From Holant To #CSP And Back: Dichotomy For Holant^c Problems

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Abstract

We explore the intricate interdependent relationship among counting problems, considered from three frameworks for such problems: Holant Problems, counting CSP and weighted H-colorings. We consider these problems for general complex valued functions that take boolean inputs. We show that results from one framework can be used to derive results in another, and this happens in both directions. Holographic reductions discover an underlying unity, which is only revealed when these counting problems are investigated in the complex domain $\mathbb{C}$. We prove three complexity dichotomy theorems, leading to a general theorem for Holant^c problems. This is the natural class of Holant problems where one can assign constants 0 or 1. More specifically, given any signature grid on $G = (V, E)$ over a set $\mathscr{F}$ of symmetric functions, we completely classify the complexity to be in P or #P-hard, according to $\mathscr{F}$, of

$$\sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v(\sigma\mid_{E(v)}),$$

where $f_v \in \mathscr{F} \cup \{\mathbf{0}, \mathbf{1}\}$ ($\mathbf{0}$, $\mathbf{1}$ are the unary constant 0, 1 functions). Not only is holographic reduction the main tool, but also the final dichotomy can be only naturally stated in the language of holographic transformations. The proof goes through another dichotomy theorem on boolean complex weighted #CSP.

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1 Introduction

In order to study the complexity of counting problems, several interesting frameworks have been proposed. One is called counting Constraint Satisfaction Problems (#CSP) [1, 2, 3, 15, 20]. Another well studied framework is called H-coloring or Graph Homomorphism, which can be viewed as a special case of #CSP problems [4, 5, 6, 16, 17, 18, 19, 22, 23]. Recently, we proposed a new refined framework called Holant Problems [9, 11] inspired by Valiant's Holographic Algorithms [31, 32]. One reason such frameworks are interesting is because the language is *expressive* enough so that they can express many natural counting problems, while *specific* enough so that we can prove *dichotomy theorems* (i.e., every problem in the class is either in P or #P-hard) [12]. By a theorem of Ladner, if P $\neq$ NP, or P $\neq$ #P, then such a dichotomy for NP or #P is *false*. Many natural counting problems can be expressed in all three frameworks. This includes counting the number of vertex covers, the number of k-colorings in a graph, and many others. However, some natural and important counting problems, such as counting the number of perfect matchings in a graph, *cannot* be expressed as a graph homomorphism function [21], but can be naturally expressed as a Holant Problem. Both #CSP and Graph Homomorphisms can be viewed as special cases of Holant Problems. The Holant framework of counting problems makes a finer complexity classification. A rich mathematical structure is uncovered in the Holant framework regarding the complexity of counting problems, which is sometimes difficult even to state in #CSP. This is particularly true when we apply holographic reductions [31, 32, 9].

We give a brief description of the Holant framework here. A *signature grid* $\Omega = (G, \mathscr{F}, \pi)$ is a tuple, where $G = (V, E)$ is a graph, and π labels each $v \in V$ with a function $f_v \in \mathscr{F}$. We consider all edge assignments (in this paper 0-1 assignments). An assignment σ for every $e \in E$ gives an evaluation $\prod_{v \in V} f_v(\sigma \mid_{E(v)})$, where $E(v)$ denotes the incident edges of v, and $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma \mid_{E(v)}).$$

For example, consider the Perfect Matching problem on G. This problem corresponds to attaching the Exact-One function at every vertex of G, and then consider all 0-1 edge assignments. In this case, Holant_Ω counts the number of perfect matchings. If we use the At-Most-One function at every vertex, then we count all (not necessarily perfect) matchings. We use the notation Holant($\mathscr{F}$) to denote the class of Holant problems where all functions are given by $\mathscr{F}$.

To see that Holant is a more expressive framework, we show that every #CSP problem can be directly simulated by a Holant problem. Represent an instance of a #CSP problem by a bipartite graph where LHS are labeled by variables and RHS are labeled by constraints. Now the signature grid Ω on this bipartite graph is as follows: Every variable node on LHS is labeled with an Equality function, every constraint node on RHS is labeled with the given constraint function. Then Holant_Ω is exactly the answer to the counting CSP problem. In effect, the Equality function on a node in LHS forces the incident edges to take the same value; this effectively reduces to a vertex assignment on LHS as in #CSP. We can show that #CSP is equivalent to Holant problems where Equality functions of k variables, for arbitrary k (denoted by $=_k$), are freely and implicitly available as constraints. However, this process provably cannot be reversed in general (if P $\neq$ #P). While #CSP is the same as adding all $=_k$ to Holant, the effect of this is non-trivial. From the lens of holographic transformations, $=_3$ is a full-fledged non-degenerate symmetric function of arity 3.

Meanwhile, starting from the Holant framework, rather than assuming Equality functions are free, one can consider new classes of counting problems which are difficult to express as #CSP problems. One such class, called Holant* Problems [11], is the class of Holant Problems where all unary functions are freely available. If we allow only two special unary functions $\mathbf{0}$ and $\mathbf{1}$ as freely available, then we obtain the family of counting problems called Holant^c Problems, which is even more appealing. This is the class of all Holant Problems (on boolean variables) where one can set any particular edge (variable) to 0 or 1 in an input graph.

Previously we proved a dichotomy theorem for Holant$^*(\mathscr{F})$, where $\mathscr{F}$ is any set of complex-valued symmetric functions [11]. It is used to prove a dichotomy theorem for #CSP in [11]. For Holant$^c(\mathscr{F})$ we could only prove a dichotomy theorem for real-valued functions. In this paper we manage to traverse in the other direction, going from #CSP to Holant Problems. First we establish a dichotomy theorem for a special Holant class. Second we prove a more general dichotomy for bipartite Holant Problems. Finally by going through #CSP, we prove a dichotomy theorem for complex-valued Holant^c Problems. Now we describe our results in more detail.

A symmetric function $f : \{0,1\}^k \to \mathbb{C}$ will be written as $[f_0, f_1, \dots, f_k]$, where f_j is the value of f on inputs of Hamming weight j. Our first main result (in Section 3) is a dichotomy theorem for Holant($\mathscr{F}$), where $\mathscr{F}$ contains a single ternary function $[x_0, x_1, x_2, x_3]$. More generally, as proved by holographic reductions, we get a dichotomy theorem for Holant($[y_0, y_1, y_2] | [x_0, x_1, x_2, x_3]$) defined on 2-3 regular bipartite graphs. Here the notation indicates that every vertex of degree 2 on LHS has label $[y_0, y_1, y_2]$ and every vertex of degree 3 on RHS has label $[x_0, x_1, x_2, x_3]$. This is the foundation of the remaining two dichotomy results in this paper. Previously we proved a dichotomy theorem for Holant($[y_0, y_1, y_2] | [x_0, x_1, x_2, x_3]$), when all x_i, y_j take values in $\{0, 1\}$ [9]. Kowalczyk extended this to $\{-1, 0, 1\}$ in [26]. In [10], we gave a dichotomy theorem for Holant($[y_0, y_1, y_2] | [1, 0, 0, 1]$), where y_0, y_1, y_3 take arbitrary real values. Finally this last result was extended to arbitrary complex numbers [27]. Our result here is built upon these results, especially [27].

Our second result (Section 4) is a dichotomy theorem, under a mild condition, for bipartite Holant problems Holant($\mathscr{F}_1 | \mathscr{F}_2$). To prove that, we first use holographic reductions to transform it to Holant($\mathscr{F}_1' | \mathscr{F}_2'$), where we transform some non-degenerate function $[x_0, x_1, x_2, x_3] \in \mathscr{F}_2$ to the Equality function $(=_3) = [1, 0, 0, 1] \in \mathscr{F}_2'$. Then we prove that we can "realize" the binary Equality function $(=_2) = [1, 0, 1]$ in the left side and reduce the problem to #CSP($\mathscr{F}_1' \cup \mathscr{F}_2'$). This is a new proof approach. Previously in [11], we reduced a #CSP problem to a Holant problem and obtained results for #CSP. Here, we go the opposite way, using results for #CSP to prove dichotomy theorems for Holant problems. This is made possible by our complete dichotomy theorem for boolean complex weighted #CSP [11]. We note that proving this over $\mathbb{C}$ is crucial, as holographic reductions naturally go beyond $\mathbb{R}$. We also note that our dichotomy theorem here does not require the functions in $\mathscr{F}_1$ or $\mathscr{F}_2$ to be symmetric. This will be useful in the future.

Our third main result, also the initial motivation of this work, is a dichotomy theorem for symmetric complex Holant^c problems. This improves our previous result in [11]. We made a conjecture in [11] that the dichotomy theorem stated as Theorem 6.2 is also true for symmetric complex functions. It turns out that this conjecture is not correct as stated. For example, Holant^c($[1, 0, i, 0]$) is tractable (according to our new theorem), but not included in the tractable cases by the conjecture. After isolating these new tractable cases we prove *everything else* is #P-hard. Generally speaking, non-trivial and previously unknown tractable cases are what make dichotomy theorems particularly interesting, but at the same time make them more difficult to prove (especially for hardness proofs, which must "carve out" exactly what's left.). The proof approach here is also different from that of [11]. In [11], the idea is to interpolate all unary functions and then use the results for Holant* Problems. Here we first prove that we can realize some non-degenerate ternary function, for which we can use the result of our first dichotomy theorem. Then we use our second dichotomy theorem to further reduce the problem to #CSP and obtain a dichotomy theorem for Holant^c.

