CS 540 Introduction to Artificial Intelligence

Neural Networks (II)

University of Wisconsin-Madison

Fall 2022

Today's outline

- Single-layer Perceptron Review
- Multi-layer Perceptron
  - Single output
  - Multiple output
- How to train neural networks
  - Gradient descent

Review: Perceptron

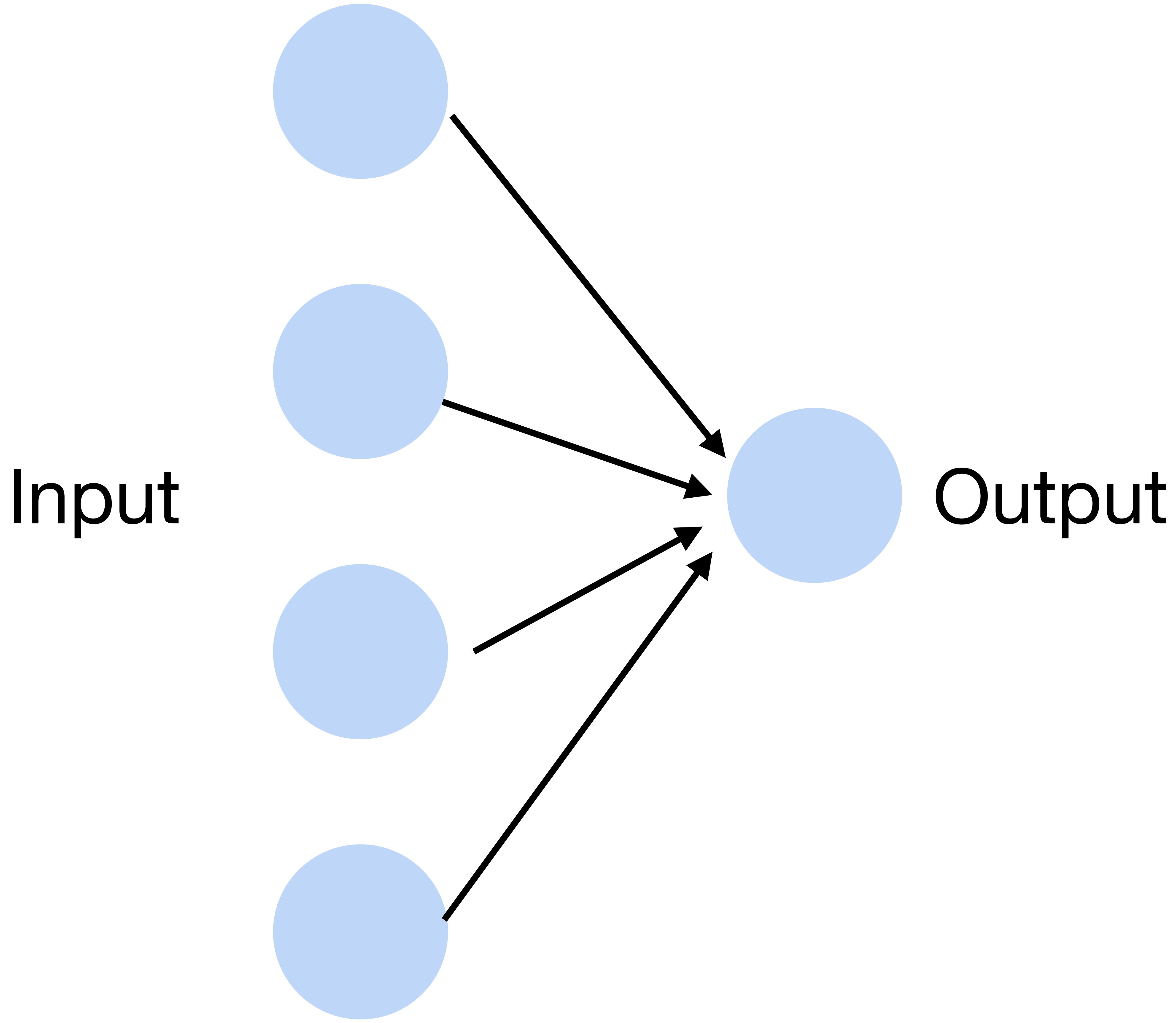
- Given input $\mathbf{x}$, weight $\mathbf{w}$ and bias b, perceptron outputs:

$$o = \sigma\left(\langle \mathbf{w}, \mathbf{x} \rangle + b\right)$$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

