Autonomous Robotics

Extended Kalman Filters

Josiah Hanna
University of Wisconsin — Madison

Announcements

- Programming assignment due 1 minute ago.
  - Any questions or comments?
- Reading assignment for next week will be posted soon.
- Homework 2 will be released Thursday (practice filters).
- Asynchronous lecture (no in-person class) on 2/19.

Learning Outcomes

After today's lecture, you will:

- Be able to specify the key assumptions underlying the extended Kalman filters.
- Understand the extended Kalman filter as an approximation of the Kalman filter.
- Understand the strengths and limitations of extended Kalman filters.

Review: Linear Gaussian Systems

We make the following assumptions on the robot's environment:

- States, controls, and observations are vectors: $x \in \mathbf{R}^d$ and $u \in \mathbf{R}^k$ and $z \in \mathbf{R}^m$.
- State transition and observation function are linear Gaussians:
  - $x_t = Ax_{t-1} + Bu_t + w_t$ where $w_t \sim \mathcal{N}(0, Q)$, $A \in \mathbf{R}^{d \times d}$, $B \in \mathbf{R}^{d \times k}$ and $Q \in \mathbf{R}^{d \times d}$. $\implies p(x_t \mid x_{t-1}, u_t) = \mathcal{N}(x; Ax_{t-1} + Bu_t, Q)$
  - $z_t = Hx_t + v_t$ where $v_t \sim \mathcal{N}(0, R)$, $H \in \mathbf{R}^{m \times d}$, and $R \in \mathbf{R}^{m \times m}$. $\implies g(z_t \mid x_t) = \mathcal{N}(z; Hx_t, R)$

Review: Kalman Filter

- The Kalman filter is a Bayes filter that represents $\texttt{bel}(x_t)$ with a Gaussian distribution, $\mathcal{N}(\mu_t, \Sigma_t)$.

- The initial belief is Gaussian: $\texttt{bel}(x_0) = \mathcal{N}(x_0; \mu_0, \Sigma_0)$.

- Under our assumptions, the posterior remains a Gaussian distribution using the updates from the Bayes filter:

$$p(x_t \mid z_{1:t}, u_{1:t}) = \mathcal{N}(x_t; \mu_t, \Sigma_t)$$

- Intuition for correctness: plug Gaussian beliefs and linear Gaussian system state transitions and observations into Bayes filter updates.

Review: Kalman Filter as a Bayes Filter

- Initialize belief:

$$\texttt{bel}(x_0) = \mathcal{N}(x_0, \mu_0, \Sigma_0)$$

- Prediction:

$$\overline{\texttt{bel}}(x_t) = \int p(x_t \,|\, x_{t-1}, u_t)\texttt{bel}(x_{t-1})dx_{t-1}$$

$$\bar{\mu}_t = A\mu_{t-1} + Bu_t$$

$$\bar{\Sigma}_t = A^T \Sigma A + Q$$

- Correction:

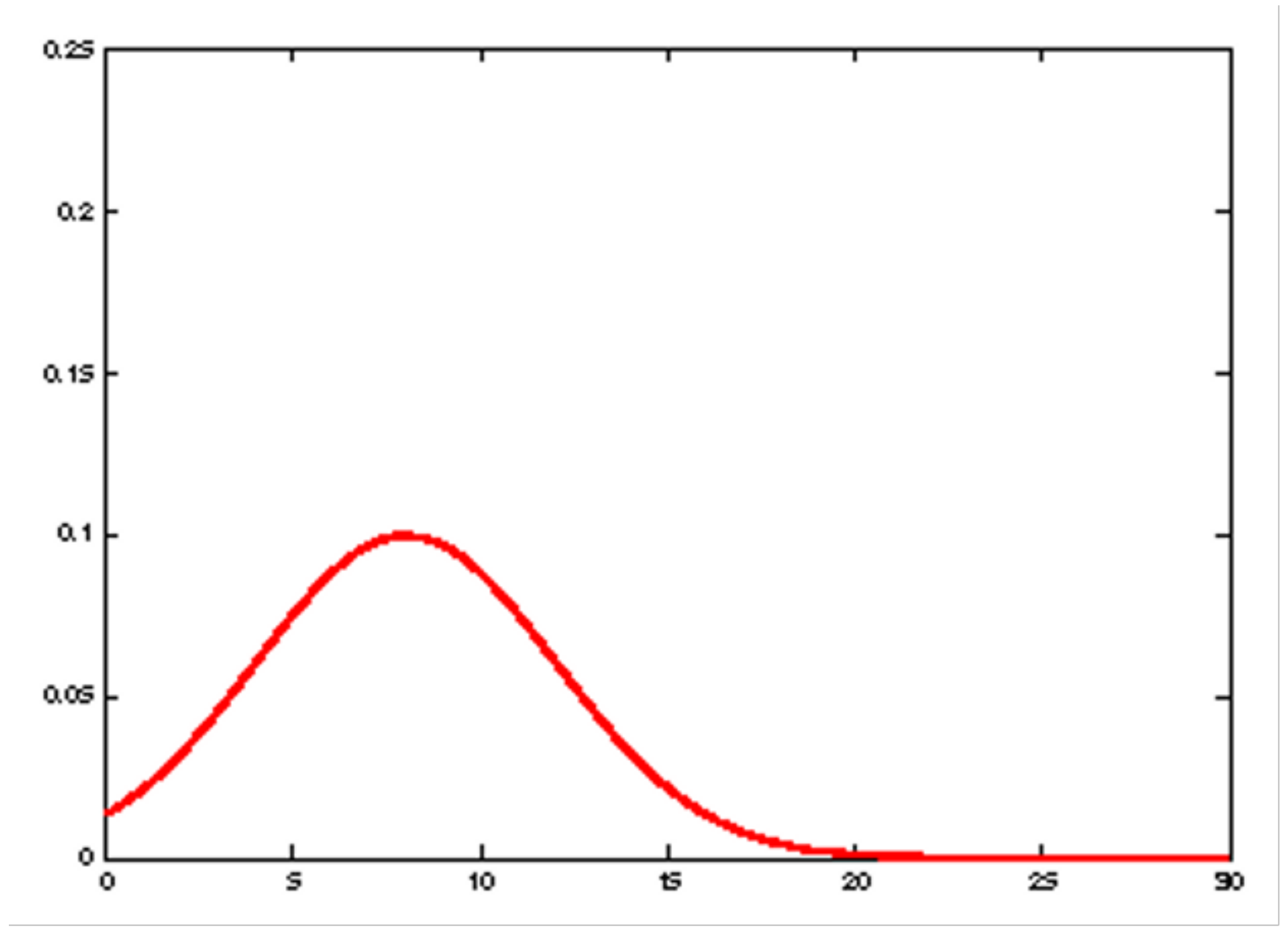
$$\mu_t = \bar{\mu}_t + K_t(z_t - H\bar{\mu}_t)$$