The study of Holant Problems is strongly influenced by the development of holographic algorithms [31, 32, 7, 9]. Holographic reduction is a primary technique in the proof of these dichotomies, both for the tractability part and the hardness part. More than that—and this seems to be the first instance—holographic reduction even provides the correct language for the *statements* of these dichotomies. Without using holographic reductions, it is not easy to even fully describe what are the tractable cases in the dichotomy theorem. Another interesting observation is that by employing holographic reductions, complex numbers appear naturally (as *eigenvalues*) in an essential way. Even if one is only interested in integer or real valued counting problems, in the complex domain $\mathbb{C}$ the picture becomes whole. "It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one." —Jacques Hadamard.

2 Preliminaries

Our functions take values in $\mathbb{C}$ by default. Strictly speaking complexity results should be restricted to computable numbers in the Turing model; but it is more convenient to express this over $\mathbb{C}$. We say a problem is tractable if it is computable in P. The framework of Holant Problems is defined for functions mapping any $[q]^k \to \mathbb{C}$ for a finite q. Our results in this paper are for the Boolean case $q = 2$.

Let $\mathscr{F}$ be a set of functions. A *signature grid* $\Omega = (G, \mathscr{F}, \pi)$ is a tuple, where $G = (V, E)$ is a graph, and π labels each $v \in V$ with a function $f_v \in \mathscr{F}$. The Holant problem on instance Ω is to compute

$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v(\sigma\mid_{E(v)})$, a sum over all 0-1 edge assignments, of the products of the function evaluations at each vertex. A function f_v can be represented as a truth table. It will be more convenient to denote it as a tensor in $(\mathbb{C}^2)^{\otimes \deg(v)}$, or a vector in $\mathbb{C}^{2^{\deg(v)}}$, when we perform holographic tranformations. We also call it a *signature*. We denote by $=_k$ the EQUALITY signature of arity k. A symmetric function f on k Boolean variables can be expressed by $[f_0, f_1, \ldots, f_k]$, where f_j is the value of f on inputs of Hamming weight j. Thus, for example, $\mathbf{0} = [1, 0]$, $\mathbf{1} = [0, 1]$ and $(=_k) = [1, 0, \ldots, 0, 1]$ (with $(k-1)$ 0's).

Definition 2.1. *Given a set of signatures $\mathscr{F}$, we define a counting problem* $\text{Holant}(\mathscr{F})$*:*
Input: A signature grid $\Omega = (G, \mathscr{F}, \pi)$;
Output: Holant_Ω.

We would like to characterize the complexity of Holant problems in terms of its signature set $\mathscr{F}$. Some special families of Holant problems have already been widely studied. For example, if $\mathscr{F}$ contains all EQUALITY signatures $\{=_1, =_2, =_3, \cdots\}$, then this is exactly the weighted #CSP problem. In [11], we also introduced the following two special families of Holant problems by assuming some signatures are freely available.

Definition 2.2. *Let $\mathscr{U}$ denote the set of all unary signatures. Then* $\text{Holant}^*(\mathscr{F}) = \text{Holant}(\mathscr{F} \cup \mathscr{U})$.

Definition 2.3. *Given a set of signatures $\mathscr{F}$, we use* $\text{Holant}^c(\mathscr{F})$ *to denote* $\text{Holant}(\mathscr{F} \cup \{\mathbf{0}, \mathbf{1}\})$.

Replacing a signature $f \in \mathscr{F}$ by a constant multiple cf, where $c \neq 0$, does not change the complexity of $\text{Holant}(\mathscr{F})$. An important property of a signature is whether it is degenerate.

Definition 2.4. *A signature is degenerate iff it is a tensor product of unary signatures. In particular, a symmetric signature in $\mathscr{F}$ is degenerate iff it can be expressed as $\lambda[x, y]^{\otimes k}$.*

To introduce the idea of holographic reductions, it is convenient to consider bipartite graphs. This is without loss of generality. For any general graph, we can make it bipartite by adding an additional vertex on each edge, and giving each new vertex the EQUALITY function $=_2$ on 2 inputs.

We use $\text{Holant}(\mathscr{G}|\mathscr{R})$ to denote all counting problems, expressed as Holant problems on bipartite graphs $H = (U, V, E)$, where each signature for a vertex in U or V is from $\mathscr{G}$ or $\mathscr{R}$, respectively. An input instance for the bipartite Holant problem is a bipartite signature grid and is denoted as $\Omega = (H, \mathscr{G}|\mathscr{R}, \pi)$. Signatures in $\mathscr{G}$ are denoted by column vectors (or contravariant tensors); signatures in $\mathscr{R}$ are denoted by row vectors (or covariant tensors) [14].

One can perform (contravariant and covariant) tensor transformations on the signatures. We will define a simple version of holographic reductions, which are invertible. Suppose $\text{Holant}(\mathscr{G}|\mathscr{R})$ and $\text{Holant}(\mathscr{G}'|\mathscr{R}')$ are two Holant problems defined for the same family of graphs, and $T \in \mathbf{GL}_2(\mathbb{C})$ is a basis. We say that there is an (invertible) holographic reduction from $\text{Holant}(\mathscr{G}|\mathscr{R})$ to $\text{Holant}(\mathscr{G}'|\mathscr{R}')$, if the *contravariant* transformation $G' = T^{\otimes g} G$ and the *covariant* transformation $R = R' T^{\otimes r}$ map $G \in \mathscr{G}$ to $G' \in \mathscr{G}'$ and $R \in \mathscr{R}$ to $R' \in \mathscr{R}'$, and vice versa, where G and R have arity g and r respectively. (Notice the reversal of directions when the transformation $T^{\otimes n}$ is applied. This is the meaning of *contravariance* and *covariance*.)

Theorem 2.5 (Valiant's Holant Theorem)**.** *Suppose there is a holographic reduction from $\#\mathscr{G}|\mathscr{R}$ to $\#\mathscr{G}'|\mathscr{R}'$ mapping signature grid Ω to Ω', then* $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

In particular, for invertible holographic reductions from $\text{Holant}(\mathscr{G}|\mathscr{R})$ to $\text{Holant}(\mathscr{G}'|\mathscr{R}')$, one problem is in P iff the other one is in P, and similarly one problem is #P-hard iff the other one is also #P-hard.

In the study of Holant problems, we will often transfer between bipartite and non-bipartite settings. When this does not cause confusion, we do not distinguish signatures between column vectors (or contravariant tensors) and row vectors (or covariant tensors). Whenever we write a transformation as $T^{\otimes n} F$ or $T\mathscr{F}$, we view the signatures as column vectors (or contravariant tensors); whenever we write a transformation as $F T^{\otimes n}$ or $\mathscr{F} T$, we view the signatures as row vectors (or covariant tensors).

3 Dichotomy Theorem for Ternary Signatures

In this section, we consider the complexity of $\text{Holant}([x_0, x_1, x_2, x_3])$. If $[x_0, x_1, x_2, x_3]$ is degenerate, it is trivially tractable, since a degenerate signature factors as a tensor product and the signature grid simply decomposes into isolated edges. In the following we always assume that it is non-degenerate. Given a non-degenerate signature $[x_0, x_1, x_2, x_3]$, we can find a non-zero tuple (a, b, c) (unique up to a scalar factor), such that $ax_0+bx_1+cx_2 = 0$ and $ax_1+bx_2+cx_3 = 0$. If $c \neq 0$, the sequence $[x_0, x_1, x_2, x_3]$ is a second order linear recurrence sequence. Its characteristic equation is $a+b\lambda+c\lambda^2 = 0$. We can write down an expression for this sequence $[x_0, x_1, x_2, x_3]$ by the eigenvalues. When $c = 0$ and $a \neq 0$, we can consider its reverse sequence. The case $a = c = 0$ can be viewed as a limiting case. Statedly succinctly, the sequence $[x_0, x_1, x_2, x_3]$ can always be expressed by one of the following three categories (with the convention that $\alpha^0 = 1$, and $k\alpha^{k-1} = 0$ if $k = 0$, even when $\alpha = 0$):

1. $x_k = \alpha_1^{3-k}\alpha_2^k + \beta_1^{3-k}\beta_2^k$, where $\det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \neq 0$;
2. $x_k = Ak\alpha^{k-1} + B\alpha^k$, where $A \neq 0$;
3. $x_k = A(3-k)\alpha^{2-k} + B\alpha^{3-k}$, where $A \neq 0$.

The first category corresponds to the case when the characteristic equation $a+b\lambda+c\lambda^2 = 0$ has two distinct roots and we call it the *generic* case. The second category corresponds to the case when it has a double root (and $c \neq 0$) and we call it the *double-root* case. Category 3 is also a *double-root* case, and is only needed for a very special case $b = c = 0$. It can be viewed as the reversal of category 2, and we always omit the formal proof for this category since it is similar to category 2.

For the *generic* case, we can apply a holographic reduction using $T = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$. Then we have the following reductions (readers may wish to take a look at Section 7 in Appendix):

$$\begin{aligned} \text{Holant}([x_0, x_1, x_2, x_3]) &\equiv_{\text{T}} \text{Holant}([1, 0, 1] | [x_0, x_1, x_2, x_3]) \\ &\equiv_{\text{T}} \text{Holant}([1, 0, 1]T^{\otimes 2} | (T^{-1})^{\otimes 3}[x_0, x_1, x_2, x_3]) \\ &\equiv_{\text{T}} \text{Holant}([y_0, y_1, y_2] | [1, 0, 0, 1]), \end{aligned}$$

where $[y_0, y_1, y_2] = [1, 0, 1]T^{\otimes 2}$. (We note that $[x_0, x_1, x_2, x_3] = T^{\otimes 3}[1, 0, 0, 1]$.)

