Some Results on Matchgates and Holographic Algorithms

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#P

Counting problems:

#SAT: How many satisfying assignments are there to a Boolean formula?

#PerfMatch: How many perfect matchings are there in a graph?

#P is at least as powerful as NP, and in fact subsumes the entire polynomial time hierarchy $\cup_i \Sigma_i^p$ [Toda].

#P-completeness: #SAT, #PerfMatch, Permanent, etc.

Some Surprises

Most #P-complete problems are counting versions of NP-complete decision problems.

But the following problems are solvable in P:

- **Whether there exists a Perfect Matching in a general graph.**
- **Count the number of Perfect Matchings in a planar graph.**

Note that the problem of counting the number of (not necessarily perfect) *matchings* **in a planar graph is still #P-complete [Jerrum].**

Some problems

#PL-3-NAE-ICE

Input: **A planar graph $G = (V, E)$ of maximum degree 3.**

Output: **The number of orientations such that no node has all edges directed towards it or away from it.**

Some more examples of problems

PL-MAXCUT

Input: **A planar graph $G = (V, E)$ of maximum degree 3.**

Output: **The Size of the MAXCUT partitioning G into a bipartite graph.**

PL-NODE-BIPARTITION

Input: **A planar graph $G = (V, E)$ of maximum degree 3.**

Output: **The minimum number of nodes to be removed to become bipartite.**

One more...

PL-NODE-EDGE-BIPARTITION

Input: **A planar graph $G = (V, E)$ of maximum degree 3. A non-negative integer $k \leq |V|$.**
Output: **The minimal l such that deletion of at most k nodes (including all of their incident edges) and l more edges results in a bipartite graph.**

Graphs and Pfaffians

$G = (V, E, w)$ **is a weighted undirected graph.**

Skew-symmetric adjacency matrix M **of** G**:**

$M(i,j) = w(i,j)$ **if** $i < j$**,** $M(i,j) = -w(i,j)$ **if** $i > j$**.**

$$\mathrm{Pf}(M) = \sum_{\pi} \epsilon_\pi w(i_1, i_2) w(i_3, i_4) \ldots w(i_{2k-1}, i_{2k})$$

$\pi : i_1 < i_2,\ i_3 < i_4,\ \ldots,\ i_{2k-1} < i_{2k}$ **and** $i_1 < i_3 < \ldots < i_{2k-1}$**.**

π **corresponds to a Perfect Matching.**

$\epsilon_\pi = \mathrm{sign}(\pi) \in \{-1, 1\}$**.**

Equivalently ϵ_π **counts the parity of the number of overlapping pairs:** $i_{2r-1} < i_{2s-1} < i_{2r} < i_{2s}$ **or** $i_{2s-1} < i_{2r-1} < i_{2s} < i_{2r}$**.**

Fisher-Kasteleyn-Temperley

There is a beautiful algorithm due to Fisher, Kasteleyn and Temperley, called FKT method, which computes the number of Perfect Matchings in a planar graph in P.

Given G, the FKT method orients every edge of G, affixing ± 1 to the weights, then computes the Pfaffian.

For skew symmetric M, the Pfaffian is computable in polynomial time. In particular $(\mathrm{Pf}(M))^2 = \det(M)$.

Valiant's new theories

Recently Valiant has introduced a novel methodology in algorithm design in the following papers:

- *Quantum circuits that can be simulated classically in polynomial time*, **SIAM J. Computing, 31(4): 1229-1254 (2002).**
- *Holographic Algorithms*, **Electronic Colloquium on Computational Complexity Report TR05-099.**

A basic component of both new theories is a Matchgate.

Matchgate

A matchgate Γ is a finite graph with input/output nodes, and is associated with a character matrix, defined in terms of Pfaffians.

Matchcircuit

A **matchcircuit** is a way of combining matchgates to form a circuit for computation.