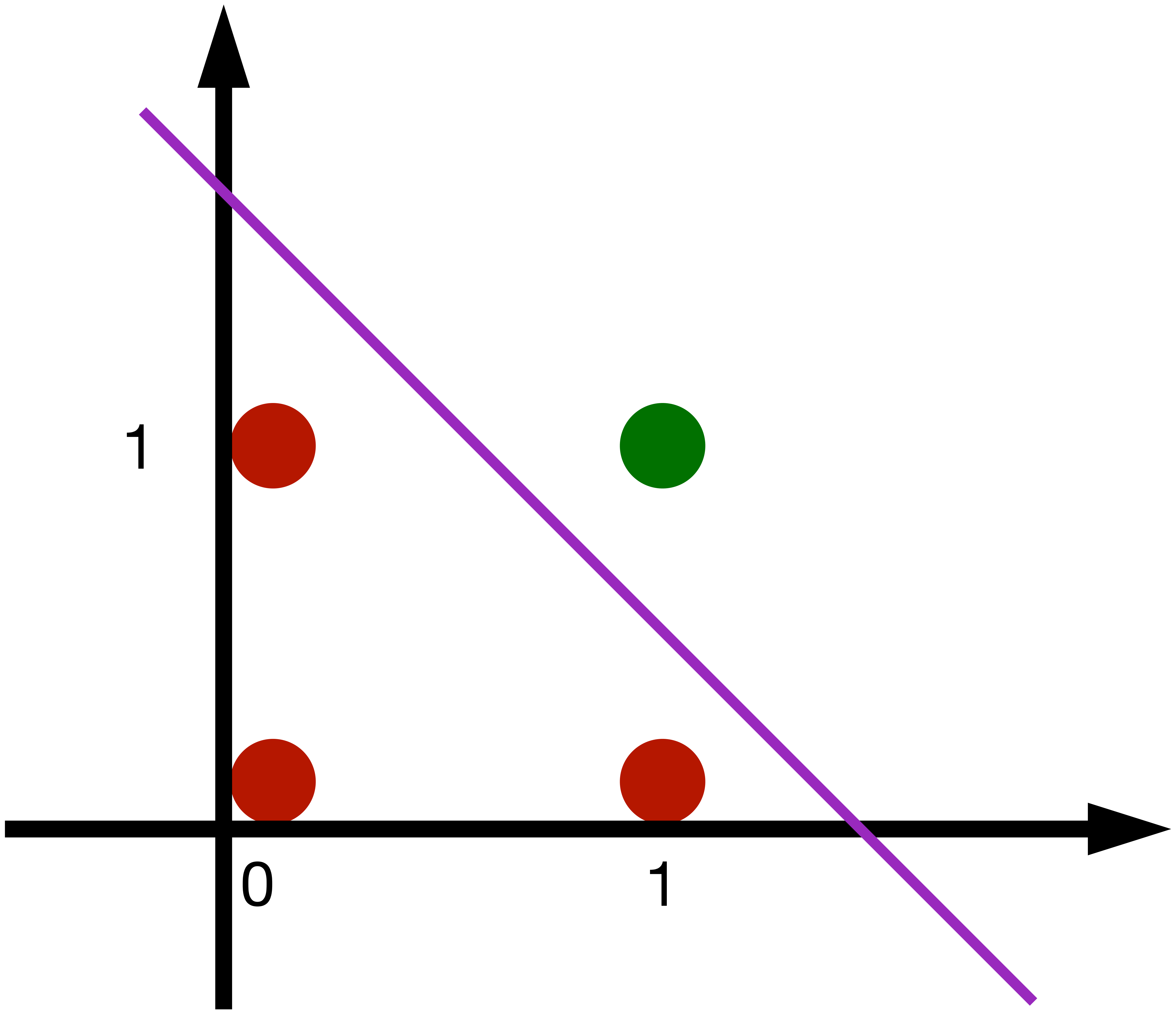
Activation function

Cats vs. dogs?