So for the *generic* case, we only need to give a dichotomy for $\text{Holant}([y_0, y_1, y_2] | [1, 0, 0, 1])$, which has been proved in [27]; we quote the theorem here.

Theorem 3.1. ([27]) *The problem* $\text{Holant}([y_0, y_1, y_2] | [1, 0, 0, 1])$ *is #P-hard for all* $y_0, y_1, y_2 \in \mathbb{C}$ *except in the following cases, for which the problem is in P: (1)* $y_1^2 = y_0 y_2$*; (2)* $y_0^{12} = y_1^{12}$ *and* $y_0 y_2 = -y_1^2$ *(* $y_1 \neq 0$*); (3)* $y_1 = 0$*; or (4)* $y_0 = y_2 = 0$*.*

To get a complete dichotomy for $\text{Holant}([x_0, x_1, x_2, x_3])$, we next deal with the *double-root* case.

Lemma 3.2. *Let* $x_k = Ak\alpha^{k-1} + B\alpha^k$*, where* $A \neq 0$ *and* $k = 0, 1, 2, 3$. $\text{Holant}([x_0, x_1, x_2, x_3])$ *is #P-hard unless* $\alpha^2 = -1$*. On the other hand, if* $\alpha = \pm i$*, then the problem is in P.*

Proof. If $\alpha = \pm i$, the signature $[x_0, x_1, x_2, x_3]$ satisfies the recurrence relation $x_{k+2} = \pm 2i x_{k+1} + x_k$, where $k = 0, 1$. This is a generalized Fibonacci signature (see [9]). Thus we know that it is in P by holographic algorithms [9] using Fibonacci gates.

Now we assume that $\alpha \neq \pm i$. We first apply an *orthogonal* holographic transformation. The crucial observation is that we can view $\text{Holant}([x_0, x_1, x_2, x_3])$ as the bipartite $\text{Holant}([1, 0, 1] | [x_0, x_1, x_2, x_3])$ and an orthogonal transformation $T \in \mathbf{O}_2(\mathbb{C})$ keeps $(=_2) = [1, 0, 1]$ invariant: $[1, 0, 1]T^{\otimes 2} = [1, 0, 1]$. By a suitable orthogonal transformation T, we can transform $[x_0, x_1, x_2, x_3]$ to $[v, 1, 0, 0]$ for some $v \in \mathbb{C}$, up to a scalar. (Details are in Appendix.) So the complexity of $\text{Holant}([x_0, x_1, x_2, x_3])$ is the same as $\text{Holant}([v, 1, 0, 0])$.

Next we prove that $\text{Holant}([v, 1, 0, 0])$ is #P-hard for all $v \in \mathbb{C}$. First, for $v = 0$, $\text{Holant}([0, 1, 0, 0])$ is #P-hard, because it is the problem of counting all perfect matchings on 3-regular graphs [13]. Second, let $v \neq 0$. We can realize $[v^3+3v, v^2+1, v, 1]$ by connecting $[v, 1, 0, 0]$'s as illustrated in Figure 1, so it is enough to prove that $\text{Holant}([v^3 + 3v, v^2 + 1, v, 1])$ is #P-hard. In tensor product notation this signature is

$$[v^3 + 3v, v^2 + 1, v, 1]^{\text{T}} = \frac{1}{2}\left(\begin{bmatrix} v+1 \\ 1 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} v-1 \\ 1 \end{bmatrix}^{\otimes 3}\right).$$

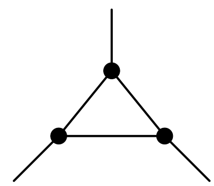

Figure 1: All vertex signatures are $[v, 1, 0, 0]$.

Then the following reduction chain holds:

$$\begin{aligned}\text{Holant}([v^3+3v, v^2+1, v, 1]) &\equiv_{\text{T}} \text{Holant}([1,0,1]|[v^3+3v, v^2+1, v, 1]) \\ &\equiv_{\text{T}} \text{Holant}([v^2+2v+2, v^2, v^2-2v+2]|[1,0,0,1])\end{aligned}$$

where the second step is a holographic reduction using $\begin{bmatrix} v+1 & v-1 \\ 1 & 1 \end{bmatrix}$. We can apply Theorem 3.1 to Holant($[v^2+2v+2, v^2, v^2-2v+2]|[1,0,0,1]$), by checking against the four exceptions. (1) $[v^2+2v+2, v^2, v^2-2v+2]$ is non-degenerate. (2) There is no solution for $(v^2-2v+2)^{12} = v^{24}$ and $(v^2+2v+2)(v^2-2v+2)+v^4 = 0$. (3) $v^2 \neq 0$. (4) It cannot be the case that $v^2+2v+2 = v^2-2v+2 = 0$. Therefore Holant($[v^3+3v, v^2+1, v, 1]$) is #P-hard, and so is Holant($[v, 1, 0, 0]$) for all $v \in \mathbb{C}$. □

By Theorem 3.1 and Lemma 3.2, we have a complete dichotomy theorem for Holant($[x_0, x_1, x_2, x_3]$). From this, we can get a further dichotomy for all bipartite Holant problems Holant($[y_0, y_1, y_2]|[x_0, x_1, x_2, x_3]$). The reduction is standard. For any non-degenerate $[y_0, y_1, y_2]$, we can find a transformation T, such that $[y_0, y_1, y_2]T^{\otimes 2} = [1, 0, 1]$. Then the bipartite problem Holant($[y_0, y_1, y_2]|[x_0, x_1, x_2, x_3]$) is transformed to the equivalent problem Holant($(T^{-1})^{\otimes 3}[x_0, x_1, x_2, x_3]$), for which we have a dichotomy theorem. The following dichotomy theorems are both effective dichotomies. They use the function families $\mathscr{A}$ and $\mathscr{P}$, called *affine* functions and functions of a *product type*. (See Section 6 for more details.)

Theorem 3.3. Holant($[x_0, x_1, x_2, x_3]$) *is #P-hard unless* $[x_0, x_1, x_2, x_3]$ *satisfies one of the following conditions, in which case the problem is in P:*

1. $[x_0, x_1, x_2, x_3]$ *is degenerate;*
2. *There is a* 2×2 *matrix* T *such that* $[x_0, x_1, x_2, x_3] = T^{\otimes 3}[1, 0, 0, 1]$ *and* $[1, 0, 1]T^{\otimes 2}$ *is in* $\mathscr{A} \cup \mathscr{P}$*;*
3. *For* $\alpha \in \{2i, -2i\}$*,* $x_2 + \alpha x_1 - x_0 = 0$ *and* $x_3 + \alpha x_2 - x_1 = 0$*.*

Theorem 3.4. Holant($[y_0, y_1, y_2]|[x_0, x_1, x_2, x_3]$) *is #P-hard unless* $[x_0, x_1, x_2, x_3]$ *and* $[y_0, y_1, y_2]$ *satisfy one of the following conditions, in which case the problem is in P:*

1. $[x_0, x_1, x_2, x_3]$ *is degenerate;*
2. *There is a* 2×2 *matrix* T *such that* $[x_0, x_1, x_2, x_3] = T^{\otimes 3}[1, 0, 0, 1]$ *and* $[y_0, y_1, y_2]T^{\otimes 2}$ *is in* $\mathscr{A} \cup \mathscr{P}$*;*
3. *There is a* 2×2 *matrix* T *such that* $[x_0, x_1, x_2, x_3] = T^{\otimes 3}[1, 1, 0, 0]$ *and* $[y_0, y_1, y_2]T^{\otimes 2}$ *is of form* $[0, *, *]$*;*
4. *There is a* 2×2 *matrix* T *such that* $[x_0, x_1, x_2, x_3] = T^{\otimes 3}[0, 0, 1, 1]$ *and* $[y_0, y_1, y_2]T^{\otimes 2}$ *is of form* $[*, *, 0]$*.*

4 Reductions Between Holant and #CSP

In this section, we extend the dichotomies in Section 3 for a single ternary signature to a set of signatures. We are going to give a dichotomy for Holant($[x_0, x_1, x_2, x_3] \cup \mathscr{F}$), or more generally for Holant($[y_0, y_1, y_2] \cup \mathscr{G}_1|[x_0, x_1, x_2, x_3] \cup \mathscr{G}_2$), where $[y_0, y_1, y_2]$ and $[x_0, x_1, x_2, x_3]$ are non-degenerate. In this section, we focus on the generic case of $[x_0, x_1, x_2, x_3]$, and the double root case will be handled in the next section in Lemma 5.2.

