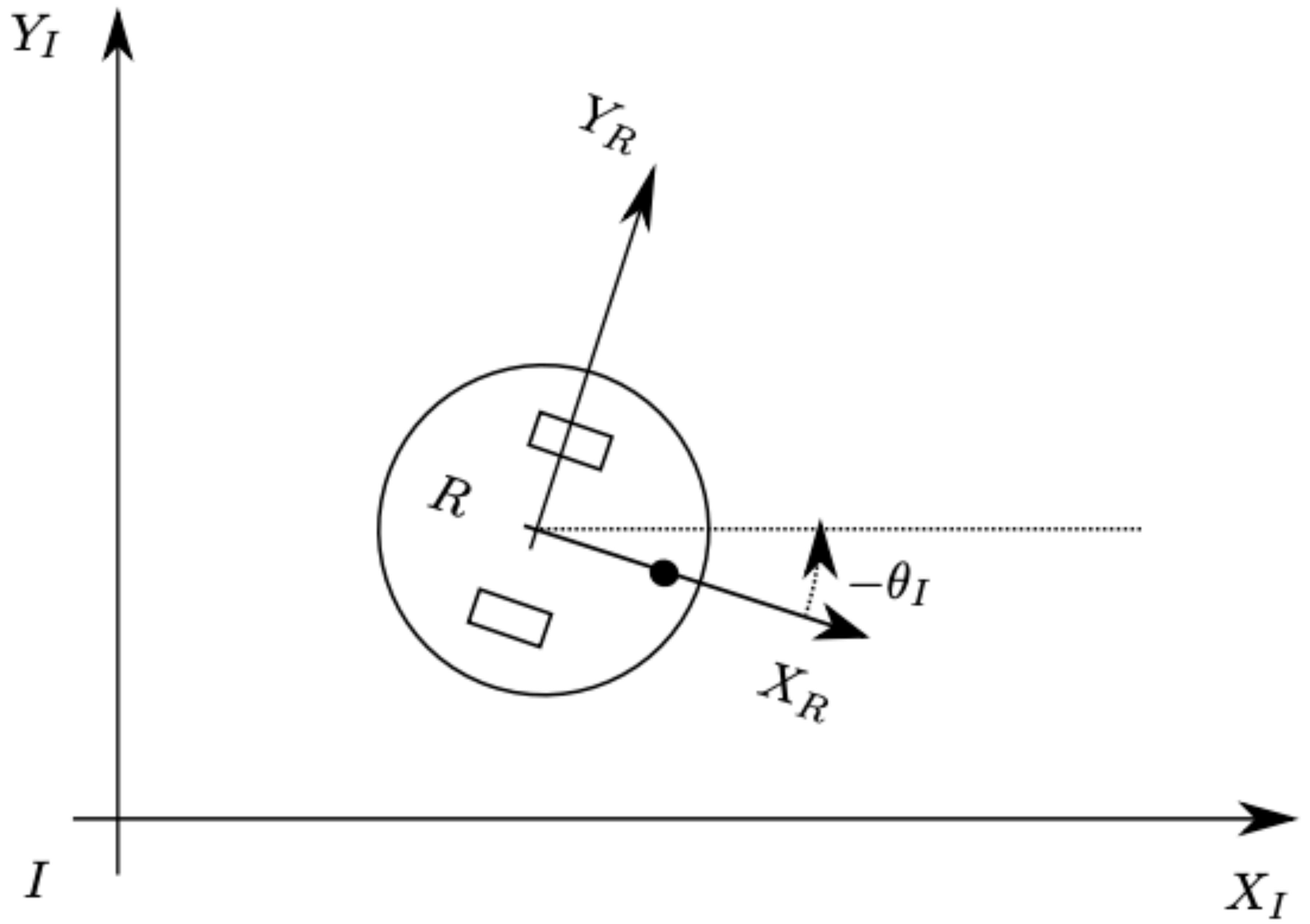
$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{d} & -\frac{r}{d} \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} = \begin{bmatrix} \frac{\partial x_R}{\partial \dot{\phi}_l} & \frac{\partial x_R}{\partial \dot{\phi}_r} \\ \frac{\partial y_R}{\partial \dot{\phi}_l} & \frac{\partial y_R}{\partial \dot{\phi}_r} \\ \frac{\partial \theta}{\partial \dot{\phi}_l} & \frac{\partial \theta}{\partial \dot{\phi}_l} \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix}$$

From wheel velocities to inertial velocities

- Now we can put this all together to obtain change in the inertial frame in terms of wheel speed.

$$\begin{bmatrix} \dot{x_I} \\ \dot{y_I} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi_l}}{2} + \frac{r\dot{\phi_r}}{2} \\ 0 \\ \frac{\dot{\phi_r} r}{d} - \frac{\dot{\phi_l} r}{d} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{d} & -\frac{r}{d} \end{bmatrix} \begin{bmatrix} \dot{\phi_l} \\ \dot{\phi_r} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_R}{\partial \dot{\phi_l}} & \frac{\partial x_R}{\partial \dot{\phi_r}} \\ \frac{\partial y_R}{\partial \dot{\phi_l}} & \frac{\partial y_R}{\partial \dot{\phi_r}} \\ \frac{\partial \theta}{\partial \dot{\phi_l}} & \frac{\partial \theta}{\partial \dot{\phi_l}} \end{bmatrix} \begin{bmatrix} \dot{\phi_l} \\ \dot{\phi_r} \end{bmatrix}$$



r: wheel radius

$\dot{\phi}_l$: left wheel velocity

$\dot{\phi}_r$: right wheel velocity

d: distance between wheels

Odometry

- Calculating pose of robot in inertial frame.

$$\begin{bmatrix} x_I(T) \\ y_I(T) \\ \theta(T) \end{bmatrix} = \int_0^T \begin{bmatrix} \dot{x}_I(t) \\ \dot{y}_I(t) \\ \dot{\theta}(t) \end{bmatrix} dt \approx \sum_{k=0}^{k=T} \begin{bmatrix} \Delta x_I(k) \\ \Delta y_I(k) \\ \Delta \theta(k) \end{bmatrix} \Delta t$$

- Approximation leads to errors in calculation ("drift").