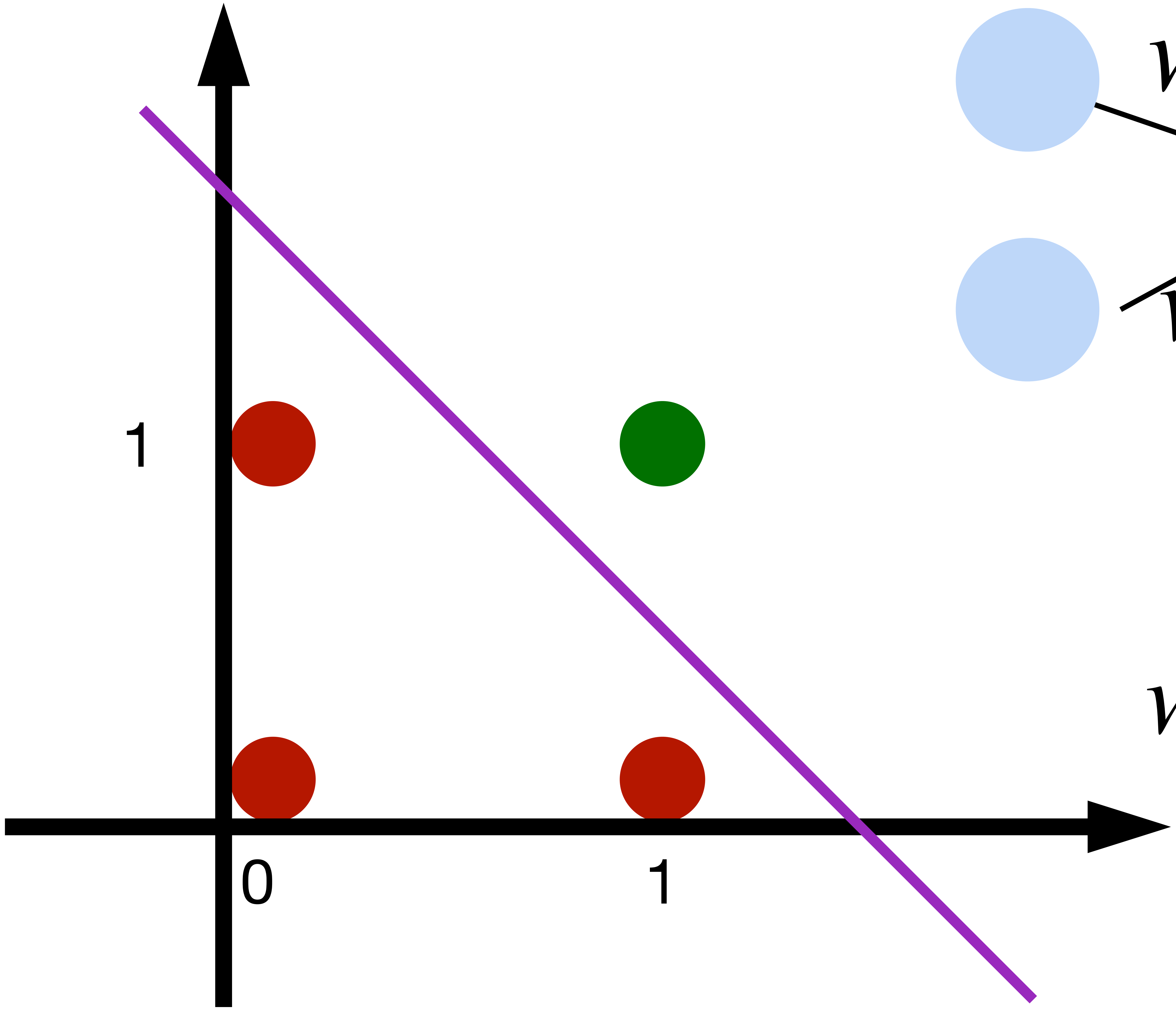
Learning AND function using perceptron

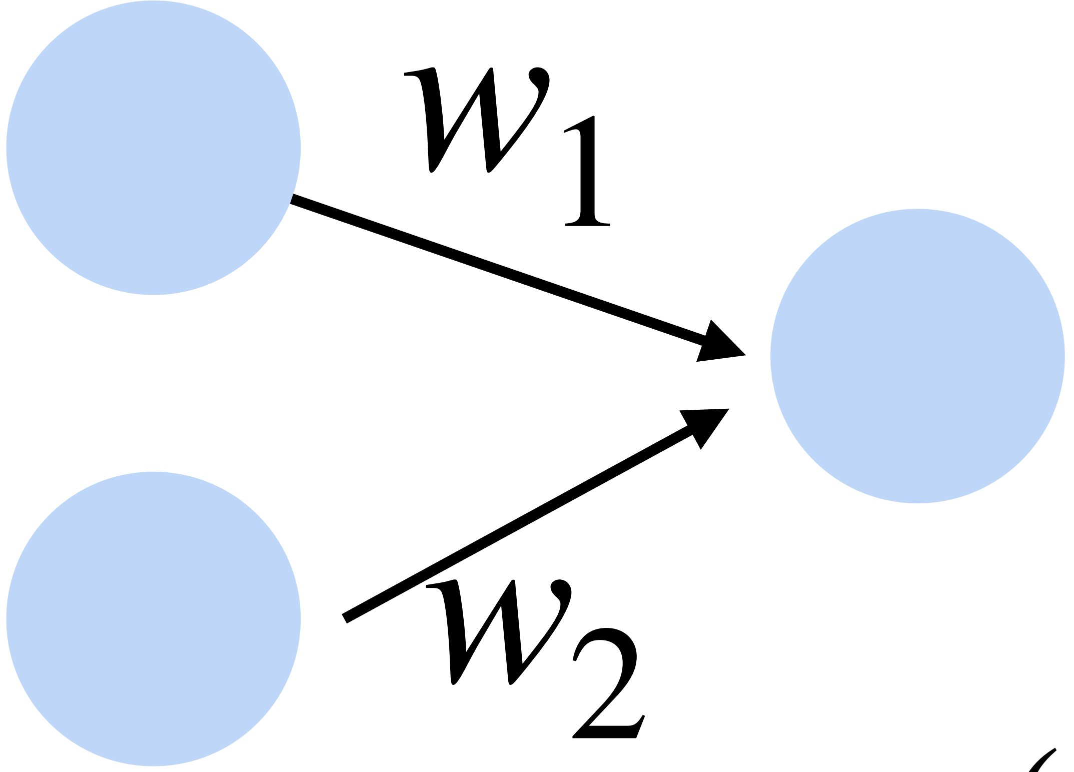
The perceptron can learn an AND function



Learning AND function using perceptron

The perceptron can learn an AND function





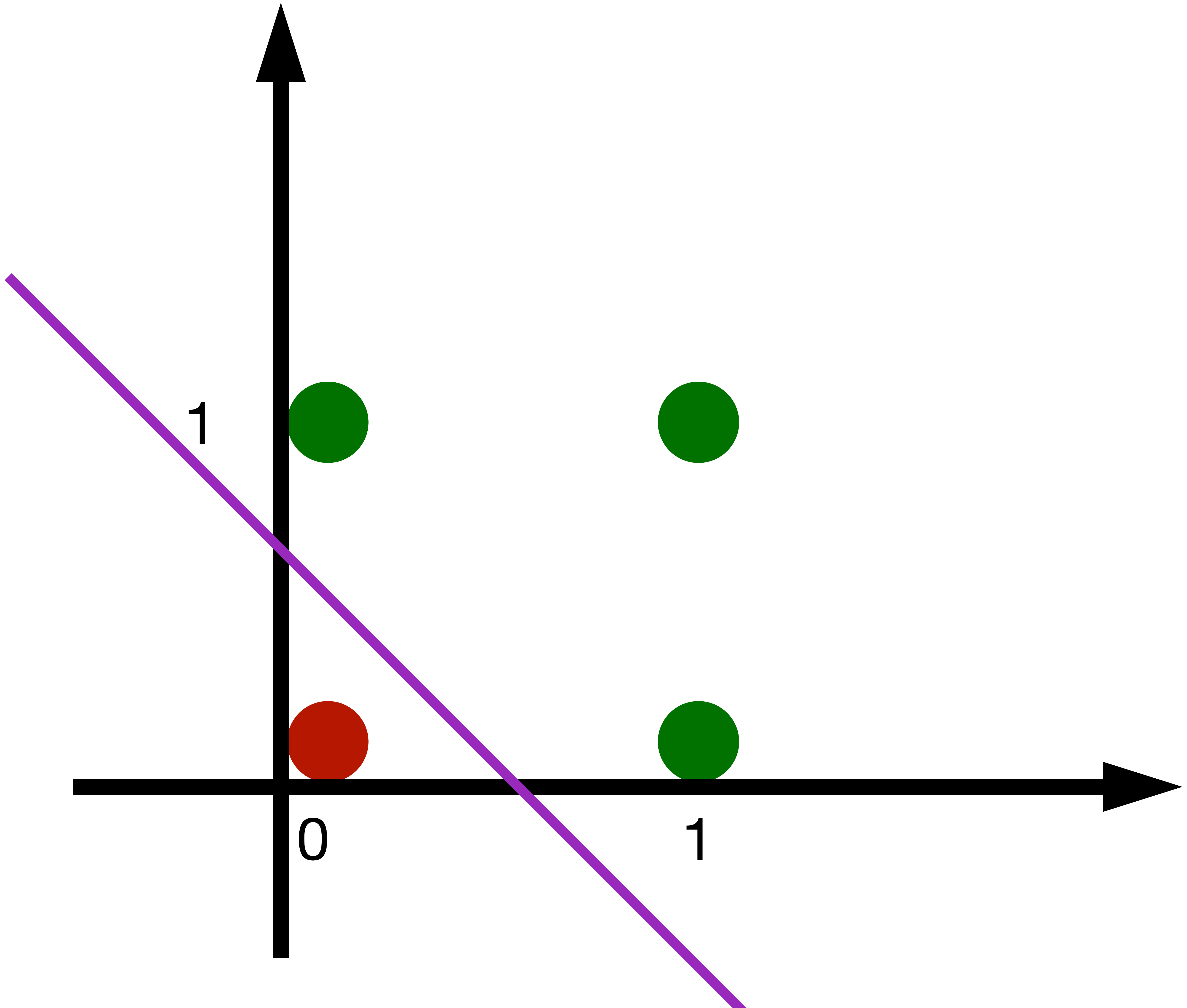
Output $\sigma(x_1 w_1 + x_2 w_2 + b)$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1 = 1, w_2 = 1, b = -1.5$$