For the generic case, we can apply a holographic reduction to transform $[x_0, x_1, x_2, x_3]$ to $[1, 0, 0, 1]$. Therefore we only need to give a dichotomy for Holant problems of the form $\text{Holant}([y_0, y_1, y_2] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$, where $[y_0, y_1, y_2]$ is non-degenerate. We make one more observation: The ternary equality signature $[1, 0, 0, 1]$ is invariant under the following transformations:

$$\mathscr{T}_3 \triangleq \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix} \right\},$$

where $\omega = \omega_3 = e^{2\pi i/3}$. For any $T \in \mathscr{T}_3$,

$$\text{Holant}([y_0, y_1, y_2] | [1, 0, 0, 1] \cup \mathscr{F}) \equiv_{\text{T}} \text{Holant}([y_0, y_1, y_2] T^{\otimes 2} | [1, 0, 0, 1] \cup T^{-1} \mathscr{F}).$$

As a result, we can normalize $[y_0, y_1, y_2]$ by a holographic reduction with any $T \in \mathscr{T}_3$. In particular, we call a symmetric binary signature $[y_0, y_1, y_2]$ *normalized* if $y_0 = 0$ or it is not the case that y_2 is y_0 times a t-th primitive root of unity, and $t = 3t'$ where $\gcd(t', 3) = 1$. If $[y_0, y_1, y_2]$ is not normalized, then $y_2 = y_0 \omega_t^s$, where $\omega_t = e^{2\pi i/t}$ and $\gcd(s, t) = 1$. Write $1 = 3u + t'v$ for some integers u and v, then $\omega_t = \omega^v \omega_{t'}^u$ and $\omega_t^s = \omega^k \omega_{t'}^l$, where $k \equiv sv \bmod 3$, and $\gcd(k, 3) = 1$. Hence $k = 1$ or 2. After applying the transformation $\begin{bmatrix} 1 & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathscr{T}_3$, we get a new signature $[y_0, y_1 \omega^k, y_0 \omega_{t'}^l]$, which is normalized. So in the following, we only deal with normalized $[y_0, y_1, y_2]$. In one case, we also need to normalized a unary signature $[x_0, x_1]$, namely $x_0 = 0$ or x_1 is not a multiple of x_0 by a t-th primitive root of unity, and $t = 3t'$ where $\gcd(t', 3) = 1$. Again we can normalize the unary signature by a suitable $T \in \mathscr{T}_3$. We note that a normalized signature is still normalized after a scalar multiple.

Theorem 4.1. *Let $[y_0, y_1, y_2]$ be a normalized and non-degenerate signature. And in the case of $y_0 = y_2 = 0$, we further assume that $\mathscr{G}_1$ contains a unary signature $[a, b]$, which is normalized and $ab \neq 0$. Then*

$$\text{Holant}([y_0, y_1, y_2] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2) \equiv_{\text{T}} \#\text{CSP}([y_0, y_1, y_2] \cup \mathscr{G}_1 \cup \mathscr{G}_2).$$

More specifically, $\text{Holant}([y_0, y_1, y_2] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$ *is #P-hard unless* $[y_0, y_1, y_2] \cup \mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{P}$ *or* $[y_0, y_1, y_2] \cup \mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{A}$*, in which cases the problem is in P.*

This dichotomy is an important reduction step in the proof of our dichotomy theorem for Holant^c. It is also interesting in its own right as a connection between Holant and #CSP. The assumption on signature normalization in the statement of the theorem is without loss of generality. For a non-normalized signature, we can apply a normalization and then apply the dichotomy criterion in the theorem. The additional assumption of the existence of a non-zero unary signature circumvents a technical difficulty, and finds a circuitous route to the proof of our dichotomy theorem for Holant^c, the main objective in this paper. For Holant^c, the needed unary signature will be produced from $[1, 0]$ and $[0, 1]$. We also note that we do not require the signatures in $\mathscr{G}_1$ and $\mathscr{G}_2$ to be symmetric, so the theorem could have applications in dichotomy theorems for general Holant problems over non-symmetric signatures.

One direction in Theorem 4.1, from Holant to #CSP, is straightforward. Thus our main claim is a reduction from #CSP to these bipartite Holant problems. The approach is to construct the binary equality gate $[1, 0, 1] = (=_2)$ in LHS in the Holant problem. As soon as we have $[1, 0, 1]$ in LHS, together with $[1, 0, 0, 1] = (=_3)$ in RHS, we get equality gates of all arities $(=_k)$ in RHS. Also with the help of $[1, 0, 1]$ in LHS we can transfer $\mathscr{G}_2$ to LHS. Then we have all of $\#\text{CSP}([y_0, y_1, y_2] \cup \mathscr{G}_1 \cup \mathscr{G}_2)$.

If the problem $\text{Holant}([y_0, y_1, y_2] | [1, 0, 0, 1])$ is already #P-hard, then for any $\mathscr{G}_1$ and $\mathscr{G}_2$, it is #P-hard. So we only need to consider the cases, where $\text{Holant}([y_0, y_1, y_2] | [1, 0, 0, 1])$ is not #P-hard. For this, we again use Theorem 3.1 from [27]. The first tractable case $y_1^2 = y_0 y_2$ is degenerate, which does not apply here. The following three lemmas deal with the remaining three tractable cases respectively.

For the case $y_0^{12} = y_1^{12}$ and $y_0 y_2 = -y_1^2$ ($y_1 \neq 0$), we can scale it to $[a, 1, b]$, where $a^{12} = 1$ and $ab = -1$. Since $[a, 1, b]$ is normalized, it follows that $a^4 = 1$.

Lemma 4.2. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two sets of signatures. For all pairs of a and b satisfying $a^4 = 1$ and $ab = -1$,* $\text{Holant}([a, 1, b] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$ *is #P-hard unless $\mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{A}$, in which case it is in P.*

Proof. We first prove that when $a^4 = 1$ and $ab = -1$,

$$\text{Holant}([a, 1, b] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2) \equiv_T \#\text{CSP}([a, 1, b] \cup \mathscr{G}_1 \cup \mathscr{G}_2).$$

To get this, it is sufficient to construct $[1, 0, 1]$ in LHS.

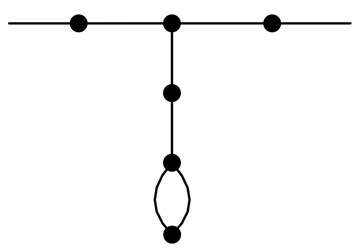

Figure 2: A gadget construction for the binary disequality gate $(\neq_2)$ on LHS. Degree 3 vertices have signature $=_3$, degree 2 vertices have signature $[1, \pm i, 1]$.

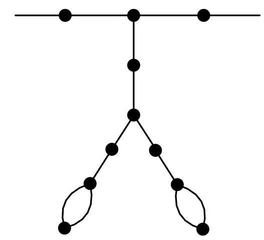

Figure 3: Another gadget construction for $(\neq_2)$ on LHS. Degree 3 vertices have signature $=_3$, degree 2 vertices signature $[1, \pm 1, -1]$.

Case 1: $a = \pm i$.

It is equivalent to consider $\text{Holant}([1, \pm i, 1] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$ because they only differ by a constant factor. We can construct $[1, 1]$ on RHS by connecting the two edges of a $[1, \pm i, 1]$ gate on the LHS with two edges of a $[1, 0, 0, 1] = (=_3)$ on the RHS. With the gadget in Figure 2, we can construct the binary disequality gate $[0, 1, 0] = (\neq_2)$ on LHS. Together with the $=_3$ on RHS, we can have $=_3$ on LHS. Connecting this LHS $=_3$ with a $[1, 1]$ on RHS, we can obtain the binary equality gate $[1, 0, 1] = (=_2)$ on LHS.

Case 2: $a = \pm 1$.

It is equivalent to consider $\text{Holant}([1, \pm 1, -1] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$. With the gadget in Figure 3, we can construct $[0, 1, 0]$ on LHS, and thus $[1, 0, 0, 1]$ on LHS. Furthermore, we can construct $[1, -1]$ and $[1, 0, -1]$ on both sides. By connecting $[1, -1]$ with $[1, 0, -1]$, we can realize $[1, 1]$ on both sides, and consequently $[1, 0, 1]$ on both sides.

By [11], $\#\text{CSP}([a, 1, b] \cup \mathscr{F})$ is #P-hard unless $[a, 1, b] \cup \mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{P}$ or $[a, 1, b] \cup \mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{A}$. Since $[a, 1, b] \in \mathscr{A} - \mathscr{P}$, we conclude that the only possible case which is not #P-hard is $\mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{A}$. This is also sufficient for tractability. The proof is complete. □

For the tractable case $y_1 = 0$ in Theorem 3.1, by non-degeneracy, we can scale it to be $[1, 0, a]$, where $a \neq 0$. Then we have the following lemma:

Lemma 4.3. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two sets of signatures, and let $a \neq 0$ be a complex number. We assume $[1, 0, a]$ is normalized. Then we have the following dichotomy:*

- *If $a^4 = 1$, then* $\text{Holant}([1, 0, a] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$ *is #P-hard unless $\mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{P}$ or $\mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{A}$, in which cases it is in P.*
- *if $a^4 \neq 1$, then* $\text{Holant}([1, 0, a] \cup \mathscr{G}_1 | [1, 0, 0, 1] \cup \mathscr{G}_2)$ *is #P-hard unless $\mathscr{G}_1 \cup \mathscr{G}_2 \subseteq \mathscr{P}$, in which case it is in P.*

Proof. As above, it is sufficient to show that we can construct $[1, 0, 1]$ in LHS. We will use the gadget in Figure 4 in our proof. We can use it to realize $[1, 0, a^{3k+1}]$ for any $k \in \mathbb{N}$ on LHS.

If a is not a root of unity, then we will be able to interpolate all signatures of the form $[1, 0, x]$ where $x \in \mathbb{C}$ on LHS. This uses a Vandermonde system and we omit the details. In particular, we will be able to interpolate $[1, 0, 1]$ on LHS. So we are done.

Now we can assume that a is a t-th primitive root of unity, that is $a = \omega_t^b$ for some b relatively prime to t, where $\omega_t = e^{2\pi i/t}$. If t is not a multiple of 3, then we can find an integer k, such that $3k + 1 \equiv 0 \pmod{t}$. Therefore, we can realize $[1, 0, 1]$ on LHS, and carry out the same reduction as above.

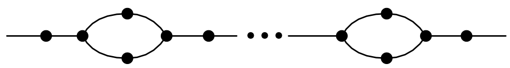

Figure 4: A recursive gadget we use to construct $[1,0,a^{3k+1}]$ on LHS. Ternary signatures are $[1,0,0,1]$, and binary signatures are $[1,0,a]$.