$$\texttt{bel}(x_t) = \eta g(z_t \,|\, x_t)\overline{\texttt{bel}}(x_t)$$

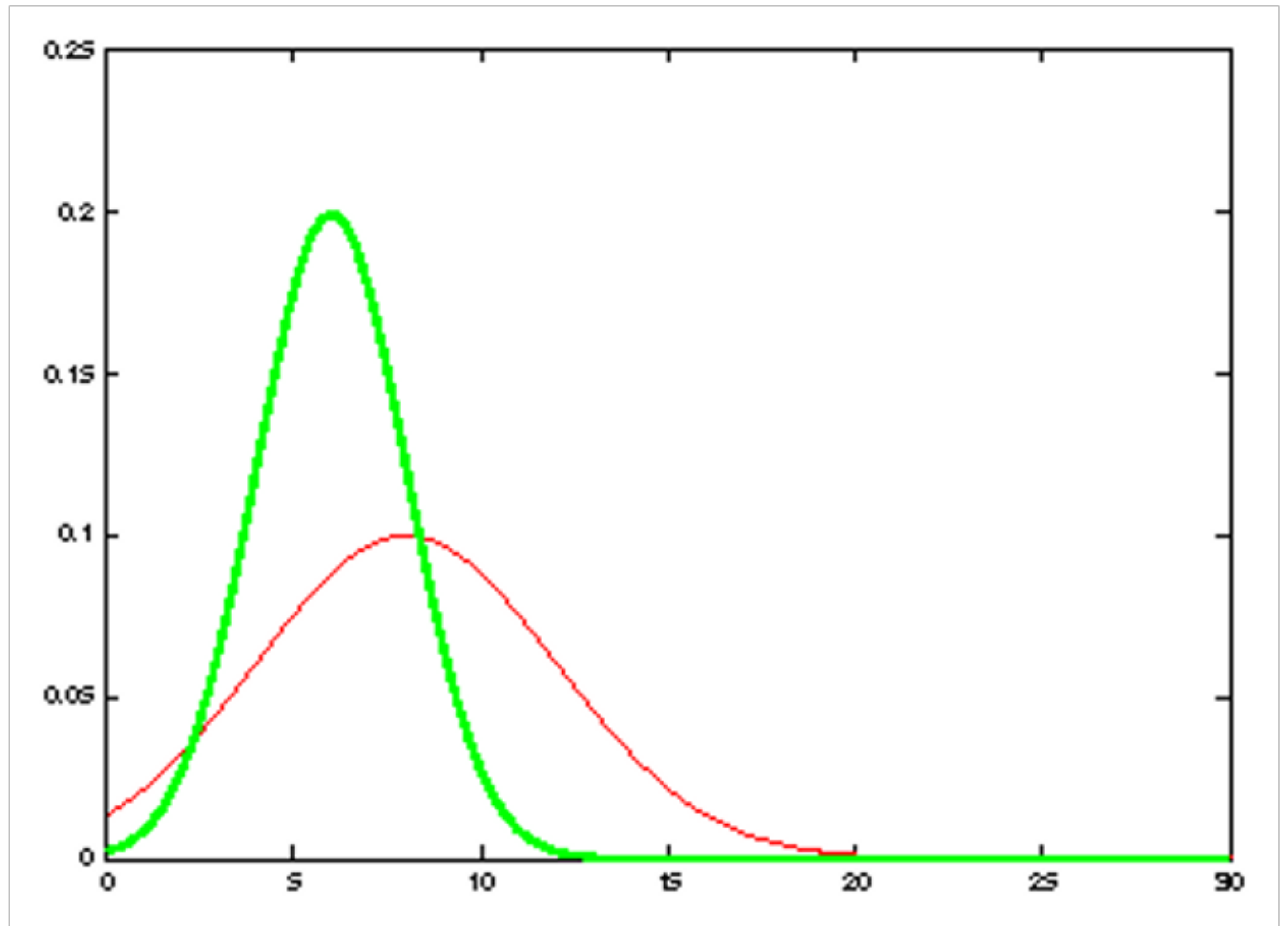
$$\Sigma_t = (I - K_t H)\bar{\Sigma}_t$$