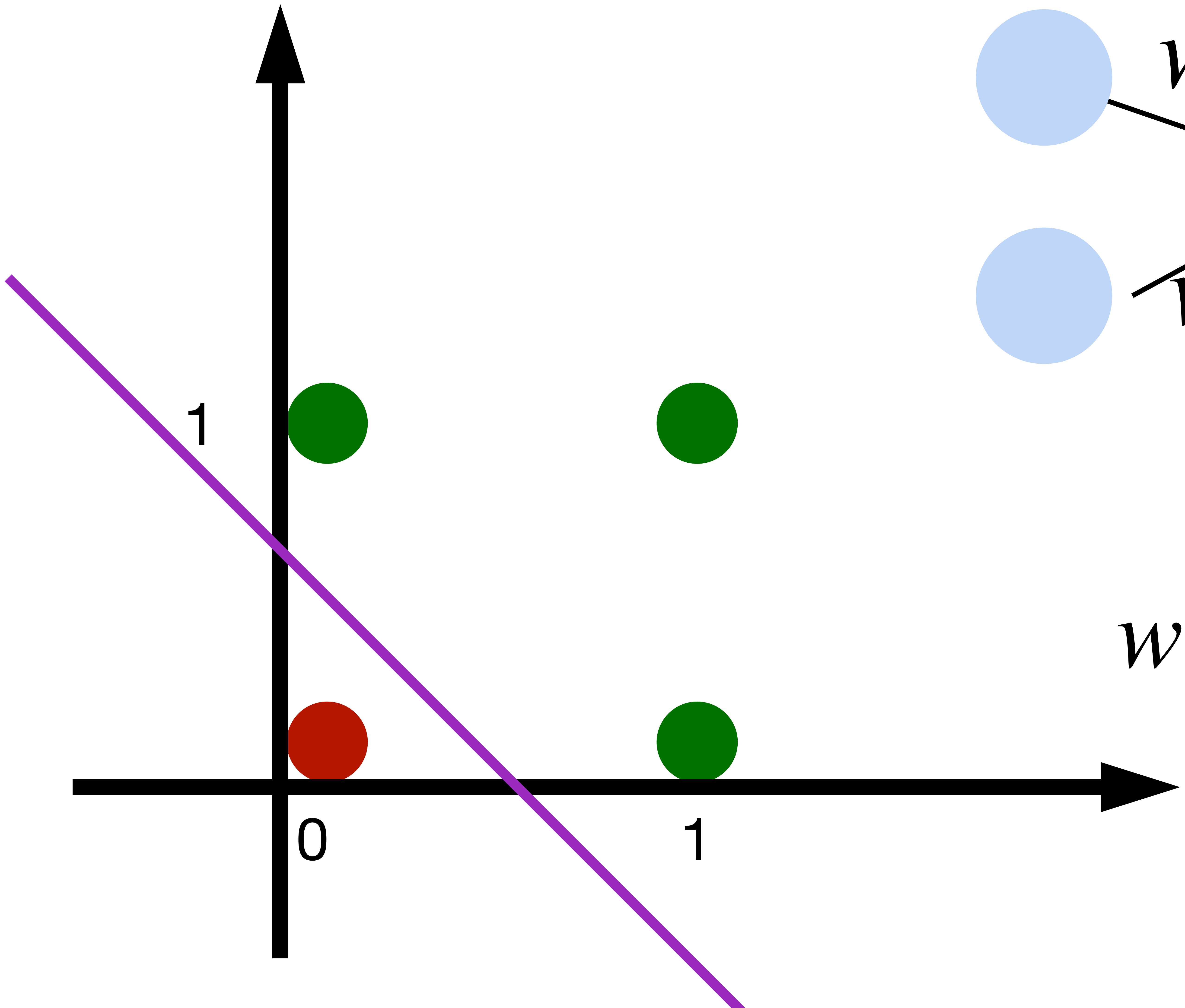
Learning OR function using perceptron

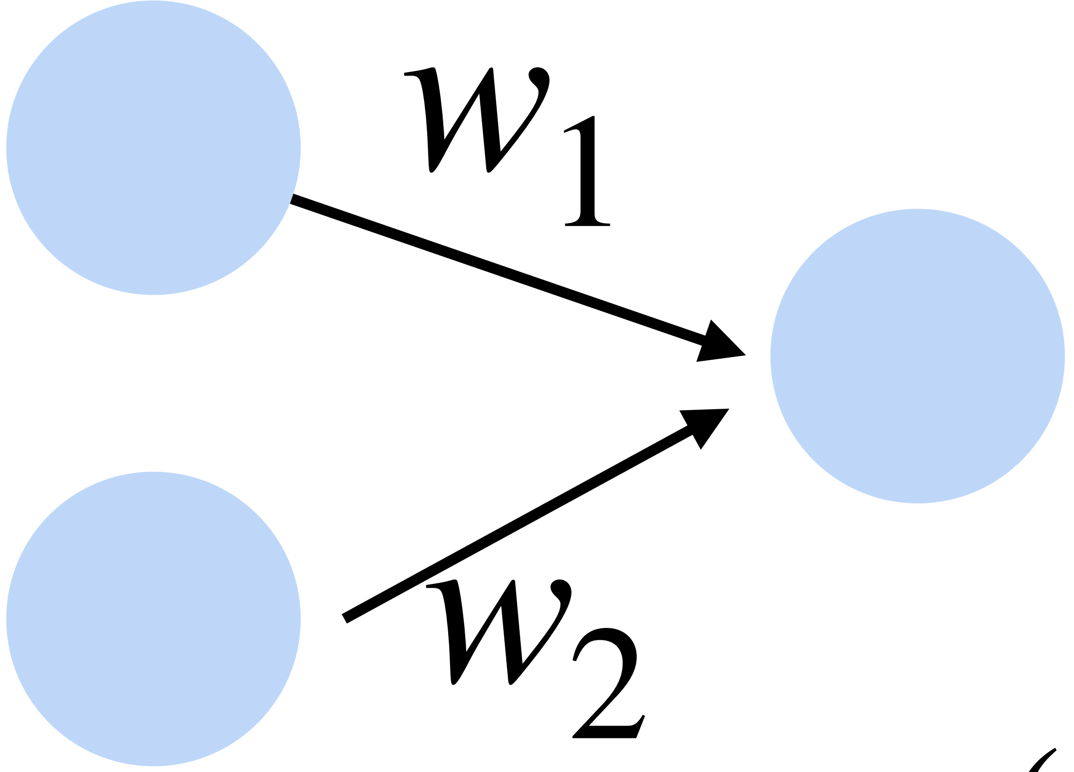
The perceptron can learn an OR function



Learning OR function using perceptron

The perceptron can learn an OR function





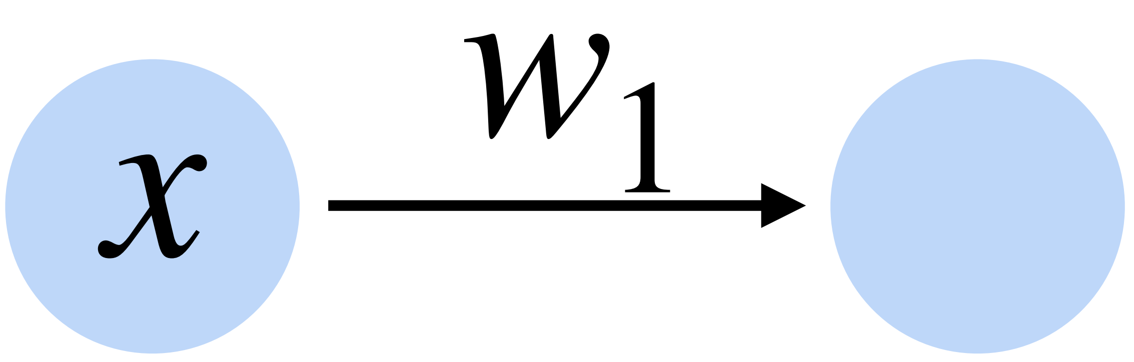
Output $\sigma(x_1 w_1 + x_2 w_2 + b)$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1 = 1, w_2 = 1, b = -0.5$$