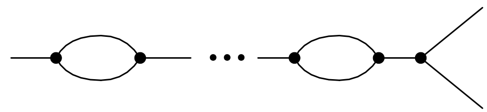

Figure 5: A recursive gadget we use to realize $[1,0,0,\omega_{3^{(l-1)}}^{mbk_0}]$ where m is an odd integer. All ternary gadgets are $[1,0,0,\omega_{3^{(l-1)}}^{bk_0}]$.

Now we consider the case of $t = 3^l t'$, where $l \geq 1$ and $\gcd(t',3) = 1$. Since $[1,0,a]$ is normalized, we have a further condition that $l > 1$. For this case, we do not know how to construct $[1,0,1]$ in LHS directly. Instead we will further apply a holographic reduction. Also in this case, we have $a^4 \neq 1$, so we want to prove that $\operatorname{Holant}([1,0,a]\cup\mathscr{G}_1|[1,0,0,1]\cup\mathscr{G}_2)$ is #P-hard unless $\mathscr{G}_1\cup\mathscr{G}_2 \subseteq \mathscr{P}$. The fact that the problem is in P when $\mathscr{G}_1\cup\mathscr{G}_2 \subseteq \mathscr{P}$ is obvious by Theorem 6.4, since $[1,0,a] \in \mathscr{P}$.

Since $2t'$ is not a multiple of 3, there exist some integers k and k_0, such that $3k+1 = 2k_0t'$. Since

$$a^{3k+1} = a^{2k_0t'} = \omega_{3^l t'}^{2bk_0t'} = \omega_{3^l}^{2bk_0}$$

we can realize $[1,0,\omega_{3^l}^{2bk_0}]$ on LHS. So

$$\operatorname{Holant}([1,0,\omega_{3^l}^{2bk_0}]\cup\mathscr{G}_1|\{=_3\}\cup\mathscr{G}_2) \leq_T \operatorname{Holant}([1,0,a]\cup\mathscr{G}_1|[1,0,0,1]\cup\mathscr{G}_2).$$

Therefore it is sufficient to prove that $\operatorname{Holant}([1,0,\omega_{3^l}^{2bk_0}]\cup\mathscr{G}_1|\{=_3\}\cup\mathscr{G}_2)$ is #P-hard if $\mathscr{G}_1\cup\mathscr{G}_2 \not\subseteq \mathscr{P}$.

We apply a holographic reduction under the basis $T = \begin{bmatrix} 1 & 0 \\ 0 & \omega_{3^l}^{-bk_0} \end{bmatrix}$, and get

$$\operatorname{Holant}([1,0,\omega_{3^l}^{2bk_0}]\cup\mathscr{G}_1|\{=_3\}\cup\mathscr{G}_2) \equiv_{\mathrm{T}} \operatorname{Holant}([1,0,1]\cup\mathscr{G}_1T|[1,0,0,\omega_{3^{(l-1)}}^{bk_0}]\cup T^{-1}\mathscr{G}_2).$$

We then use the gadget in Figure 5 to realize $[1,0,0,\omega_{3^{(l-1)}}^{3^{l-1}bk_0}] = [1,0,0,1] = (=_3)$ in RHS. Together with $[1,0,1] = (=_2)$ in LHS this gives all equality gates. As a result

$$\#\mathrm{CSP}([1,0,0,\omega_{3^{(l-1)}}^{bk_0}]\cup\mathscr{G}_1T\cup T^{-1}\mathscr{G}_2) \leq_T \operatorname{Holant}([1,0,1]\cup\mathscr{G}_1T|[1,0,0,\omega_{3^{(l-1)}}^{bk_0}]\cup T^{-1}\mathscr{G}_2).$$

Since $l > 1$, $[1,0,0,\omega_{3^{(l-1)}}^{bk_0}] \notin \mathscr{A}$. So the problem is #P-hard unless $\mathscr{G}_1T\cup T^{-1}\mathscr{G}_2 \subseteq \mathscr{P}$. Since T and T^{-1} are diagonal matrices, it is equivalent to say that $\mathscr{G}_1\cup\mathscr{G}_2 \subseteq \mathscr{P}$. This completes the proof.

□

For the last tractable case $y_0 = y_2 = 0$ in Theorem 3.1, we can scale it to $[0,1,0]$.

Lemma 4.4. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two sets of signatures, and $a \neq 0$ be a complex number. We assume $[1,a]$ is normalized. Then we have the following dichotomy:*

- *If $a^4 = 1$, then $\operatorname{Holant}(\{[0,1,0],[1,a]\}\cup\mathscr{G}_1|[1,0,0,1]\cup\mathscr{G}_2)$ is #P-hard unless $\mathscr{G}_1\cup\mathscr{G}_2 \subseteq \mathscr{P}$ or $\mathscr{G}_1\cup\mathscr{G}_2 \subseteq \mathscr{A}$, in which cases it is in P.*
- *if $a^4 \neq 1$, then $\operatorname{Holant}(\{[0,1,0],[1,a]\}\cup\mathscr{G}_1|[1,0,0,1]\cup\mathscr{G}_2)$ is #P-hard unless $\mathscr{G}_1\cup\mathscr{G}_2 \subseteq \mathscr{P}$, in which case it is in P.*

Proof. By connecting a $[a,1]$ and two $[0,1,0]$'s to a $[1,0,0,1]$, we can realize $[a,0,1]$, or equivalently $[1,0,1/a]$ on LHS, and by Lemma 4.3, the proof follows. □

5 Dichotomy Theorem for Complex Holant^c Problems

In this section, we prove our main result, a dichotomy theorem for Holant^c problems with complex valued symmetric signatures, which is stated as Theorem 5.3. The proof crucially uses the dichotomies proved in the previous two sections. In order to use them, we first prove in Lemma 5.1 that we can always realize a non-degenerate ternary signature except in some trivial cases. After having a non-degenerate ternary signature, we can immediately prove #P-hard if the ternary is not of one of the tractable cases in Theorem 3.3. For the tractable ternary signatures, we use Theorem 4.1 to extend the dichotomy theorem to the whole signature set. In Theorem 4.1, we only consider the generic case of the ternary function. The double-root case is handled here in Lemma 5.2.

Lemma 5.1. *Given any set of symmetric signatures $\mathscr{F}$ which contains $[1,0]$ and $[0,1]$, we can construct a non-degenerate symmetric ternary signature $X = [x_0, x_1, x_2, x_3]$, except in the following two trivial cases:*

1. *Any non-degenerate signature in $\mathscr{F}$ is of arity at most 2;*
2. *In $\mathscr{F}$, all unary signatures are of form $[x, 0]$ or $[0, x]$; all binary signatures are of form $[x, 0, y]$ or $[0, x, 0]$; and all signatures of arity greater than 2 are of form $[x, 0, \ldots, 0, y]$.*

Proof. Suppose case 1. does not hold, and let $[x_0, x_1, \ldots, x_m] \in \mathscr{F}$ be a non-degenerate signature of arity at least 3. Since we have $[1,0]$ and $[0,1]$, we can construct all sub-signatures of any signature in $\mathscr{F}$. If there exists a ternary non-degenerate sub-signature, we are done. Now suppose all ternary sub-signatures are degenerate, and $m > 3$. Then we can show that it must be of form $[x_0, 0, \ldots, 0, x_m]$, where $x_0 x_m \neq 0$. If we have a unary signature $[a, b]$ where $ab \neq 0$ or a unary sub-signature $[a, b]$ (where $ab \neq 0$) of a binary signature, we can connect this signature to $m-3$ dangling edges of $[x_0, 0, \ldots, 0, x_m]$ to get a non-degenerate ternary signature $[x, 0, 0, y]$, and we are done. Otherwise, all unary signatures are of form $[x, 0]$ or $[0, x]$ and all binary signatures are of form $[x, 0, y]$ or $[0, x, 0]$. Therefore we are in the exceptional case 2. □

We next consider the double root case for $X = [x_0, x_1, x_2, x_3]$. By Lemma 3.2, Holant(X) is already #P-hard unless the double eigenvalue is i or $-i$. Then, $x_{k+2} + \alpha x_{k+1} - x_k = 0$ for $k = 0, 1$, where $\alpha = \pm 2i$.

Lemma 5.2. *Let $X = [x_0, x_1, x_2, x_3]$ be a complex signature satisfying $x_{k+2} + \alpha x_{k+1} - x_k = 0$ for $k = 0, 1$, where $\alpha = \pm 2i$. Let $Y = [y_0, y_1, y_2]$ be a non-degenerate binary signature. Then Holant$(\{X, Y\})$ is #P-hard unless $y_2 + \alpha y_1 - y_0 = 0$ (in which case it is in P by Fibonacci gates).*

Proof. We prove this result for $\alpha = -2i$. The other case is similar.