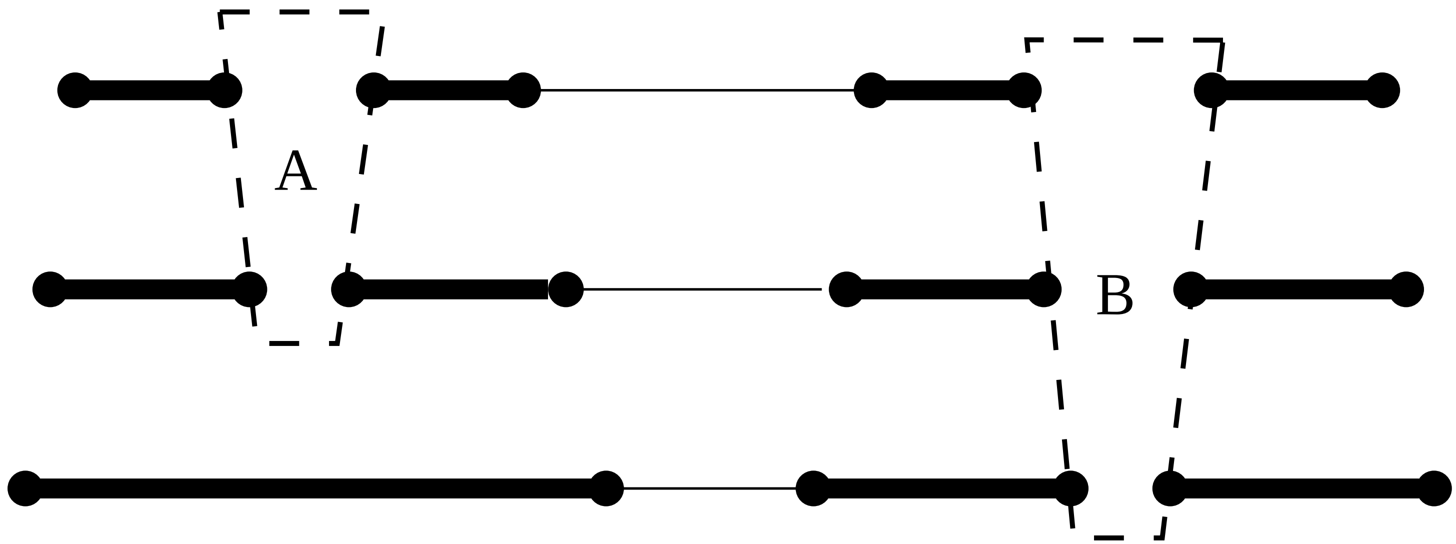

Figure 1: An example of a matchcircuit consisting of two matchgates A and B. The internal structures of A and B are not shown.

Matchcircuit computation

Some global properties of matchcircuit can be interpreted as realizing certain computations which would seem to take exponential time. However, due to algebraic properties, certain matchcircuits can be evaluated in polynomial time.

Valiant showed that a non-trivial fragment of quantum circuits can be simulated classically in P.

Planar matchgates and matchgrids

For a planar matchgate $\Gamma = (G, X, Y)$ Valiant assigns a signature matrix.

Let $\mathrm{PerfMatch}(G) = \sum_M \prod_{(i,j)\in M} w_{ij}$, where the sum is over all perfect matchings M.

The standard signature, $u = u(\Gamma)$, is the $2^{|X|} \times 2^{|Y|}$ matrix whose entry indexed by (X', Y') is $\mathrm{PerfMatch}(G - Z)$, where $Z = X' \cup Y'$.

Tensor framework

We assign to each Γ with only outputs (generator) a contravariant tensor $\mathrm{valG}(\Gamma)$.

We assign to each Γ with only inputs (recognizer) a covariant tensor $\mathrm{valR}(\Gamma)$.

Matchgrid and Holant

A matchgrid is a weighted planar graph consisting of a number of generators and recognizers that are connected by a number of connecting edges in a 1-1 fashion.

$$\mathrm{Holant}(\Omega) = \sum_{x \in \boldsymbol{\beta}^{\otimes f}} \left\{ [\Pi_{1 \le i \le g} \mathrm{valG}(A_i, x|_{A_i})] \cdot [\Pi_{1 \le j \le r} \mathrm{valR}(B_j, x|_{B_j})] \right\}.$$

Holant Theorem

Theorem (Valiant)

$$\text{Holant}(\Omega) = \text{PerfMatch}(G).$$

For a proof of this beautiful theorem based on the tensor theoretic framework:

- **Cai and V. Choudhary.**
  Valiant's Holant Theorem and Matchgate Tensors.
  TAMC 2006: 248-261. LNCS vol. 3959.
  Also available as ECCC Report TR05-118.