Learning NOT function using perceptron

The perceptron can learn NOT function (single input)



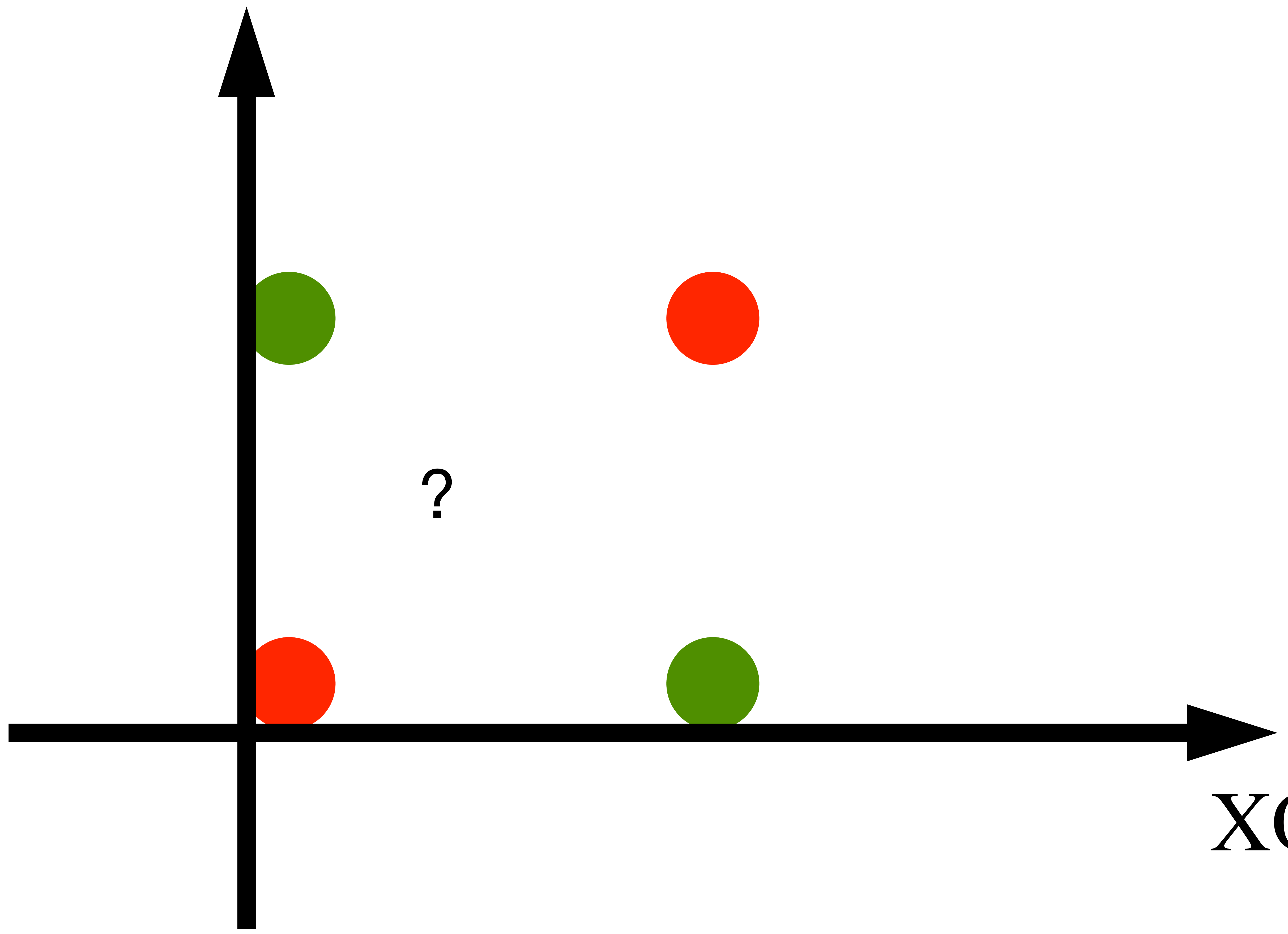
Output $\sigma(xw_1 + b)$

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1 = -1, b = 0.5$$

The limited power of a single neuron

The perceptron cannot learn an **XOR** function
(neurons can only generate linear separators)