The sequence $\{x_k\}$ can be written as follows: $x_k = Aki^{k-1} + Bi^k$, where $A \neq 0$. Thus, we have

$$X = T^{\otimes 3}[1,1,0,0]^{\mathrm{T}}, \qquad \text{where } T = \begin{bmatrix} 1 & \frac{B-1}{3} \\ i & A + \frac{B-1}{3} i \end{bmatrix}.$$

By expressing $\begin{bmatrix} y_0 & y_1 \\ y_1 & y_2 \end{bmatrix} = T_0^{\mathrm{T}} T_0$, which is always possible for some non-singular $T_0 = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, we have

$$Y = [1,0,1] T_0^{\otimes 2} = (\begin{pmatrix} 1 & 0 \end{pmatrix} T_0)^{\otimes 2} + (\begin{pmatrix} 0 & 1 \end{pmatrix} T_0)^{\otimes 2} = [a^2 + b^2, ac + bd, c^2 + d^2].$$

Thus we have the following chain of reductions

$$\text{Holant}(Y|X) \equiv_{\mathrm{T}} \text{Holant}([1,0,1]T_0^{\otimes 2} | T^{\otimes 3}[1,1,0,0]^{\mathrm{T}}) \equiv_{\mathrm{T}} \text{Holant}([1,0,1] | (T_0 T)^{\otimes 3}[1,1,0,0]^{\mathrm{T}}),$$

$$\text{where,} \quad T_0 T = \begin{bmatrix} a + ci & Ac + \frac{B-1}{3}(a + ci) \\ b + di & Ad + \frac{B-1}{3}(b + di) \end{bmatrix}, \qquad \text{and we will call it } \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Our next goal is to use an orthogonal matrix to transform $T_0 T$ to be upper-triangular. T_0 is non-singular, therefore p and r cannot both be zero. If $r = 0$ then $T_0 T$ is already upper-triangular. If $p = 0$ then the orthogonal matrix $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ makes $Q T_0 T$ upper-triangular. In general, if $p^2 + r^2 \neq 0$, then we can find a (complex) orthogonal matrix Q such that QT_0T is upper-triangular. It can be verified that $(QT_0T)^{\otimes 3}[1,1,0,0]^{\mathrm{T}}$, where QT_0T is upper-triangular, has the form $[v,u,0,0]$ for some non-zero u and v. We normalize it to $[v,1,0,0]$.

A crucial observation is that for any orthogonal matrix Q, the LHS $[1,0,1]$ is *unchanged* under the holographic transformation Q: $[1,0,1](Q^{-1})^{\otimes 2}=[1,0,1]$. This gives us

$$\mathrm{Holant}([1,0,1]|(T_0T)^{\otimes 3}[1,1,0,0]^{\mathrm{T}}) \equiv_{\mathrm{T}} \mathrm{Holant}([1,0,1]|[v,1,0,0]) \equiv_{\mathrm{T}} \mathrm{Holant}([v,1,0,0]),$$

for some v. This shows the equivalence of the original instance with $\mathrm{Holant}([v,1,0,0])$. By Lemma 3.2, $\mathrm{Holant}([v,1,0,0])$ is #P-hard.

Finally $p^2+r^2=0$ implies that $(a+ci)^2+(b+di)^2=(a^2+b^2)+2i(ac+bd)-(c^2+d^2)=0$. This is exactly $y_2-2iy_1-y_0=0$.

□

Theorem 5.3. *Let $\mathscr{F}$ be a set of complex symmetric signatures.* $\mathrm{Holant}^c(\mathscr{F})$ *is #P-hard unless $\mathscr{F}$ satisfies one of the following conditions, in which case it is tractable:*

1. $\mathrm{Holant}^*(\mathscr{F})$ *is tractable (for which we have an effective dichotomy—Theorem 6.1); or*
2. *There exists a $T\in\mathscr{T}$ such that $\mathscr{F}\subseteq T\mathscr{A}$, where $\mathscr{T}\triangleq\{T \mid [1,0,1]T^{\otimes 2},[1,0]T,[0,1]T\in\mathscr{A}\}$.*

Proof. First of all, if $\mathscr{F}$ is an exceptional case of Lemma 5.1, we know that $\mathrm{Holant}^*(\mathscr{F})$ is tractable and we are done. Now we can assume that we have a non-degenerate symmetric ternary signature $X=[x_0,x_1,x_2,x_3]$ and the problem is $\mathrm{Holant}^c(\mathscr{F}\cup\{X\})$.

As discussed in Section 3, there are three categories for X and we only need to consider the first two:

1. $x_k=\alpha_1^{3-k}\alpha_2^k+\beta_1^{3-k}\beta_2^k$;
2. $x_k=Ak\alpha^{k-1}+B\alpha^k$, where $A\neq 0$;

Case 1: $x_k=\alpha_1^{3-k}\alpha_2^k+\beta_1^{3-k}\beta_2^k$.

In this case, $X=T^{\otimes 3}[1,0,0,1]^T$, where $T=\begin{bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{bmatrix}$. So we have the following reduction chain,

$$\begin{aligned}\mathrm{Holant}^c(\mathscr{F}) &\equiv_{\mathrm{T}} \mathrm{Holant}^c(\mathscr{F}\cup\{X\}) \equiv_{\mathrm{T}} \mathrm{Holant}(\mathscr{F}\cup\{X,[1,0],[0,1]\})\\ &\equiv_{\mathrm{T}} \mathrm{Holant}(\{[1,0,1],[1,0],[0,1]\}|\mathscr{F}\cup\{X\})\\ &\equiv_{\mathrm{T}} \mathrm{Holant}(\{[1,0,1]T^{\otimes 2},[1,0]T,[0,1]T\}|[1,0,0,1]\cup T^{-1}\mathscr{F}).\end{aligned}$$

Since $[1,0,1]T^{\otimes 2}$ is a non-degenerate binary signature, we can apply Theorem 4.1. The only thing we need to verify is that in the case $[1,0,1]T^{\otimes 2}=[\alpha_1^2+\alpha_2^0,\alpha_1\beta_1+\alpha_2\beta_2,\beta_1^2+\beta_2^2]=[0,\alpha_1\beta_1+\alpha_2\beta_2,0]$, at least one of $[1,0]T=[\alpha_1,\beta_1]$ or $[0,1]T=[\alpha_2,\beta_2]$ has both entries non-zero. If not, we would have $\alpha_1\beta_1=0$ and $\alpha_2\beta_2=0$, which implies that $[1,0,1]T^{\otimes 2}=[0,\alpha_1\beta_1+\alpha_2\beta_2,0]=[0,0,0]$, a contradiction. Therefore, by Theorem 4.1, we know that the problem is #P-hard unless $[1,0,1]T^{\otimes 2}\cup T^{-1}\mathscr{F}\subseteq\mathscr{P}$ (note that unary $[1,0]T,[0,1]T$ are automatically in $\mathscr{P}$) or $\{[1,0,1]T^{\otimes 2},[1,0]T,[0,1]T\}\cup T^{-1}\mathscr{F}\subseteq\mathscr{A}$. In the first case, $\mathrm{Holant}^*(\mathscr{F})$ is tractable; in the second case, this is equivalent to having $T\in\mathscr{T}$ satisfying $\mathscr{F}\subseteq T\mathscr{A}$.

Case 2: $x_k=Ak\alpha^{k-1}+B\alpha^k$, where $A\neq 0$.

In this case, if $\alpha\neq\pm i$, the problem is #P-hard by Lemma 3.2 and we are done. Now we consider the case $\alpha=i$ (the case $\alpha=-i$ is similar). Consider the following Equation

$$z_{k+2}-2iz_{k+1}-z_k=0. \tag{1}$$

We note that $X=[x_0,x_1,x_2,x_3]$ satisfies this equation for $k=0,1$. If all non-degenerate signatures $Z=[z_0,z_1,\ldots,z_m]$ in $\mathscr{F}$ with arity $m\geq 2$ satisfy the following

Condition: Z satisfies Equation (1) for $k=0,1,\ldots,m-2$.

then, by Theorem 6.1 (tractable case 2), $\mathrm{Holant}^*(\mathscr{F})$ is tractable, and we are done. So suppose this is not the case, and $Z=[z_0,z_1,\ldots,z_m]\in\mathscr{F}$, for some $m\geq 2$, is a non-degenerate signature that does not satisfy this Condition. By Lemma 5.2, if any non-degenerate sub-signature $[z_k, z_{k+1}, z_{k+2}]$ does not satisfy Equation (1), then, together with X which does satisfy (1), we know that the problem is #P-hard and we are done. So we assume every non-degenerate sub-signature $[z_k, z_{k+1}, z_{k+2}]$ of Z satisfies (1). In particular $m \geq 3$, and there exists some binary sub-signature of Z that is degenerate and does not satisfy (1).

Subcase 1: All binary sub-signatures of Z are degenerate (but Z itself is non-degenerate).

We claim that Z has the form $[z_0, 0, \ldots, 0, z_m]$, where $z_0 z_m \neq 0$. For a contradiction suppose $z_0 = 0$, since Z is non-degenerate, there exists $k < m$ such that $z_k \neq 0$. Let k be the minimum such, then $0 < k < m$ and $[z_{k-1}, z_k, z_{k+1}]$ is non-degenerate. So $z_0 \neq 0$ and similarly $z_m \neq 0$. If there are any other $0 < k < m$ such that $z_k \neq 0$, then let k be the minimum such. If $k = 1$, then a simple induction shows that $z_k = z_0(z_1/z_0)^k$, for $0 \leq k \leq m$, and Z is degenerate. If $k > 1$, then $[z_{k-1}, z_k, z_{k+1}]$ is non-degenerate, a contradiction.

Next we claim that there exists a unary sub-signature $[x_k, x_{k+1}]$ of X with both entries non-zero. If $x_0 x_1 \neq 0$, then we set $k = 0$ and the claim is proved; if $x_0 = 0, x_1 \neq 0$, then we have $x_2 = 2ix_1 + x_0 \neq 0$; if $x_0 \neq 0, x_1 = 0$, then we have $x_2 = 2ix_1 + x_0 \neq 0$ and $x_3 = 2ix_2 + x_1 \neq 0$. (Note that $x_0 = x_1 = 0$ is impossible because $A \neq 0$.)