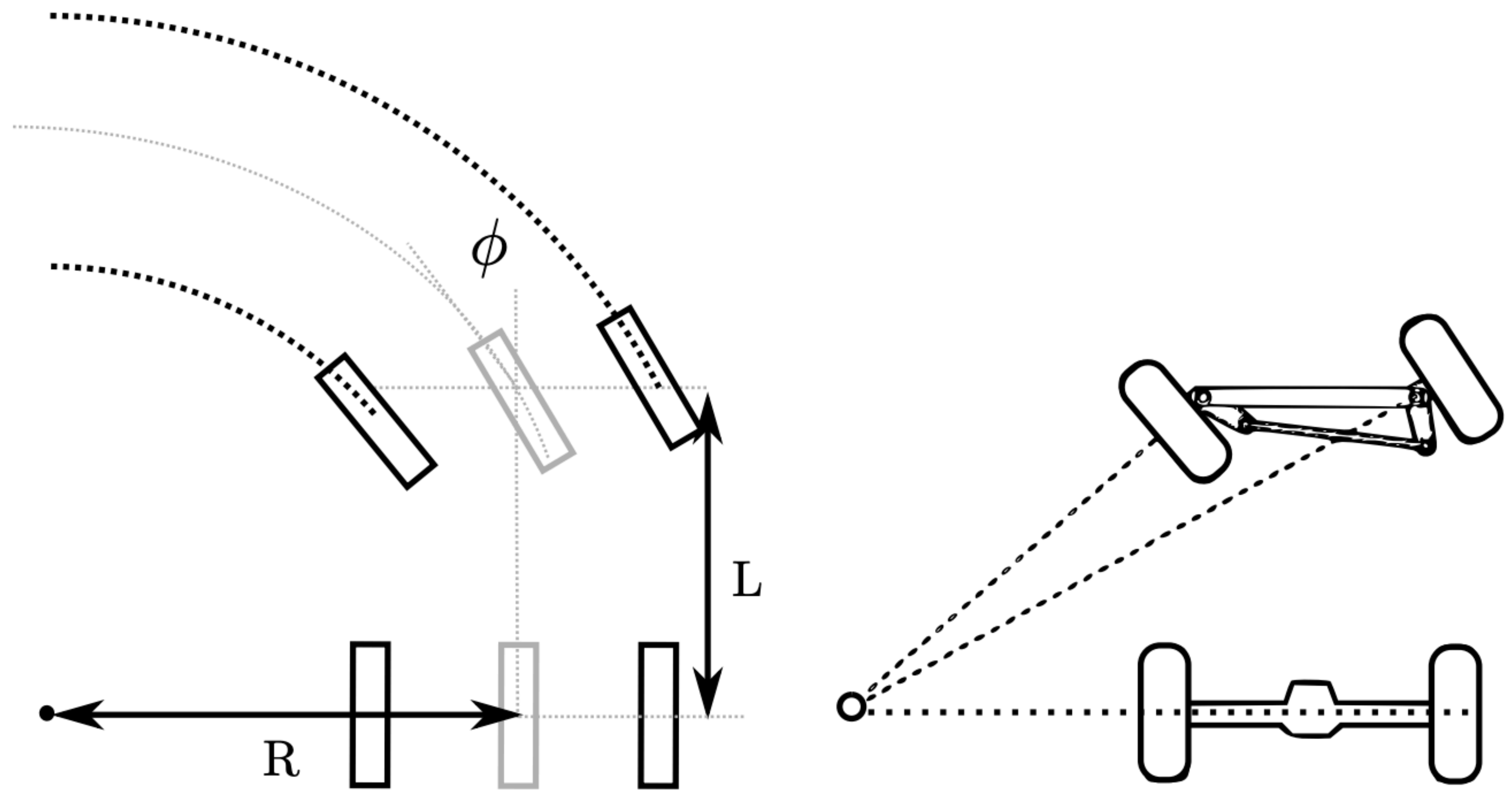
Ackerman Steering

- Ackerman steering: two wheels with 1 motor but wheels have own pivot point.
- Leads to different computation of robot speed from wheel speed.

$$\dot{x}_R = \dot{\omega} r$$

$$\dot{y}_R = 0$$

$$\dot{\theta}_R = \frac{\dot{\omega} r \tan\phi}{L}$$



Summary

- Reviewed forward kinematics.
- Introduced differential forward kinematics.
- Derived kinematics equations for simple wheeled vehicles.

Action Items

- Planning reading for next week; send a reading response by 12 pm on Monday.
- Homework #3 due in one week.
- Start studying for midterm

$$(l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2), l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2))$$