$$x_1 = 1, x_2 = 1, y = 0$$

$$x_1 = 1, x_2 = 0, y = 1$$

$$x_1 = 0, x_2 = 1, y = 1$$

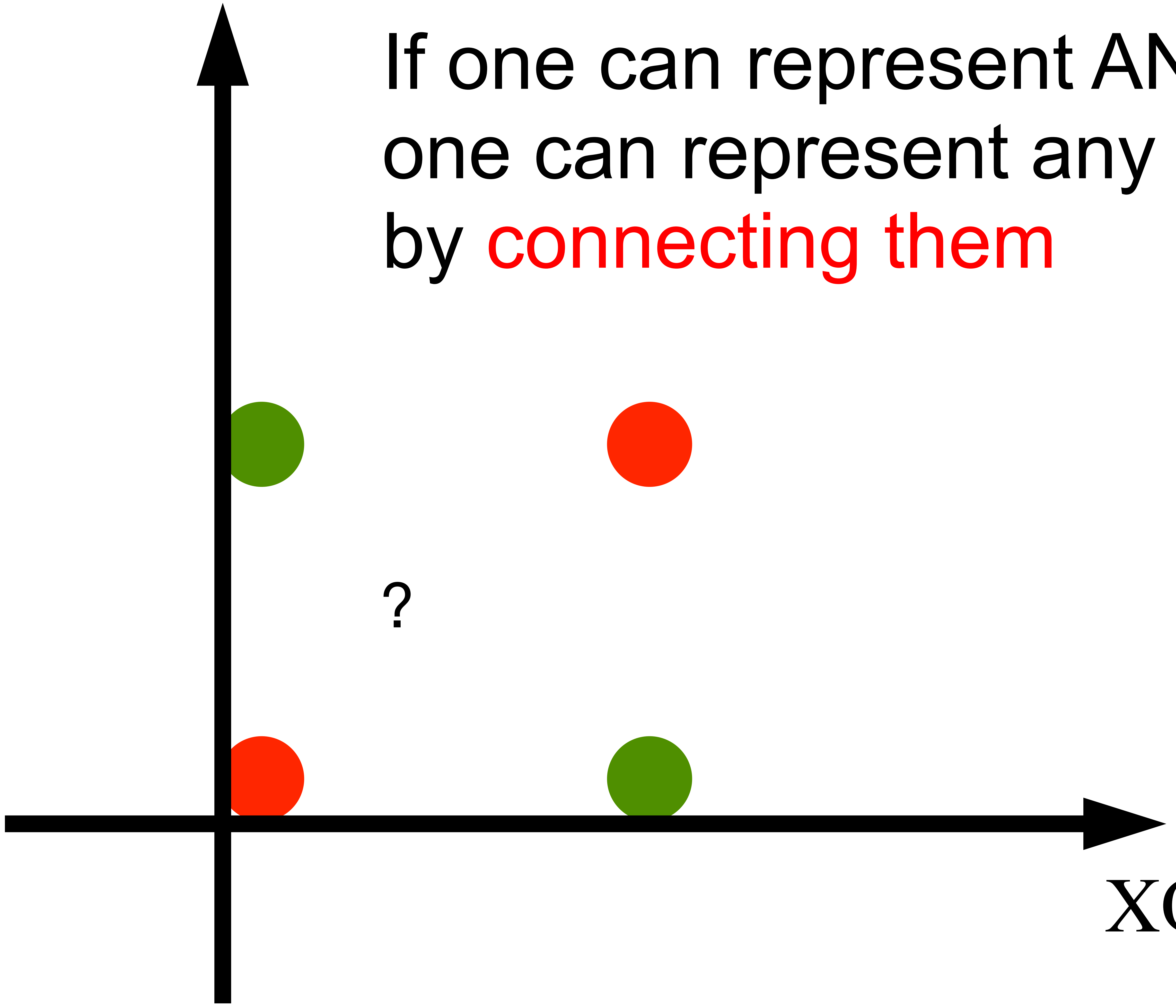
$$x_1 = 0, x_2 = 0, y = 0$$

XOR(x1,x2)=(x1∧¬x2)∨(¬x1∧x2)