Connect this unary signature to $m - 3$ dangling edges of $[z_0, 0, \ldots, 0, z_m]$, we have a ternary signature $[a, 0, 0, b]$ where $ab \neq 0$. We can use this as the non-degenerate ternary signature X and we have reduced this case to Case 1.

Subcase 2: Some binary sub-signatures of $[z_0, z_1, \ldots, z_m]$ are non-degenerate (some others are degenerate).

Then we can find a ternary sub-signature $[z_k, z_{k+1}, z_{k+2}, z_{k+3}]$ (or its reversal) where $[z_k, z_{k+1}, z_{k+2}]$ is degenerate and $[z_{k+1}, z_{k+2}, z_{k+3}]$ is non-degenerate and thus satisfies $-z_{k+1} - 2iz_{k+2} + z_{k+3} = 0$. If $\{z_k, z_{k+1}, z_{k+2}\}$ is a geometric sequence of a non-zero ratio p, we could assume that $z_k = 1$ after scaling. Then we have

$$\begin{bmatrix} 1 & 0 \\ p & (p + 2ip^2 - p^3)^{1/3} \end{bmatrix}^{\otimes 3} [1, 0, 0, 1]^{\mathtt{T}} = [z_k, z_{k+1}, z_{k+2}, z_{k+3}]^{\mathtt{T}}$$

Therefore, the ternary sub-signature $[z_k, z_{k+1}, z_{k+2}, z_{k+3}]$ is in the first category and we reduce the problem to Case 1. Otherwise, it must be that $z_k = z_{k+1} = 0$ and $z_{k+2} \neq 0$. This signature became $[0, 0, z_{k+2}, z_{k+3}]$, which is equivalent to $[0, 0, 1, v]$ for some $v \in \mathbb{C}$, and as we proved in Section 3, it is #P-hard. $\square$

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Appendix

6 Some Known Dichotomy Results

In this section, we review three dichotomy theorems from [11].

Theorem 6.1. *[11] Let $\mathscr{F}$ be a set of symmetric non-degenerate signatures over $\mathbb{C}$. Then* $\text{Holant}^*(\mathscr{F})$ *is computable in polynomial time in the following three cases. In all other cases,* $\text{Holant}^*(\mathscr{F})$ *is #P-hard.*

1. *Every signature in $\mathscr{F}$ is of arity no more than two;*
2. *There exist two constants a and b (not both zero, depending only on $\mathscr{F}$), such that for every signature $[x_0, x_1, \ldots, x_n] \in \mathscr{F}$ one of the two conditions is satisfied: (1) for every $k = 0, 1, \ldots, n-2$, we have $ax_k + bx_{k+1} - ax_{k+2} = 0$; (2) $n = 2$ and the signature $[x_0, x_1, x_2]$ is of form $[2a\lambda, b\lambda, -2a\lambda]$.*
3. *For every signature $[x_0, x_1, \ldots, x_n] \in \mathscr{F}$, one of the two conditions is satisfied: (1) For every $k = 0, 1, \ldots, n-2$, we have $x_k + x_{k+2} = 0$; (2) $n = 2$ and the signature $[x_0, x_1, x_2]$ is of form $[\lambda, 0, \lambda]$.*

Theorem 6.2. *[11] Let $\mathscr{F}$ be a set of* real *symmetric signatures, and let $\mathscr{F}_1, \mathscr{F}_2$ and $\mathscr{F}_3$ be three families of signatures defined as*

$$\begin{aligned}
\mathscr{F}_1 &= \{\lambda([1,0]^{\otimes k} + i^r[0,1]^{\otimes k}) | \lambda \in \mathbb{C}, k = 1, 2, \ldots, r = 0, 1, 2, 3\}; \\
\mathscr{F}_2 &= \{\lambda([1,1]^{\otimes k} + i^r[1,-1]^{\otimes k}) | \lambda \in \mathbb{C}, k = 1, 2, \ldots, r = 0, 1, 2, 3\}; \\
\mathscr{F}_3 &= \{\lambda([1,i]^{\otimes k} + i^r[1,-i]^{\otimes k}) | \lambda \in \mathbb{C}, k = 1, 2, \ldots, r = 0, 1, 2, 3\}.
\end{aligned}$$

Then $\text{Holant}^c(\mathscr{F})$ *is computable in polynomial time if (1) After removing unary signatures from $\mathscr{F}$, it falls in one of the three cases of Theorem 6.1 (this implies* $\text{Holant}^*(\mathscr{F})$ *is computable in polynomial time) or (2) (Without removing any unary signature) $\mathscr{F} \subseteq \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. Otherwise,* $\text{Holant}^c(\mathscr{F})$ *is #P-hard.*

We explicitly list all the signatures in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, up to an arbitrary constant multiple from $\mathbb{C}$.

1. $[1, 0, 0, \ldots, 0, \pm 1]$;
2. $[1, 0, 0, \ldots, 0, \pm i]$;
3. $[1, 0, 1, 0, \ldots, 0/1]$;
4. $[0, 1, 0, 1, \ldots, 0/1]$;
5. $[1, i, 1, i, \ldots, i/1]$;
6. $[1, -i, 1, -i, \ldots, (-i)/1]$;
7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0/1/(-1)]$;
8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1/(-1)]$;
9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0/1/(-1)]$;
10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1/(-1)]$.

Definition 6.3. *A k-ary function $f(x_1, \ldots, x_k)$ is affine if it has the form*

$$\chi_{AX=0} \cdot \sqrt{-1}^{\sum_{j=1}^{n} \langle \alpha_j, X \rangle}$$

where $X = (x_1, x_2, \ldots, x_k, 1)$, A is matrix over $\mathbb{F}_2$, α_j is a vector over $\mathbb{F}_2$, and χ is a 0-1 indicator function such that $\chi_{AX=0}$ is 1 iff $AX = 0$. Note that the inner product $\langle \alpha_j, X \rangle$ is calculated over $\mathbb{F}_2$, while the summation $\sum_{j=1}^{n}$ on the exponent of $i = \sqrt{-1}$ is evaluated as a sum mod 4 of 0-1 terms. We use $\mathscr{A}$ to denote the set of all affine functions.

We use $\mathscr{P}$ to denote the set of functions which can be expressed as a product of unary functions, binary equality functions ($[1, 0, 1]$) and binary disequality functions ($[0, 1, 0]$).

Theorem 6.4. *[11] Suppose $\mathscr{F}$ is a class of functions mapping Boolean inputs to complex numbers. If $\mathscr{F} \subseteq \mathscr{A}$ or $\mathscr{F} \subseteq \mathscr{P}$, then #CSP($\mathscr{F}$) is computable in polynomial time. Otherwise, #CSP($\mathscr{F}$) is* #P*-hard.*

As we mentioned in [11], the class $\mathscr{A}$ is a natural generalization of the symmetric signatures family $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. It is easy to show that the set of symmetric signatures in $\mathscr{A}$ is exactly $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$.

7 Some Useful Reductions

In this section, we list some useful simple reductions: reduction between Holant and #CSP, reduction between bipartite and non-bipartite settings, and holographic reduction.

Proposition 7.1. $\#\mathrm{CSP}(\mathscr{F}) \equiv_{\mathrm{T}} \mathrm{Holant}\left(\mathscr{F} \cup \bigcup_{j \geq 1}\{=_j\}\right) \equiv_{\mathrm{T}} \mathrm{Holant}(\mathscr{F} \cup \{=_3\})$.

This says that #CSP is the same as Holant problems with Equality functions given for free.

Proposition 7.2. $\mathrm{Holant}(\mathscr{F}) \equiv_{\mathrm{T}} \mathrm{Holant}([1,0,1]|\mathscr{F})$.

That is, we can transform every edge to a path of length 2 with the new vertex given $(=_2) = [1,0,1]$.

Proposition 7.3. $\mathrm{Holant}(\mathscr{G}_1 \cup [1,0,1]|\mathscr{G}_2 \cup [1,0,1]) \equiv_{\mathrm{T}} \mathrm{Holant}(\mathscr{G}_1 \cup \mathscr{G}_2)$.

Binary Equality functions on both sides allow the transfer of signatures.

Proposition 7.4. *For any $T \in \mathbf{GL}_2(\mathbb{C})$,* $\mathrm{Holant}(\mathscr{G}_1|\mathscr{G}_2) \equiv_{\mathrm{T}} \mathrm{Holant}(\mathscr{G}_1 T|T^{-1}\mathscr{G}_2)$.

This is a restatement of Valiant's Holant Theorem.

Proposition 7.5. *Let T be an orthogonal transformation ($TT^{\mathtt{T}} = I$). Then* $\mathrm{Holant}(\mathscr{F}) \equiv_{\mathrm{T}} \mathrm{Holant}(T\mathscr{F})$.

This follows from the invariance of $(=_2) = [1,0,1]$ under an orthogonal transformation, and Props. 7.2, & 7.4.

8 An Orthogonal Transformation

In this Section we give the detail of an *orthogonal* holographic transformation used in the proof of Lemma 3.2.

We are given $x_k = Ak\alpha^{k-1} + B\alpha^k$, where $A \neq 0$, and $\alpha \neq \pm i$. Let $S = \begin{bmatrix} 1 & \frac{B-1}{3} \\ \alpha & A + \frac{B-1}{3}\alpha \end{bmatrix}$, then the signature $[x_0, x_1, x_2, x_3]$ can be expressed as

$$(x_0, x_1, x_1, x_2, x_1, x_2, x_2, x_3)^{\mathtt{T}} = S^{\otimes 3}(1,1,1,0,1,0,0,0)^{\mathtt{T}}.$$

This identity can be verified by observing that

$$(1,1,1,0,1,0,0,0)^{\mathtt{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and we apply $S^{\otimes 3}$ using properties of tensor product, $S^{\otimes 3}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 3} = (S\begin{bmatrix} 1 \\ 0 \end{bmatrix})^{\otimes 3}$, etc.