$$K_t = \bar{\Sigma}_t H^\top (H\bar{\Sigma}_t H^\top + R)^{-1}$$

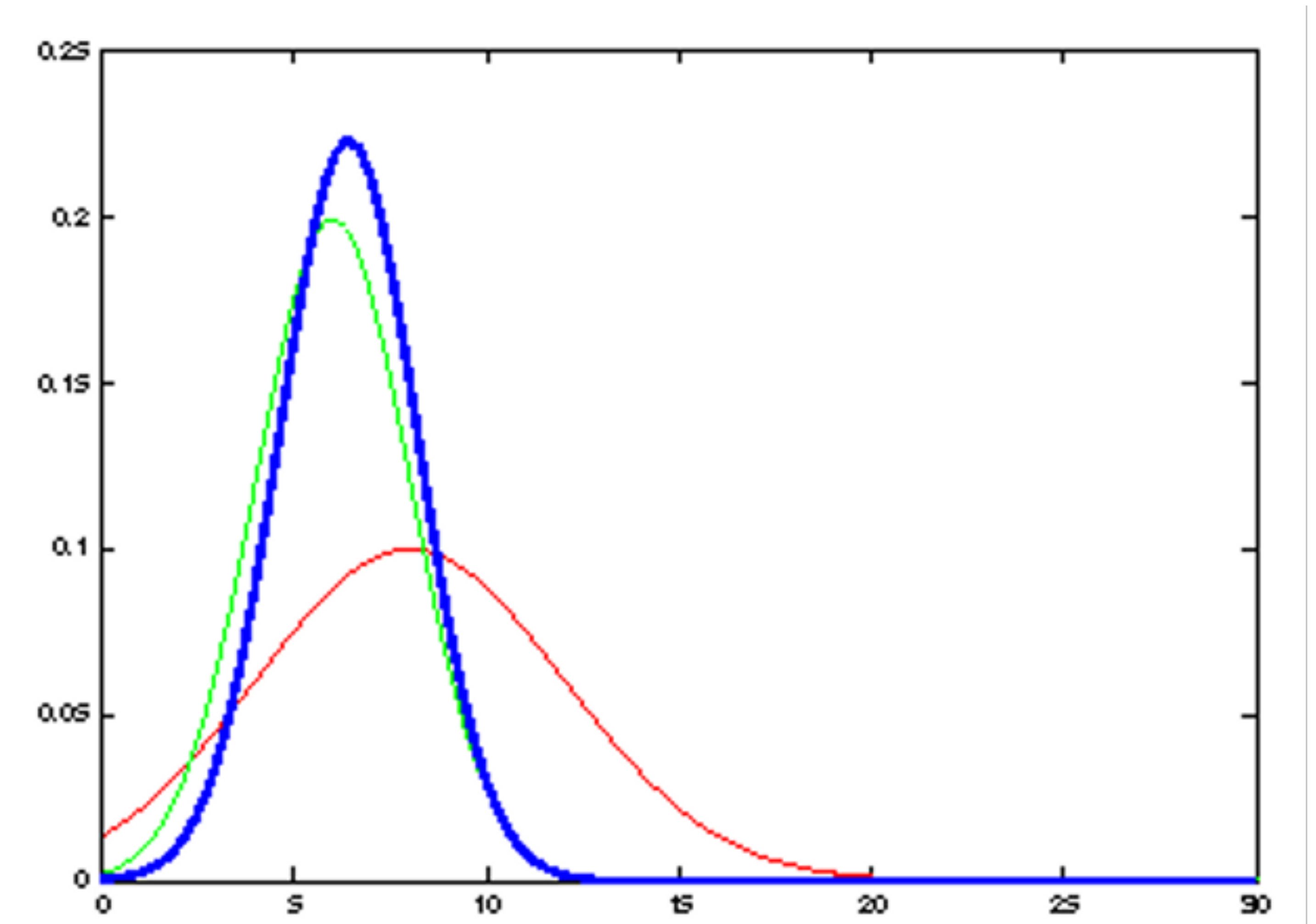
Illustration of Kalman Filter Updates



Belief after motion



Observation Probability



New belief

Advantages / Disadvantages

- Kalman filters:
  - Can be used for continuous state spaces.
  - Are optimal filters if our assumptions hold.
  - Are very efficient; polynomial in state and observation dimensionality.
- But…
  - Randomness may not be Gaussian.
  - Most robotics systems are nonlinear.

Non-linear Gaussian Systems

Let's change our assumptions to allow non-linearity:

- States, controls, and observations are vectors: $x \in \mathbf{R}^d$ and $u \in \mathbf{R}^k$ and $z \in \mathbf{R}^m$.

- State transition and observation function are non-linear Gaussians:

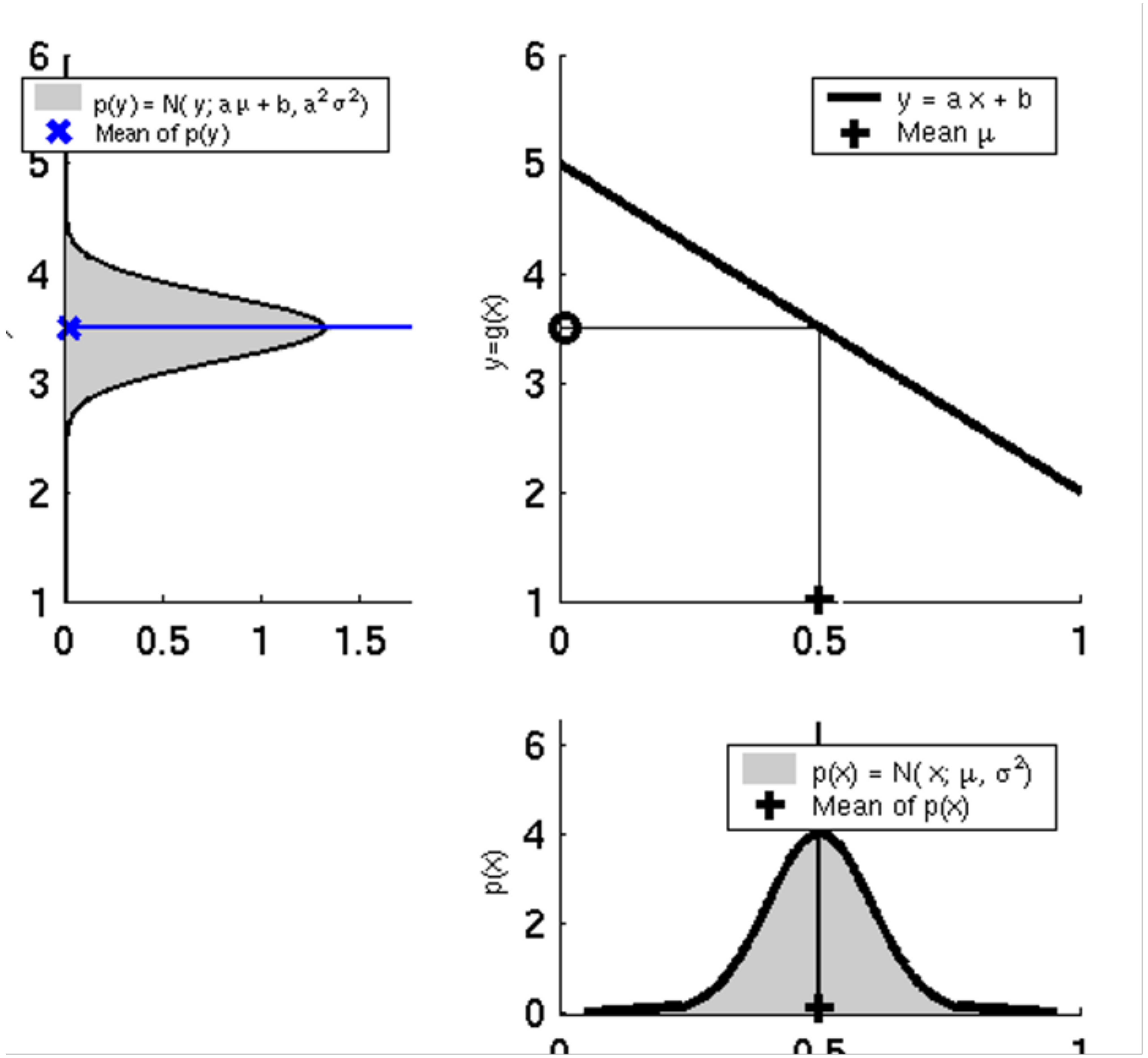
  - $x_t = f(x_{t-1}, u_t) + w_t$ where $w_t \sim \mathcal{N}(0,Q)$, $Q \in \mathbf{R}^{d\times d}$, and f is a non-linear function.