The limited power of a single neuron

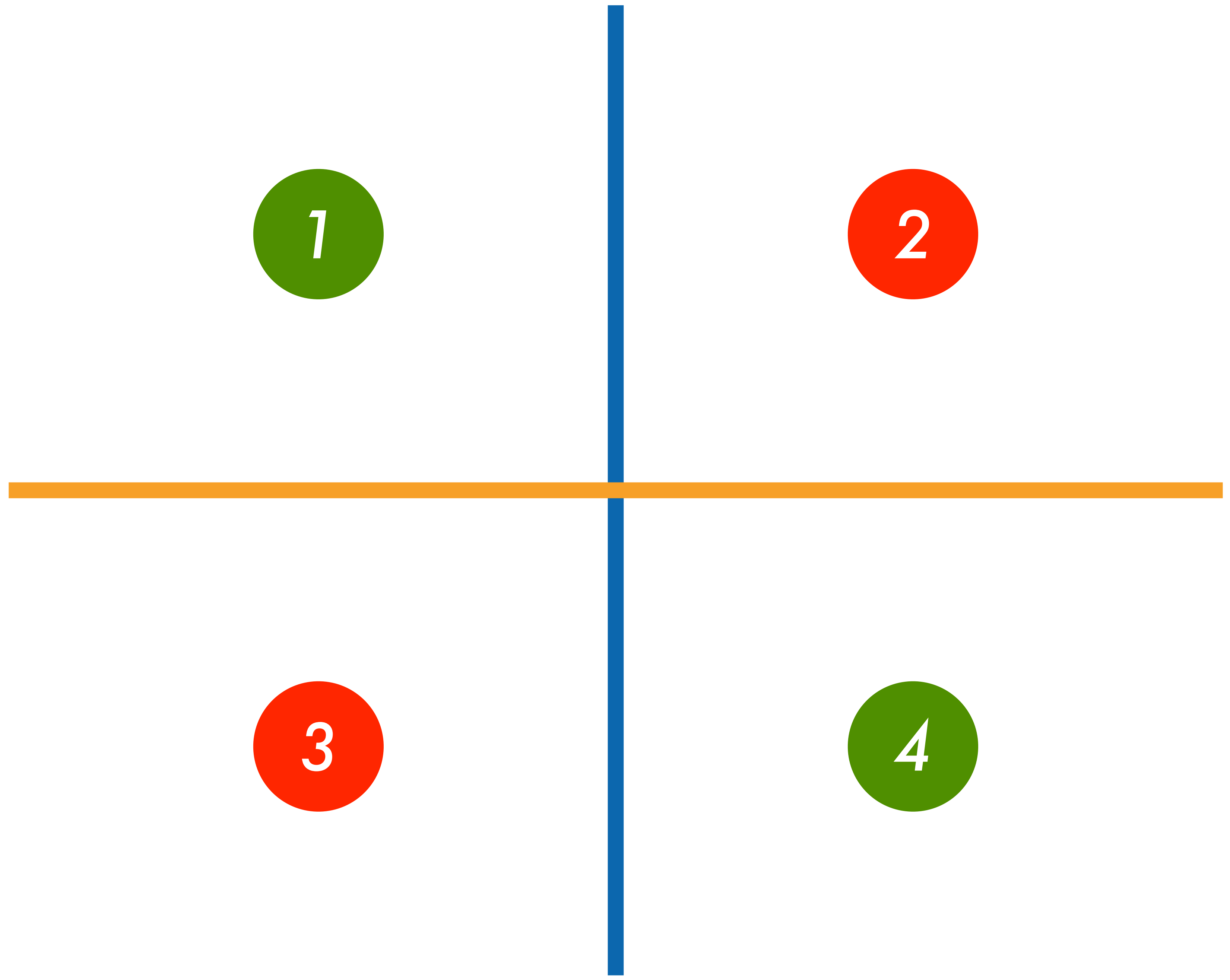
XOR problem

If one can represent AND, OR, NOT,
one can represent any logic circuit (including XOR),
by connecting them