Let $T = \frac{1}{\sqrt{1+\alpha^2}}\begin{bmatrix} 1 & \alpha \\ \alpha & -1 \end{bmatrix}$, then $T = T^{\mathtt{T}} = T^{-1} \in \mathbf{O}_2(\mathbb{C})$ is orthogonal, and $R = TS = \begin{bmatrix} u & w \\ 0 & v \end{bmatrix}$ is upper triangular, where $u = \sqrt{1+\alpha^2}$. As $\det R = \det T \det S = (-1)A \neq 0$, we have $uv \neq 0$. It follows that

$$\begin{aligned}
& T^{\otimes 3}(x_0, x_1, x_1, x_2, x_1, x_2, x_2, x_3)^{\mathtt{T}} \\
= \; & (TS)^{\otimes 3}(1,1,1,0,1,0,0,0)^{\mathtt{T}} \\
= \; & R^{\otimes 3}(1,1,1,0,1,0,0,0)^{\mathtt{T}} \\
= \; & R^{\otimes 3}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \\
= \; & \begin{bmatrix} u \\ 0 \end{bmatrix}^{\otimes 3} + \begin{bmatrix} u \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u \\ 0 \end{bmatrix} \otimes \begin{bmatrix} w \\ v \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix} \otimes \begin{bmatrix} w \\ v \end{bmatrix} \otimes \begin{bmatrix} u \\ 0 \end{bmatrix} + \begin{bmatrix} w \\ v \end{bmatrix} \otimes \begin{bmatrix} u \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u \\ 0 \end{bmatrix}
\end{aligned}$$

This can be written as a symmetric signature form $[u^3+3u^2w, u^2v, 0, 0]$. Note that the entry $u^2v \neq 0$, which we can normalize to 1, after a scalar multiplication. This gives us the form $[z, 1, 0, 0]$ for some $z \in \mathbb{C}$.

9 List of Matrices in $\mathscr{T}$

In this section, we explicitly list all the matrices in the family

$$\mathscr{T} \triangleq \{T \mid [1,0,1]T^{\otimes 2}, [1,0]T, [0,1]T \in \mathscr{A}\},$$

which is defined and used in the statement of Theorem 5.3. The following condition is given in Theorem 5.3:

$$\text{There exists a } T \in \mathscr{T} \text{ such that } \mathscr{F} \subseteq T\mathscr{A}. \tag{2}$$

Together with the condition in Theorem 6.1, this gives an effective tractability condition which is both necessary and sufficient for Holant^c problems, by Theorem 5.3.

As noted before, the set of symmetric signatures in $\mathscr{A}$ is exactly $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. We note that $[1,0,1]T^{\otimes 2}$, $[1,0]T$ and $[0,1]T$ are all symmetric, as is the requirement $T^{-1}\mathscr{F} \subseteq \mathscr{A}$ in Theorem 5.3. Thus we can replace $\mathscr{A}$ by $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ in the expression $\mathscr{T}$ above.

It is obvious that the family $\mathscr{T}$ is closed under a scalar multiplication. Thus we list them up to a scalar multiple. After a scalar multiple, symmetric binary signatures in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ are precisely

$$[1,0,\pm 1], [1,0,\pm i], [1,\pm 1,-1], [1,\pm i,1], [0,1,0].$$

Also the unary signatures in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, up to a scalar multiple, are

$$[1,\pm 1], [1,\pm i], [1,0], [0,1].$$

Before we enumerate all the possibilities, we make two observations to simplify this process:

1. If we exchange the two columns of T, the signature $[1,0,1]T^{\otimes 2}$ becomes its reversal, and the two numbers in $[1,0]T$ are interchanged. Similarly the two numbers in $[0,1]T$ are interchanged as well. The effect of exchanging the two columns of T is the same as replacing T by $T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. On the other hand, the holographic transformation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathscr{A}$ amounts exchanging input values 0 and 1 for functions in $\mathscr{A}$, and this operation keeps $\mathscr{A}$ invariant.

2. If we multiply $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ on the right side of T, then $[y_0, y_1, y_2] \triangleq [1,0,1]T^{\otimes 2}$ becomes $[y_0, -y_1, y_2]$. This operation also preserves both the set of binary and the set of unary signatures, respectively, listed for $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, up to a scalar factor. On the other hand, the effect of the holographic transformation $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathscr{A}$ is to transform an original function f to $f \cdot (-1)^{\sum_i x_i}$, and thus it is in $\mathscr{A}$ iff the original $f \in \mathscr{A}$.

   Similarly, we can multiply $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ on the right side of T. Of course the invariance under $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ implies that of $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}^2$.

What has been shown is that Condition (2) is invariant under the right action on $\mathscr{T}$ by the group generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. By $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, we may consider only those T's such that $[1,0,1]T^{\otimes 2} \in \{[1,0,1], [1,0,i], [1,i,1], [0,1,0]\}$, up to a scalar factor. If we further normalize by the reversal action $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we may consider only those T's such that $[1,0]T \in \{[1,\pm 1], [1,i], [1,0]\}$, up to a scalar factor. However this reversal action is only partially closed for $\{[1,0,1],[1,0,i],[1,i,1],[0,1,0]\}$, with the exception $[1,0,i]$ which is changed to $[1,0,-i]$. Thus we may have two extra cases to consider: $[1,0,1]T^{\otimes 2}=[1,0,i]$ and, $[1,0]T=[0,1]$ or $[1,-i]$. But for these two cases if we apply first $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, followed by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we obtain $[1,0,1]T^{\otimes 2}=[1,0,i]$ and, $[1,0]T=[1,0]$ or $[1,1]$ respectively. Hence we can eliminate these two cases.

To summarize, to enumerate all T satisfying Condition (2) we only need to consider

$$[1,0,1]T^{\otimes 2}\in\{[1,0,1],[1,0,i],[1,i,1],[0,1,0]\} \quad \text{and} \quad [1,0]T\in\{[1,\pm 1],[1,i],[1,0]\},$$

up to a scalar factor. In the following, we denote by $\alpha=(1+i)/\sqrt{2}=\sqrt{i}$.

If $[1,0,1]T^{\otimes 2}=\gamma[1,0,1]$, $[1,0]T=\lambda[1,1]$, we have

$$T=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,0,1]$, $[1,0]T=\lambda[1,-1]$, we have

$$T=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

For $[1,0,1]T^{\otimes 2}=\gamma[1,0,1]$, $[1,0]T=\lambda[1,i]$, there is no solution.
If $[1,0,1]T^{\otimes 2}=\gamma[1,0,1]$, $[1,0]T=\lambda[1,0]$, we have

$$T=\begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,0,i]$, $[1,0]T=\lambda[1,1]$, we have

$$T=\begin{bmatrix} 1 & 1 \\ \alpha^3 & \alpha \end{bmatrix},\begin{bmatrix} 1 & 1 \\ -\alpha^3 & -\alpha \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,0,i]$, $[1,0]T=\lambda[1,-1]$, we have

$$T=\begin{bmatrix} 1 & -1 \\ \alpha^3 & -\alpha \end{bmatrix},\begin{bmatrix} 1 & -1 \\ -\alpha^3 & \alpha \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,0,i]$, $[1,0]T=\lambda[1,i]$, we have

$$T=\begin{bmatrix} 1 & i \\ \alpha & -\alpha \end{bmatrix},\begin{bmatrix} 1 & i \\ -\alpha & \alpha \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,0,i]$, $[1,0]T=\lambda[1,0]$, we have

$$T=\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,i,1]$, $[1,0]T=\lambda[1,1]$, we have

$$T=\begin{bmatrix} 1 & 1 \\ -\alpha^3 & \alpha^3 \end{bmatrix},\begin{bmatrix} 1 & 1 \\ \alpha^3 & -\alpha^3 \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2}=\gamma[1,i,1]$, $[1,0]T=\lambda[1,-1]$, we have

$$T=\begin{bmatrix} 1 & -1 \\ \alpha & \alpha \end{bmatrix},\begin{bmatrix} 1 & -1 \\ -\alpha & -\alpha \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2} = \gamma[1,i,1]$, $[1,0]T = \lambda[1,i]$, we have

$$T = \begin{bmatrix} 1 & i \\ 0 & \sqrt{2} \end{bmatrix}, \begin{bmatrix} 1 & i \\ 0 & -\sqrt{2} \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2} = \gamma[1,i,1]$, $[1,0]T = \lambda[1,0]$, we have

$$T = \begin{bmatrix} \sqrt{2} & 0 \\ i & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{2} & 0 \\ -i & -1 \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2} = \gamma[0,1,0]$, $[1,0]T = \lambda[1,1]$, we have

$$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2} = \gamma[0,1,0]$, $[1,0]T = \lambda[1,-1]$, we have

$$T = \begin{bmatrix} 1 & -1 \\ -i & -i \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}.$$

If $[1,0,1]T^{\otimes 2} = \gamma[0,1,0]$, $[1,0]T = \lambda[1,i]$, we have

$$T = \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}, \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

For $[1,0,1]T^{\otimes 2} = \gamma[0,1,0]$, $[1,0]T = \lambda[1,0]$, there is no solution.