    $\implies p(x_t \,|\, x_{t-1}, u_t) = \mathcal{N}(x; f(x_{t-1}, u_t), Q)$

  - $z_t = h(x_t) + v_t$ where $v_t \sim \mathcal{N}(0,R)$, $R \in \mathbf{R}^{m\times m}$, and h is a non-linear function.

    $\implies g(z_t \,|\, x_t) = \mathcal{N}(z; h(x_t), R)$

Why do we need linearity?



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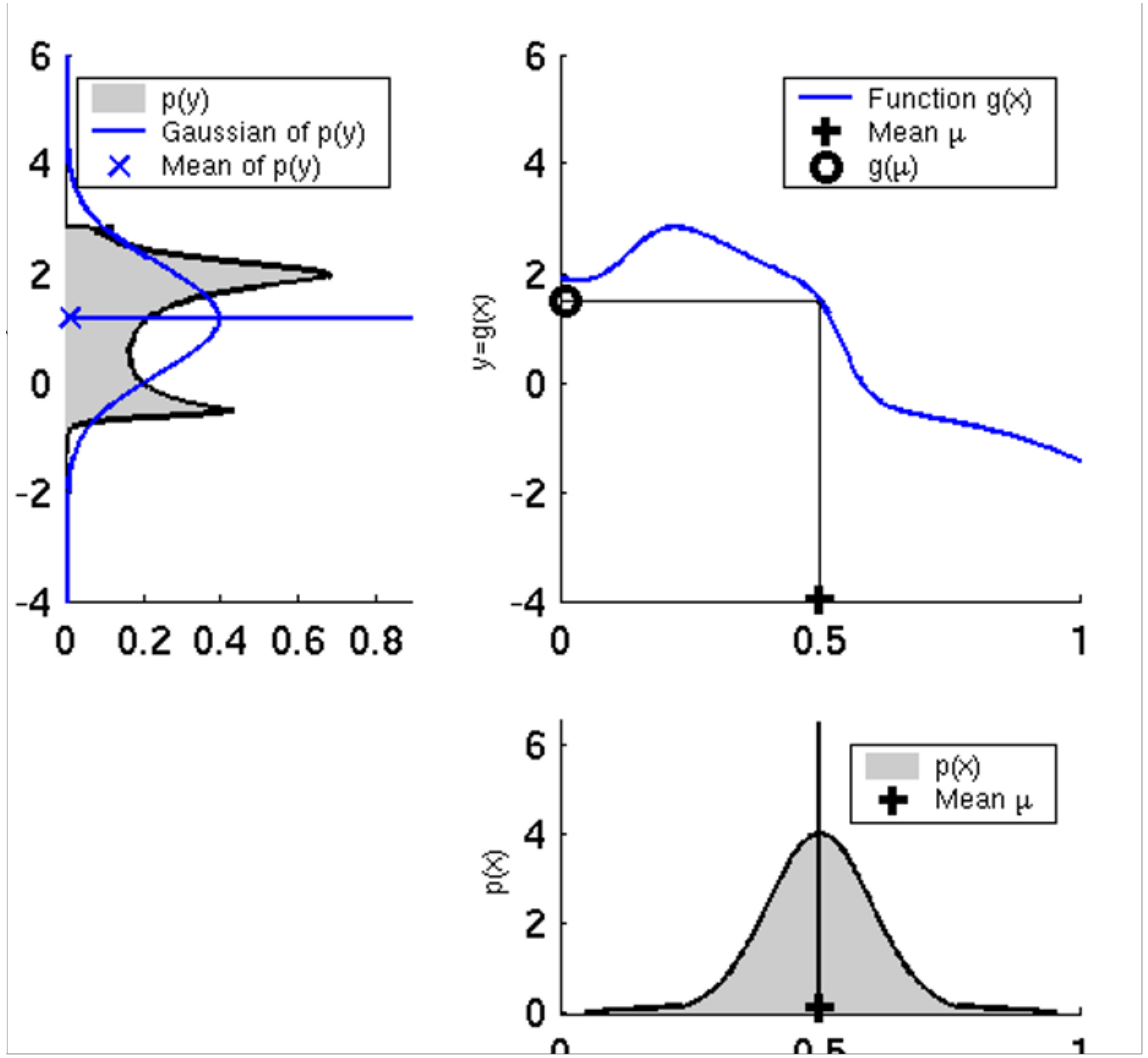
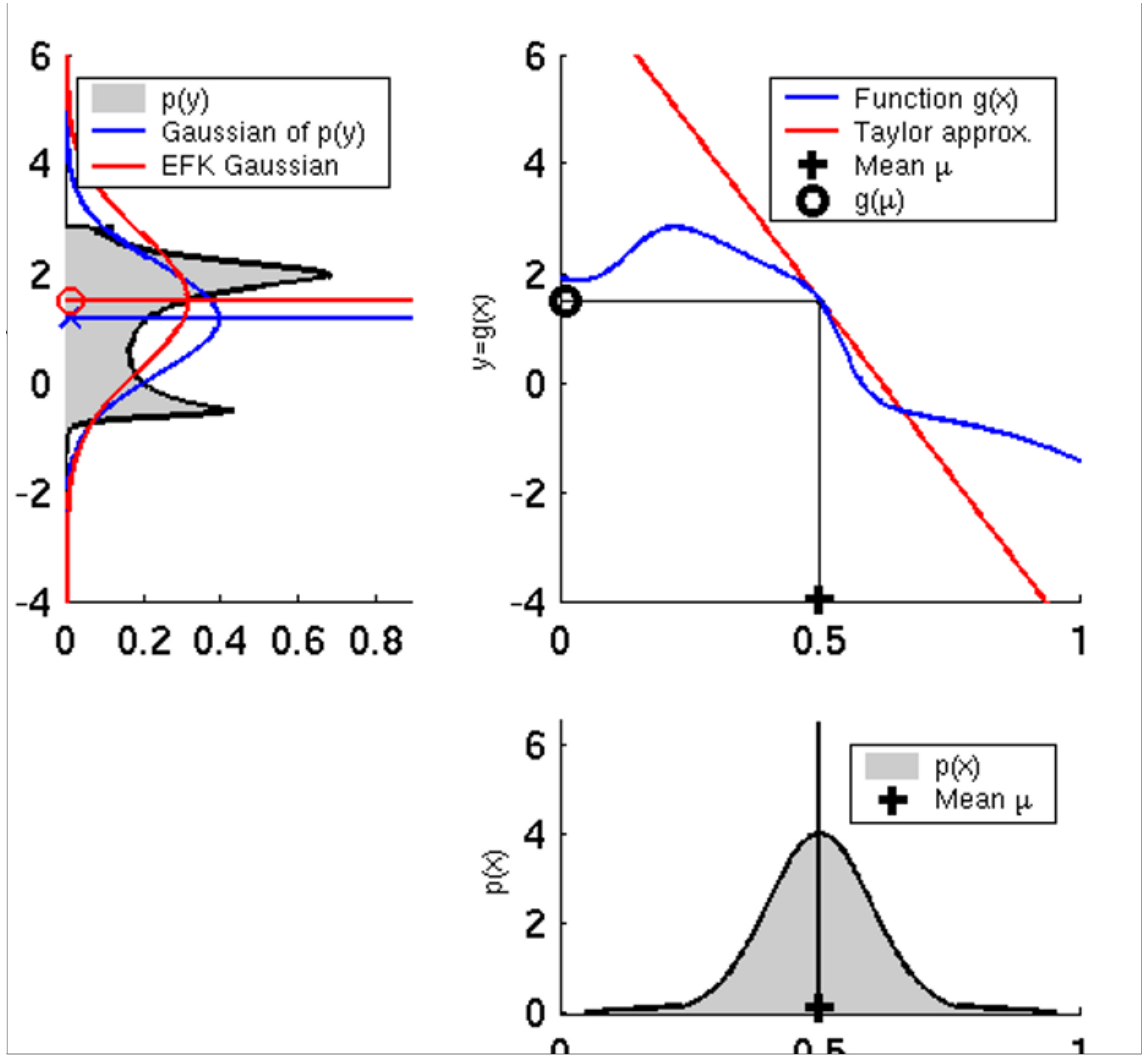