The Holographic Algorithms introduce an exponential number of solution fragments in a pattern of interference, analogous to quantum computing.

Equivalence Theorem

We establish a 1-1 correspondence between Valiant's character theory of matchgate/matchcircuit and his signature theory of planar-matchgate/matchgrid.

Proof Outline

From matchgate/matchcircuit to planar-matchgate/matchgrid:

- **We define an intermediate notion of Naked characters.**
- **A crossover gadget to accomplish local replacement.**
- **a global embedding.**

From planar-matchgate/matchgrid to matchgate/matchcircuit:

Use FKT method.

Naked characters

We define naked characters as a modification to the definition of characters.

By results from

- **Cai and V. Choudhary.**
  On the Theory of Matchgate Computations.
  ECCC TR06-018.

We can show a complete characterization theorem for naked characters as well by the Grassmann-Plücker identities.

Crossover gadget

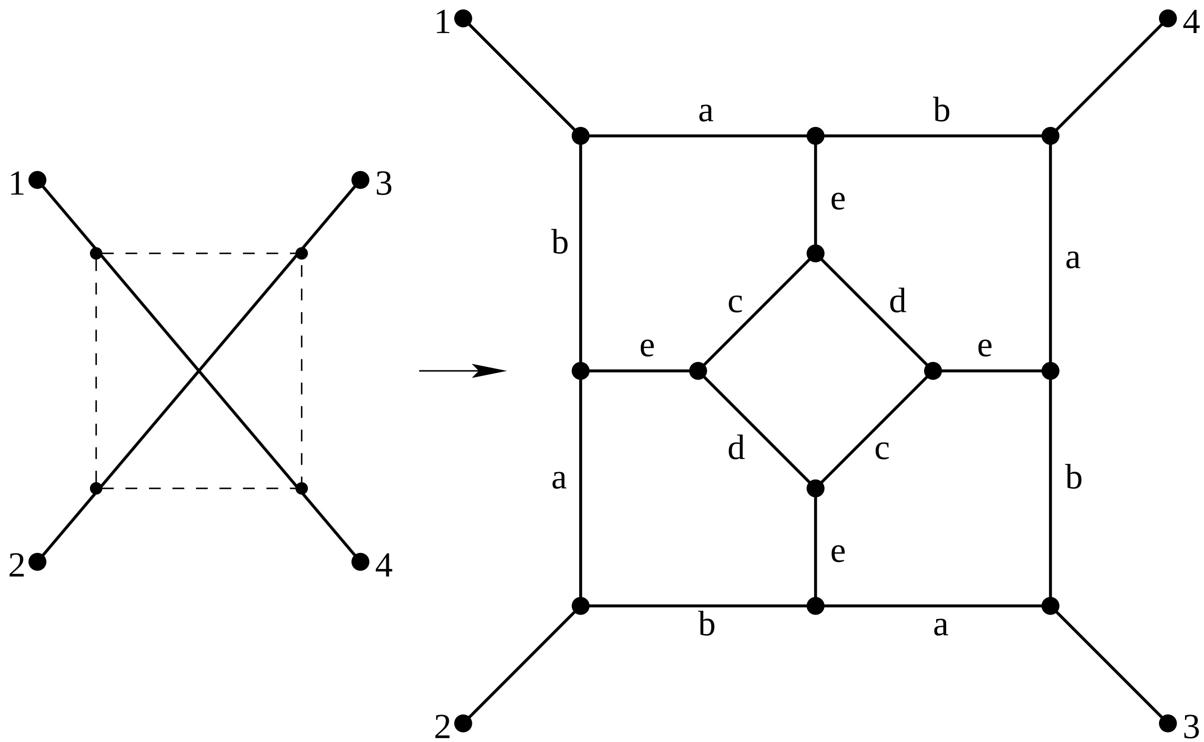

Figure 2: The gadget used to replace crossovers. Here $a = 1, b = i, c = d = -1/2, e = \sqrt{i}$**, where** $i = \sqrt{-1}$**. The gadget was by Valiant.**