Why do we need linearity?

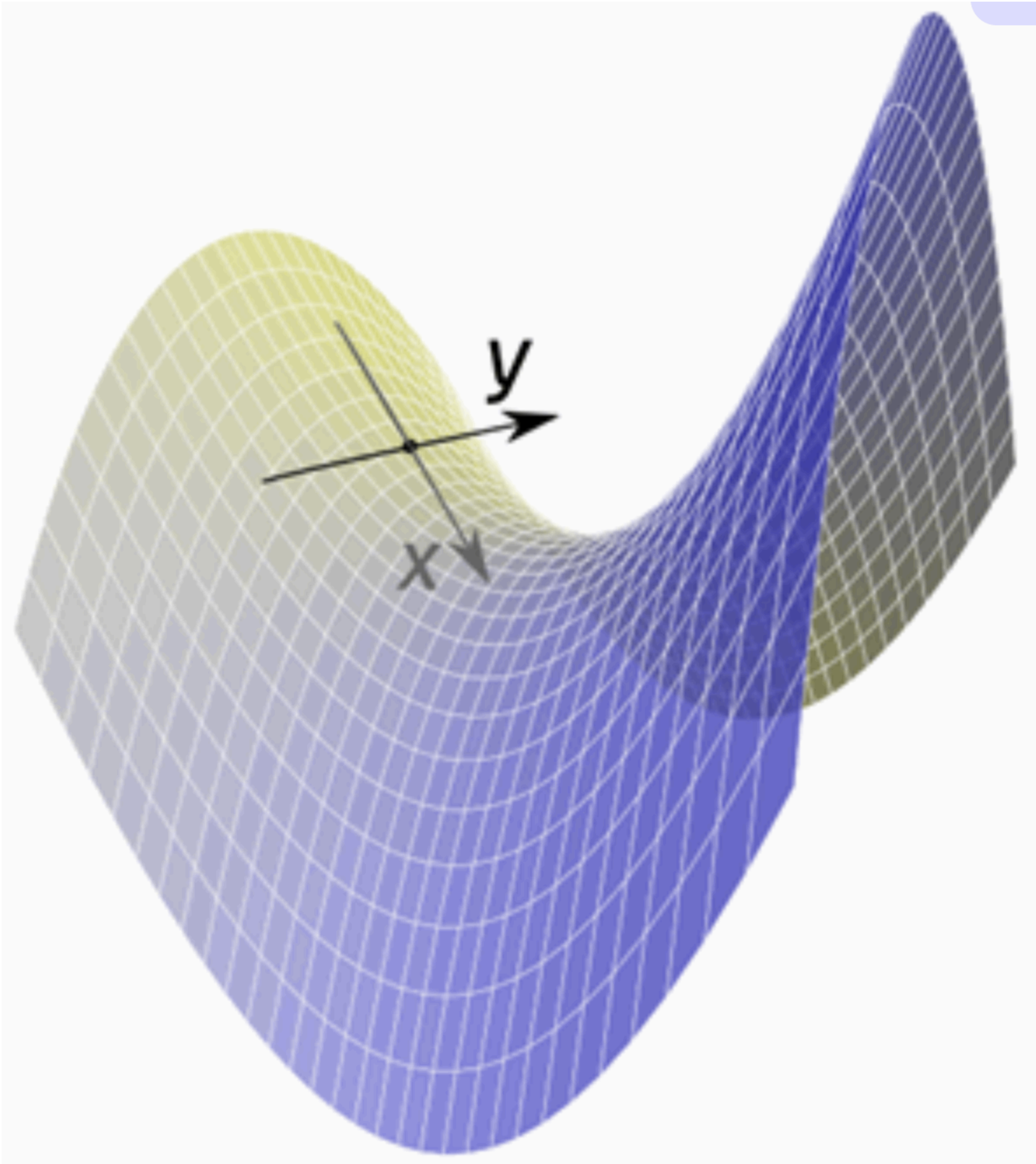


Calculus Review: Partial Derivatives

- Given a function, $f(x_1, \ldots, x_n)$.

- The partial derivative $\frac{\partial f}{\partial x_i}$ captures the rate of change of f as one of the x_i increases.

- The gradient is the vector of partial derivatives: $\nabla_x f = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$



mathisfun.com

Calculus Review: Jacobian Matrix

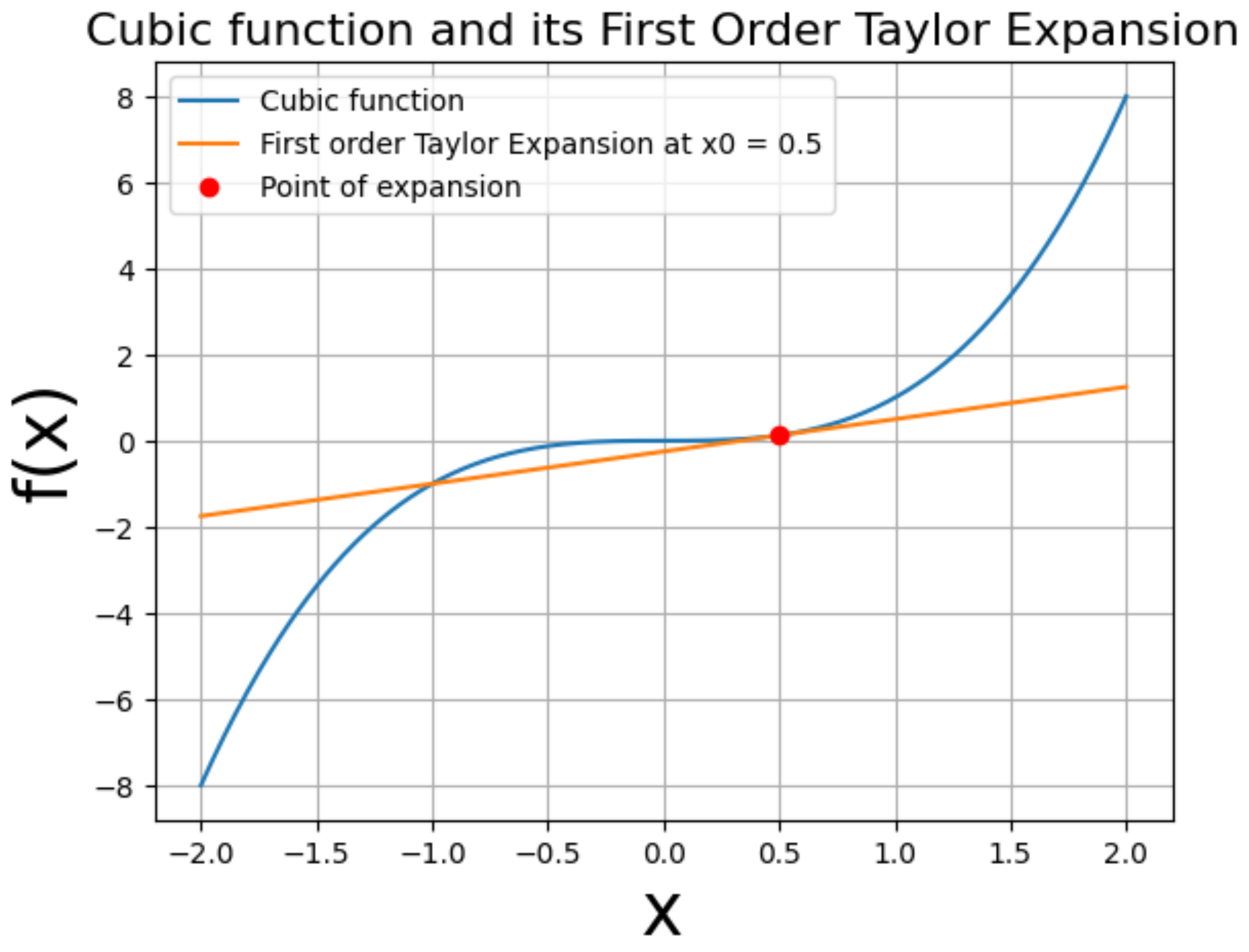
- Given a function, $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$.
  - Equivalently, $x = (x_1, ..., x_n)$ and $f(x) = (f_1(x), ..., f_m(x))$.
- The Jacobian, J, is the matrix of partial derivatives of f.
  - Entry (i, j) captures how fast $f_i(x)$ is changing as x_j increases.

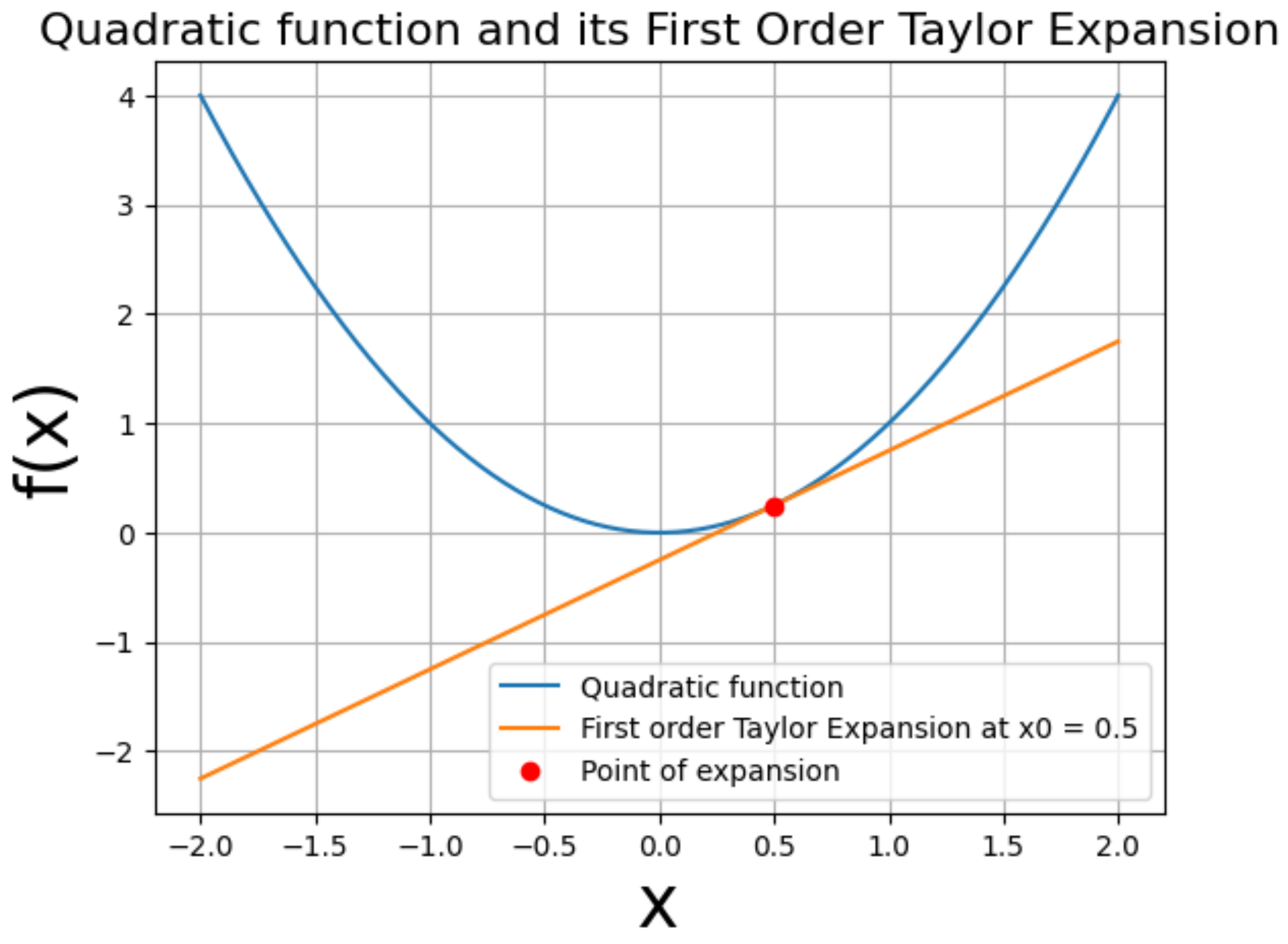
$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