Crossover with both matched edges

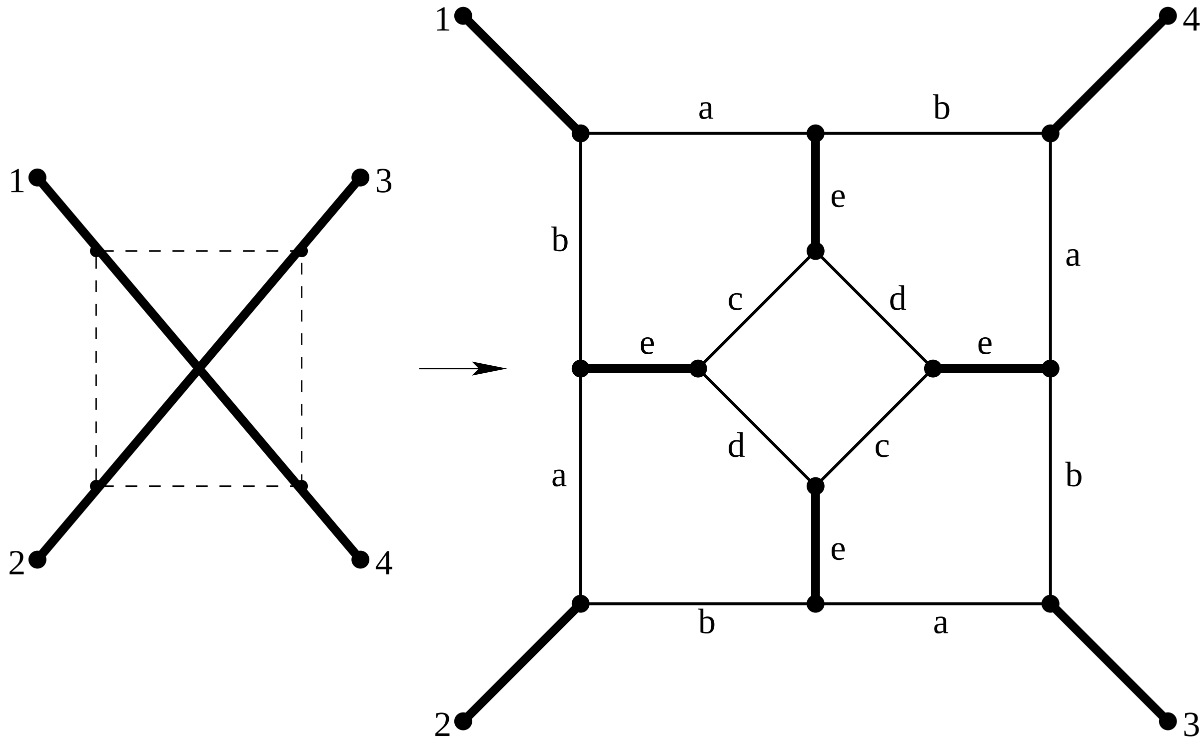

Figure 3: When both crossed edges are matched.

This gives local total weight $e^4 = -1$.

Crossover with just one matched edge

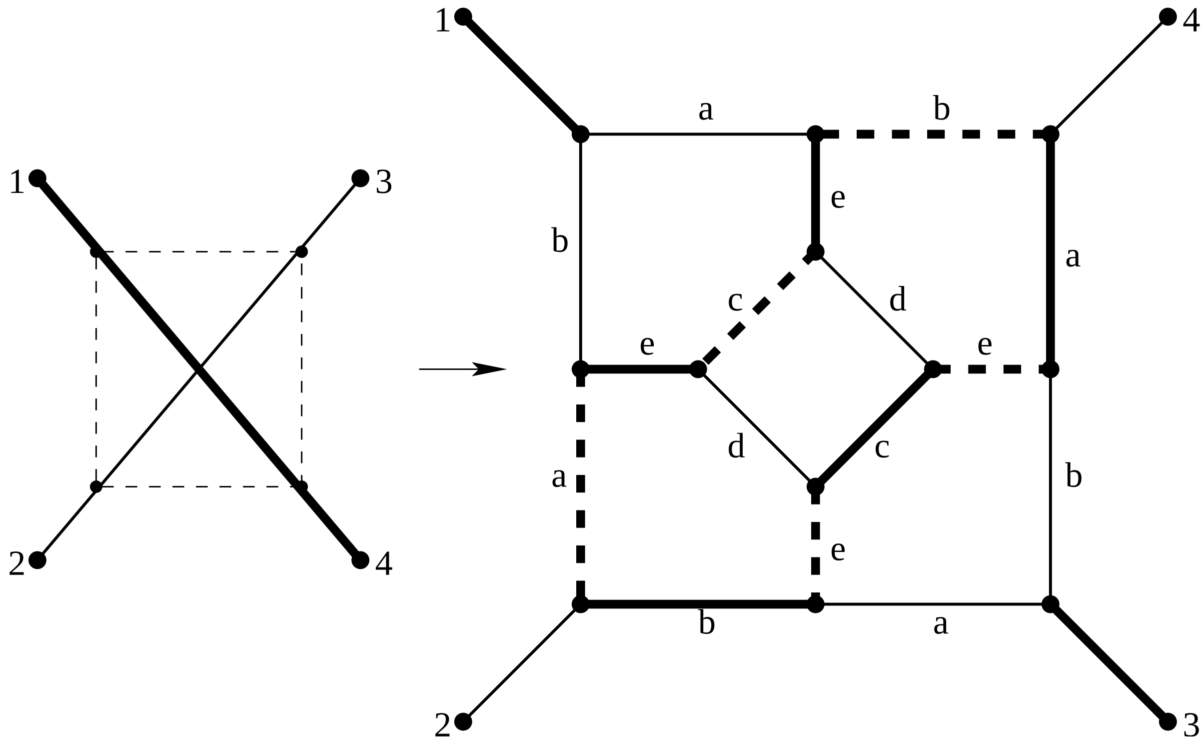

Figure 4: When exactly one edge is matched.

Note that in this case, the edge with weight d must not be included. This gives local total weight $2abce^2 = +1$.

Crossover with just no matched edge

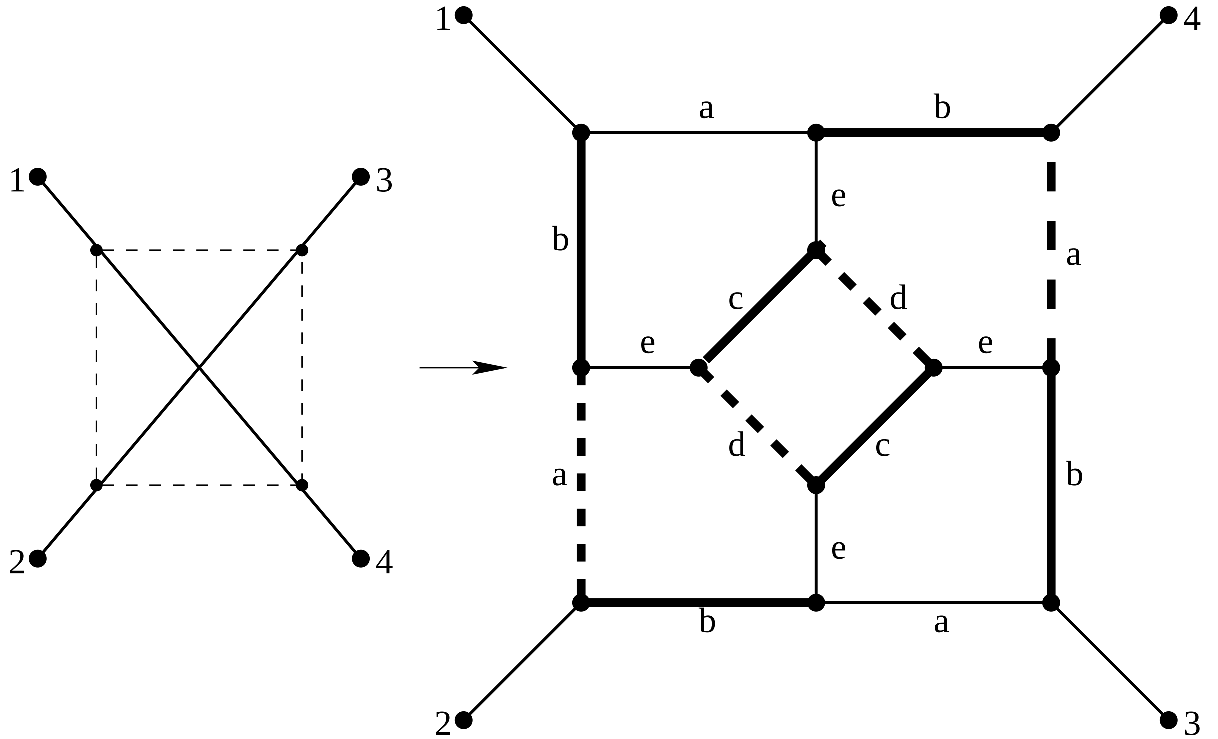

Figure 5: When both edges are unmatched.