https://discourse.julialang.org/t/visualizing-jacobians-gradients-hows-this/36799

Calculus Review: Taylor Series Expansion

- Goal is to approximate a given function f (typically non-linear) with a linear function.





f has one input and one output.

$$f(x) \approx f(x_0) + (x - x_0)\frac{\partial f}{\partial x}\bigg|_{x=x_0}$$

f has multiple inputs and outputs.

$$f(x) \approx f(x_0) + J(x - x_0)$$

Extended Kalman Filter

- Intuition for EKF: linearize the non-linear system with a first-order Taylor expansion and then apply Kalman filtering to the linearized system.

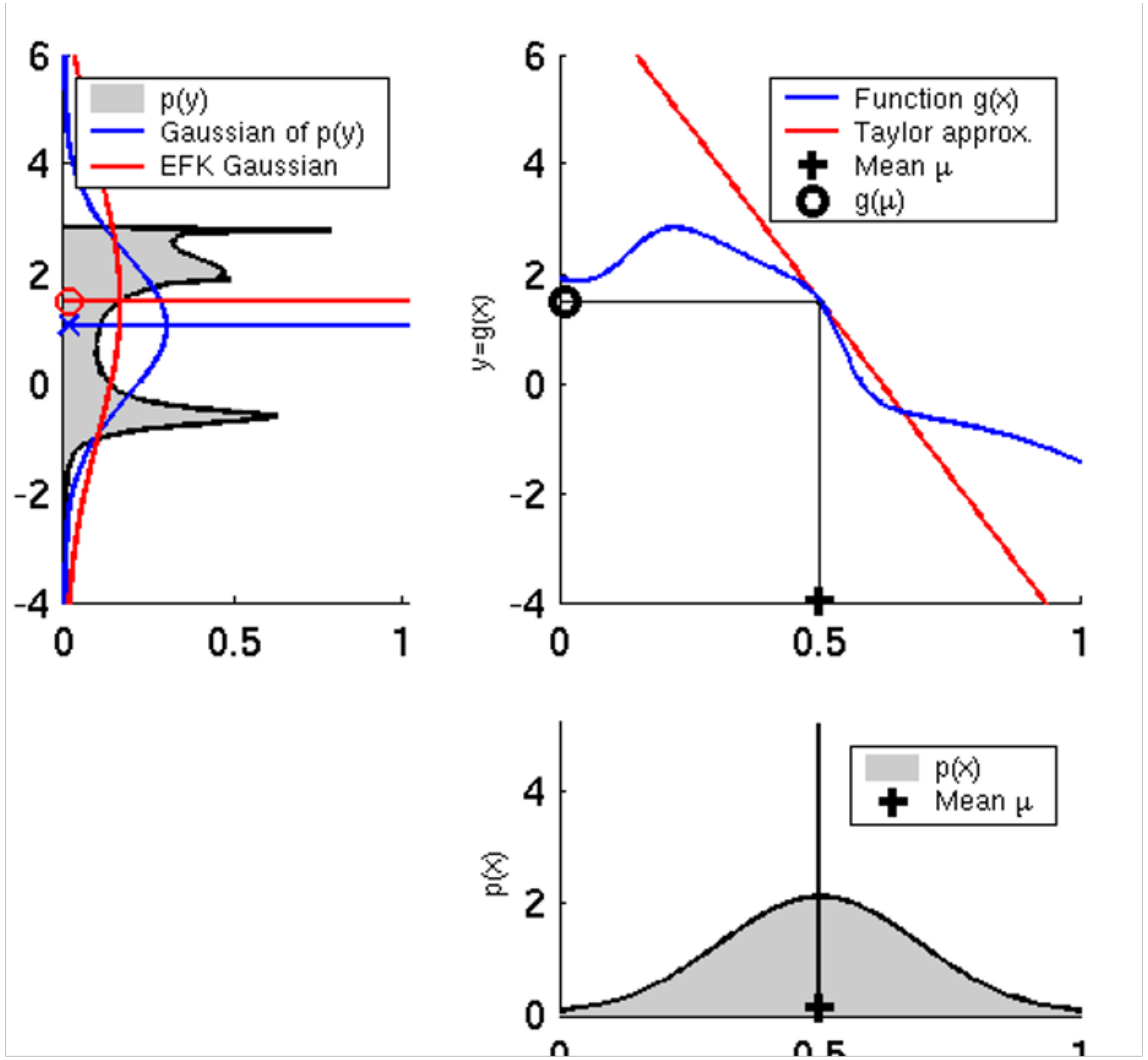
- $f(x_{t-1}, u_t) \approx f(\mu_{t-1}, u_t) + G_t(x_{t-1} - \mu_{t-1})$ where G_t is the Jacobian of f at μ_{t-1}.

- $h(x_t) \approx h(\mu_t) + H_t(x_t - \mu_t)$ where H_t is the Jacobian of h at μ_t.