E.g. the NE corner, either a or b must be included. Then e must be out. Local total weight $a^4(c^2+d^2)+b^4(c^2+d^2)=1$.

Global Embedding

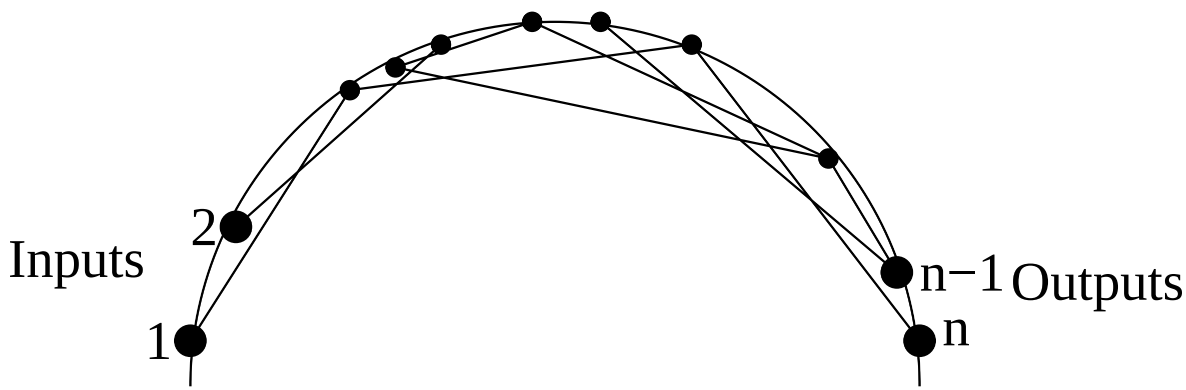

Figure 6: An example of converting a 2-input, 2-output matchgate to a planar matchgate.

Global Embedding

We arrange the vertices (with their edges) on a strictly convex curve, e.g., a upper semicircle, such that as we move clockwise from vertex 1, we encounter all the vertices in increasing order.

Any two edges (i, j) and (k, l) overlap (i.e. $i < k < j < l$ or $k < i < l < j$) iff they physically cross each other.

If any such pair of overlapping edges is present in a matching, it introduces a negative sign to the Pfaffian.

Now we can convert this graph into a planar graph by using the gadget. Replace any physical crossing by a local copy of the gadget.

Equivalence Theorem continued

Lemma

Given a planar matchgate Γ with signature u, there is a matchgate Γ' with naked character equal to u.

Using the FKT method we can prove the reverse:

Lemma

Given a matchgate Γ with naked character u, there is a planar matchgate Γ' with signature equal to u.