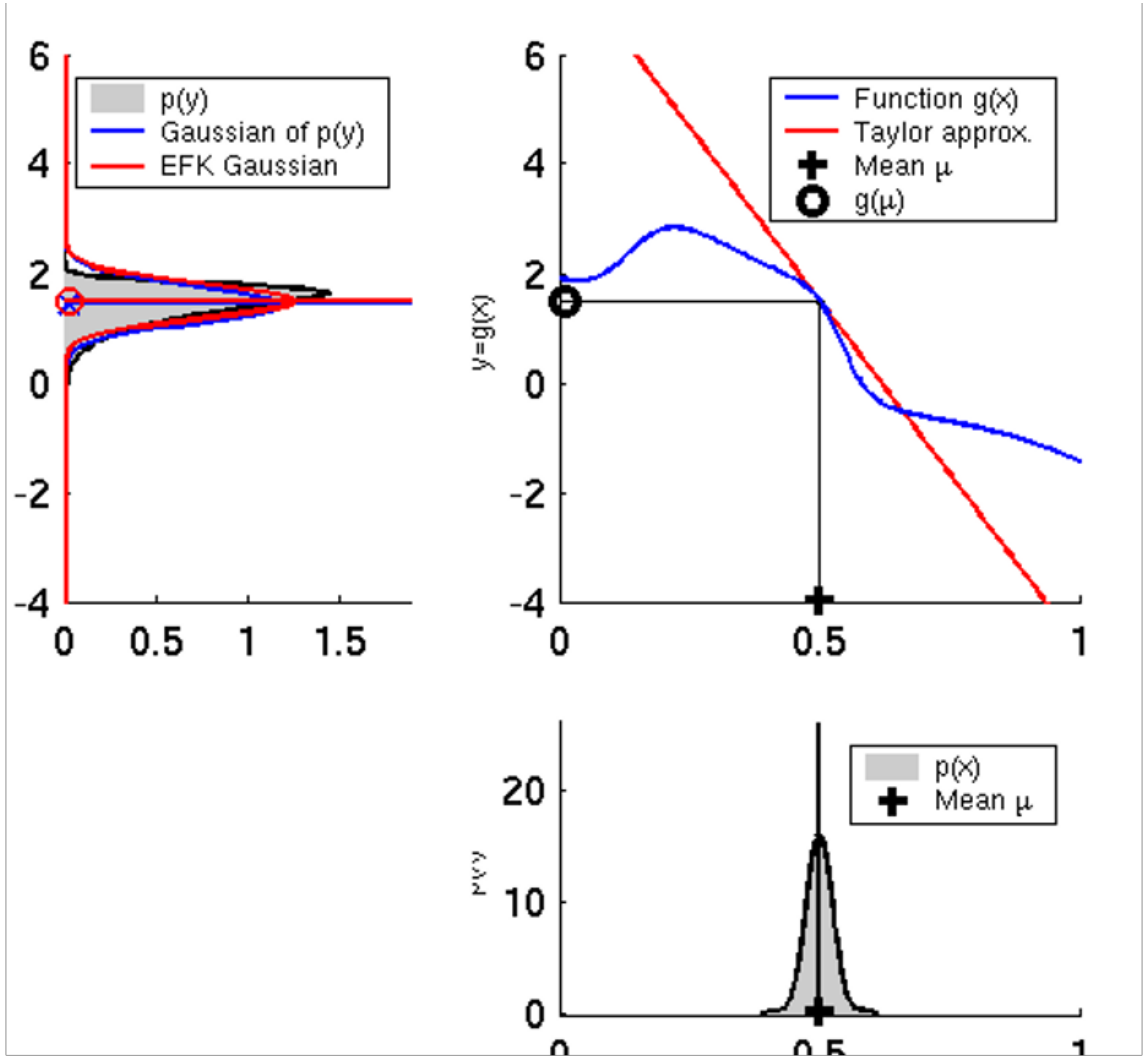
- **Note:** the expansion point is set to be the mean of the current belief.

- If f and h are linear functions, then the EKF reduces to the basic Kalman filter.

Expanding at the Mean



Expanding at the Mean



Extended Kalman Filter

- Initialize belief: $\mathcal{N}(x_0, \mu_0, \Sigma_0)$

- Prediction:

- Correction:

Kalman Filter	Extended Kalman Filter
$\bar{\mu}_t = A\mu_{t-1} + Bu_t$	$\bar{\mu}_t = f(\mu_{t-1}, u_t)$
$\overline{\Sigma}_t = A^T \Sigma A + Q$	$\overline{\Sigma}_t = G_t^T \Sigma G_t + Q$
$\mu_t = \bar{\mu}_t + K_t(z_t - H\bar{\mu}_t)$	$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
$\Sigma_t = (I - K_t H)\overline{\Sigma}_t$	$\Sigma_t = (I - K_t H_t)\overline{\Sigma}_t$

H_t is the Jacobian of h at μ_{t-1}

G_t is the Jacobian of f at μ_{t-1}

$$K_t = \overline{\Sigma}_t H_t^\top (H_t \overline{\Sigma}_t H_t^\top + R)^{-1}$$

Advantages / Disadvantages

- Extended Kalman filters:
    - Same strengths of Kalman filters (except optimality)
    - Relax the assumption of linear state transitions and observations.
    - Widely used.
- But…
    - Lose optimality guarantees.
    - Taylor expansion can be a bad approximation for highly non-linear systems.
    - If learning models, need accurate estimation of gradients.

Tracking multiple hypotheses

- Gaussians are unimodal distributions — key limitation of KF and EKF.
- One extension of KFs and EKFs is to use a Gaussian mixture model representation.
- Each possible mode is represented by a different Gaussian and updates are similar to KF/EKF updates.
  - But, must include mechanisms for splitting or pruning individual modes when they become very unlikely.

Practice

- A robot is using the following model of its environment:
  - $f(x_{t-1}, u_t) = Ax_{t-1} + Bu_t + w_t$ where w_t is Gaussian noise and A and B are matrices. The Jacobian of f is A.
  - $g(x_t) = Hx_t + v_t$ where v_t is Gaussian noise, H is a matrix, and the Jacobian of g is H.

What are the extended Kalman filter updates under this model?

Practice

- A robot is using the following model of its environment:
    - $f(x_{t-1}, u_t) = Ax_{t-1} + Bu_t + w_t$ where w_t is Gaussian noise and A and B are matrices. The Jacobian of f is A.
    - $g(x_t) = Hx_t + v_t$ where v_t is Gaussian noise, H is a matrix, and the Jacobian of g is H.

What are the extended Kalman filter updates under this model?

Note that f and g are already linear functions. Consequently, the EKF reduces to the KF.

Summary

- Extended the linear Gaussian model to the non-linear Gaussian model.
- Introduced the extended Kalman filter as a generalization of the Kalman filter for non-linear Gaussian assumption.
- Discussed pros and cons of the EKF.

Action Items

- Reading for next week (released soon); send a reading response by 12 pm on Monday.