Matchcircuit

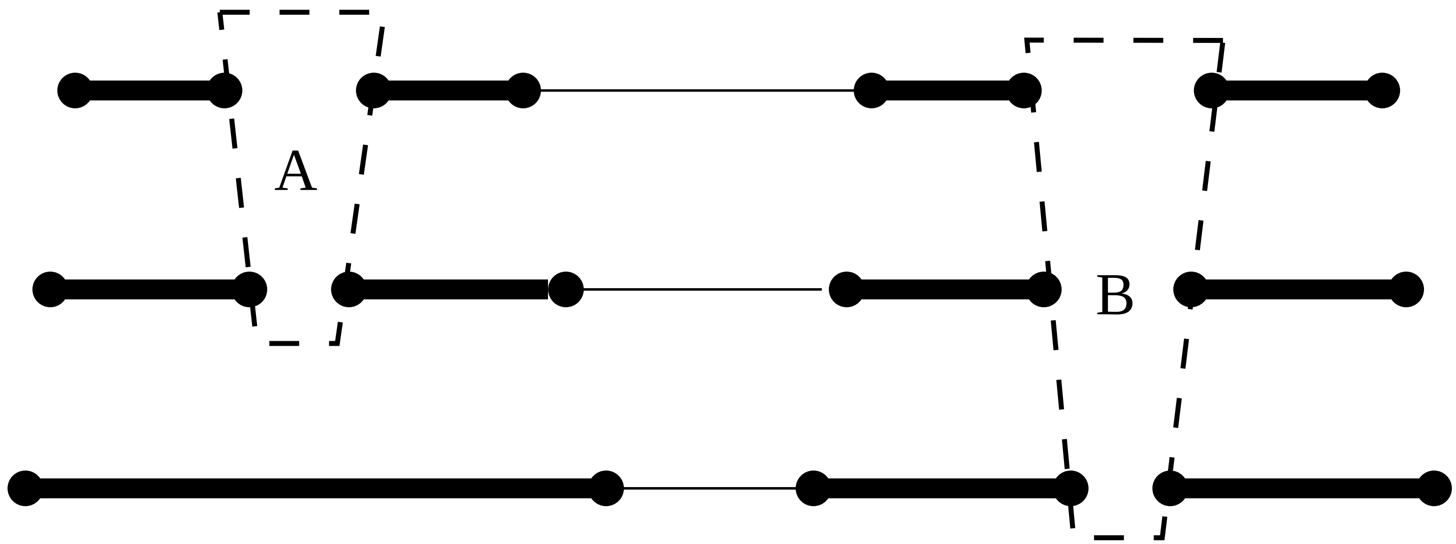

Figure 7: An example of a matchcircuit consisting of two matchgates A and B. The internal structures of A and B are not shown.

Global embedding of a matchcircuit

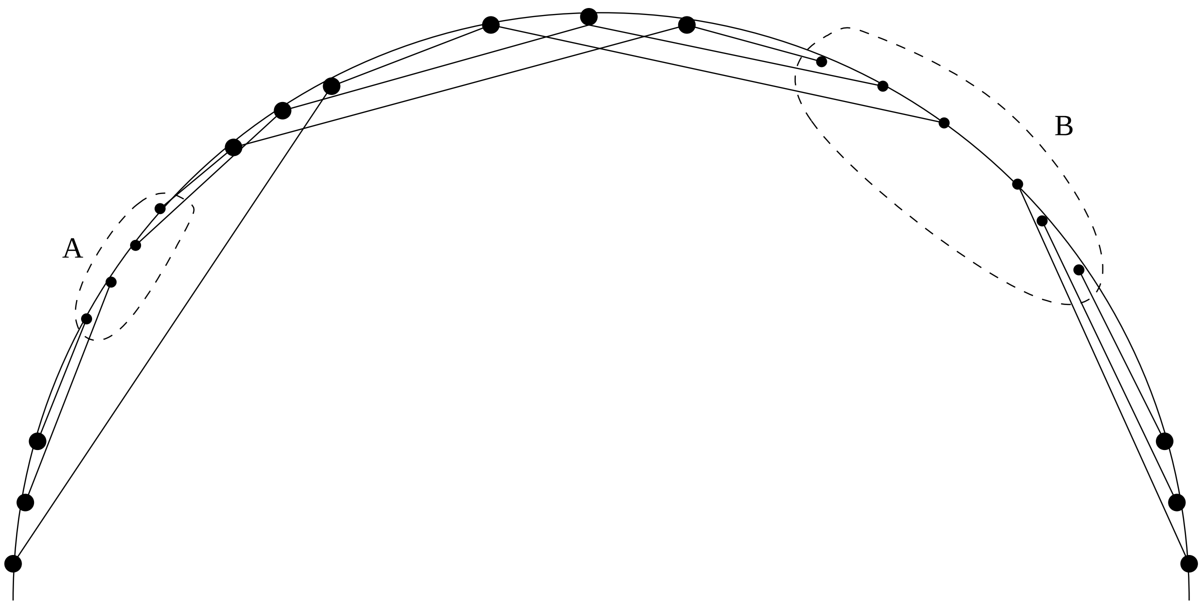

Figure 8: A planar matchgate equivalent to the matchcircuit shown in the previous slide. The dotted curves enclose the planar matchgates equivalent to the matchgates A and B from the matchcircuit.

From matchgrid to matchcircuit

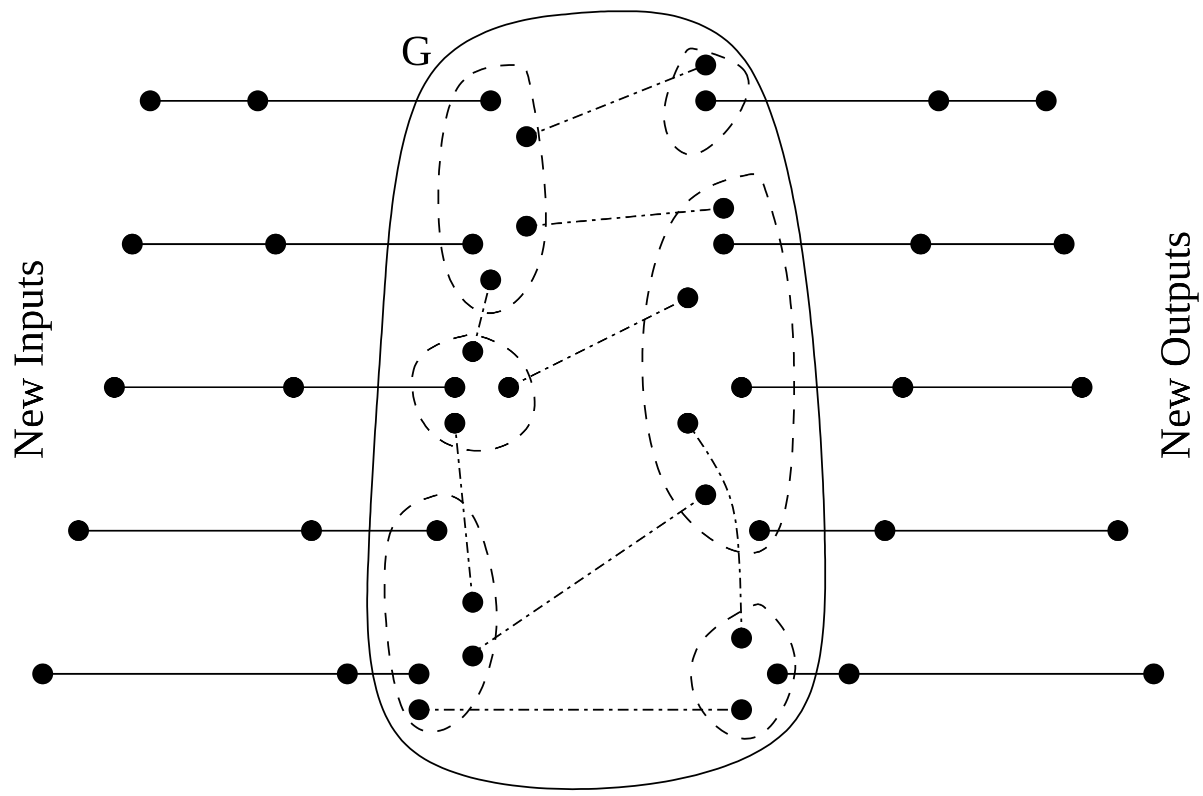

Figure 9: Constructing a matchcircuit from a planar matchgrid.

Quantum computation simulated by matchgrids

Theorem

The same fragment of quantum computation that was simulated by matchcircuits can be simulated by matchgrids.

Characterization theorems for signatures

Via the characterization theorem for characters, and then naked characters, we get a characterization theorem for signatures.

- **Cai and V. Choudhary.**
  On the Theory of Matchgate Computations.
  ECCC TR06-018.

This allows us to prove non-existence theorem for holographic algorithms.

Holographic template

We define holographic template as a restrictive class of holographic algorithms.

Theorem

There is no holographic template solving Valiant's X-Matching problem without the extra factor.

This is made possible by the equivalence theorem and the characterization of the character theory based on the useful Grassmann-Plücker identities.

More general ICE problems

Theorem

There is no holographic template to solve the ICE problem if we replace the degree bound by any $k > 3$.

Outlook

The most intriguing question is whether this new theory leads to any collapse of complexity classes.

The kinds of algorithms that are obtained by this theory are quite unlike anything before and almost exotic.

The uncertainty of its ultimate prospect makes it exciting.

Full paper

- **Cai and V. Choudhary.**
  On the Theory of Matchgate Computations.
  ECCC TR06-018.
- **Cai and V. Choudhary.**
  Some Results on Matchgates and Holographic Algorithms.
  ECCC TR06-048.