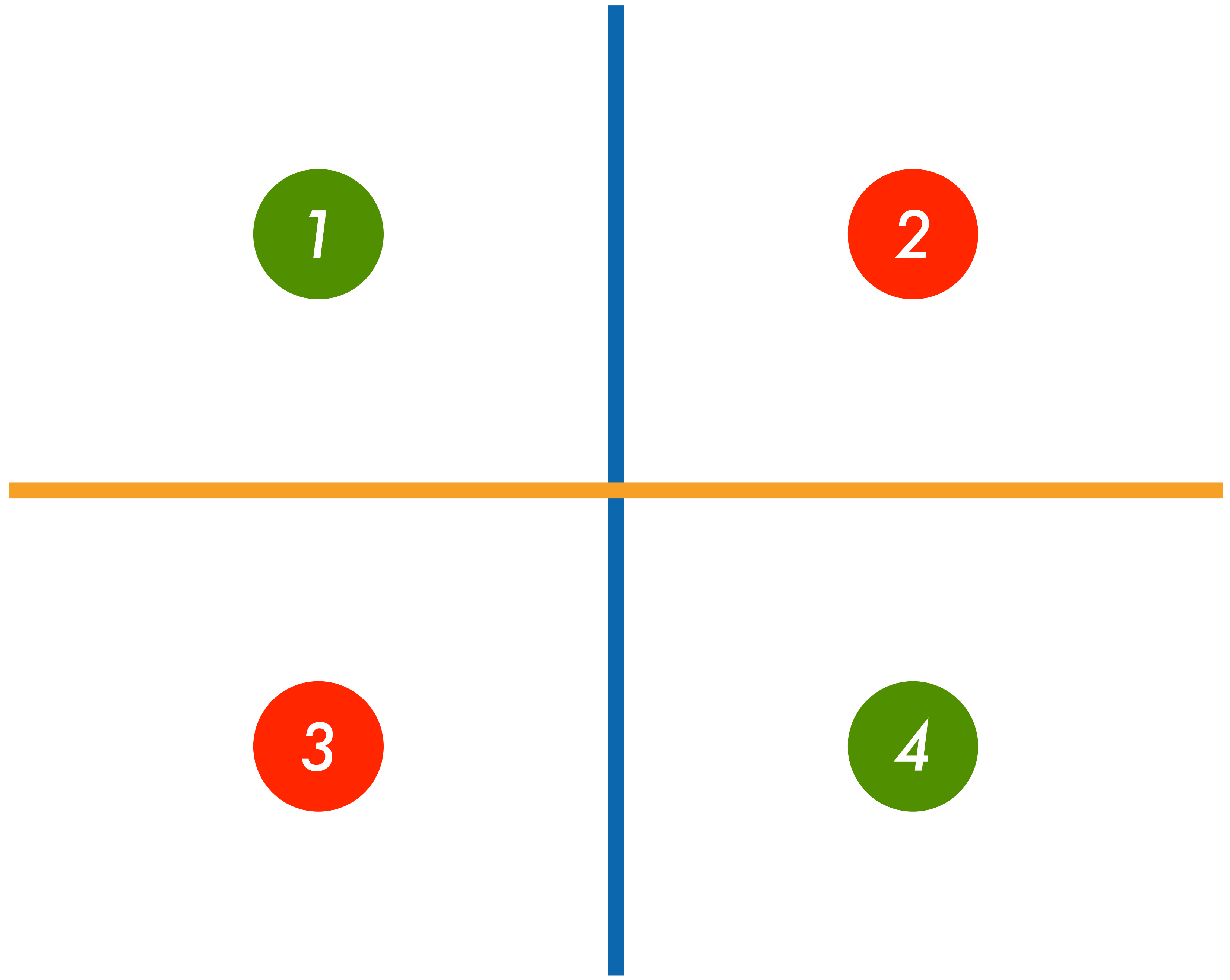


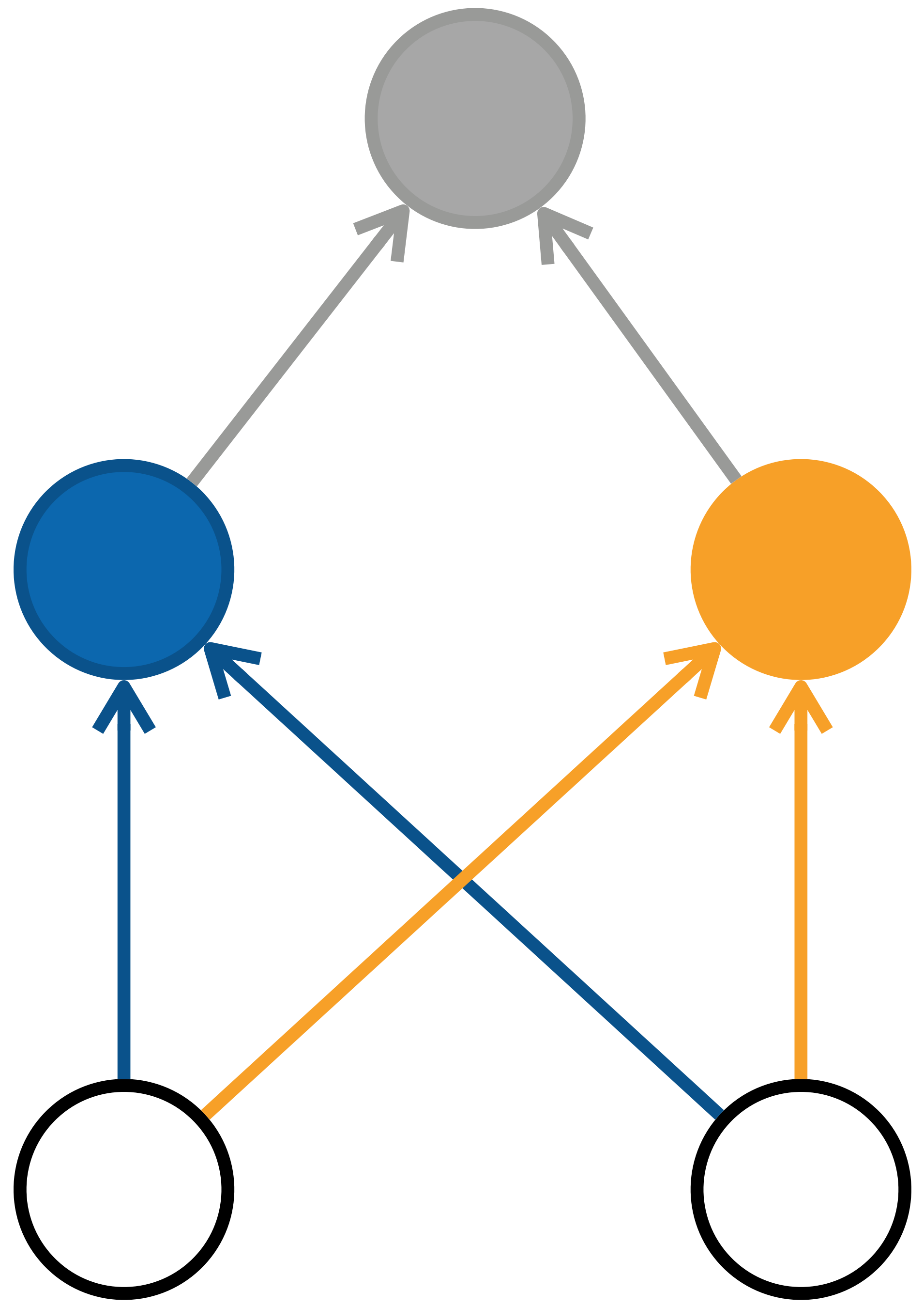
$XOR(x1,x2) = (x1 \wedge \neg x2) \vee (\neg x1 \wedge x2)$

Learning XOR

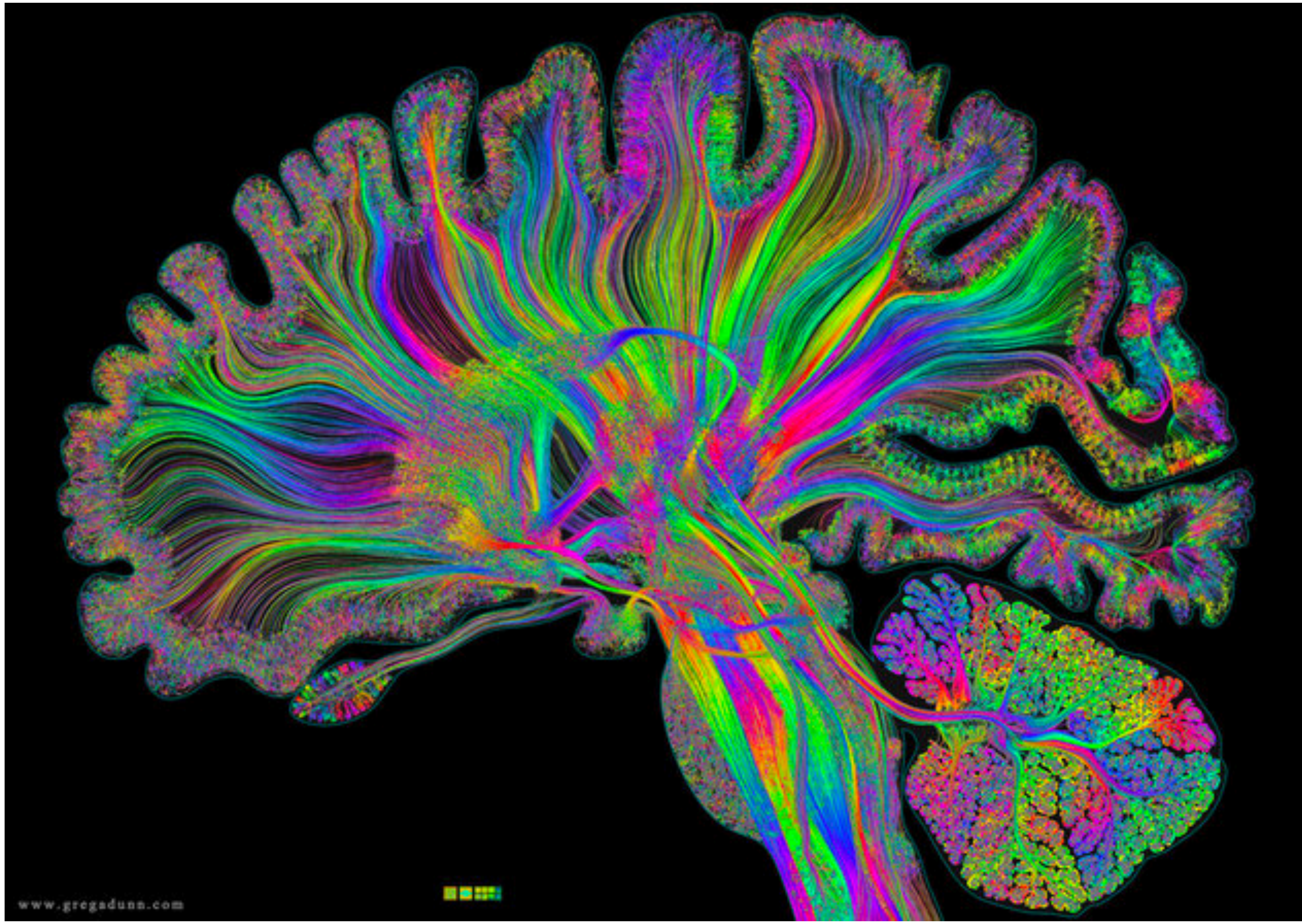


Learning XOR



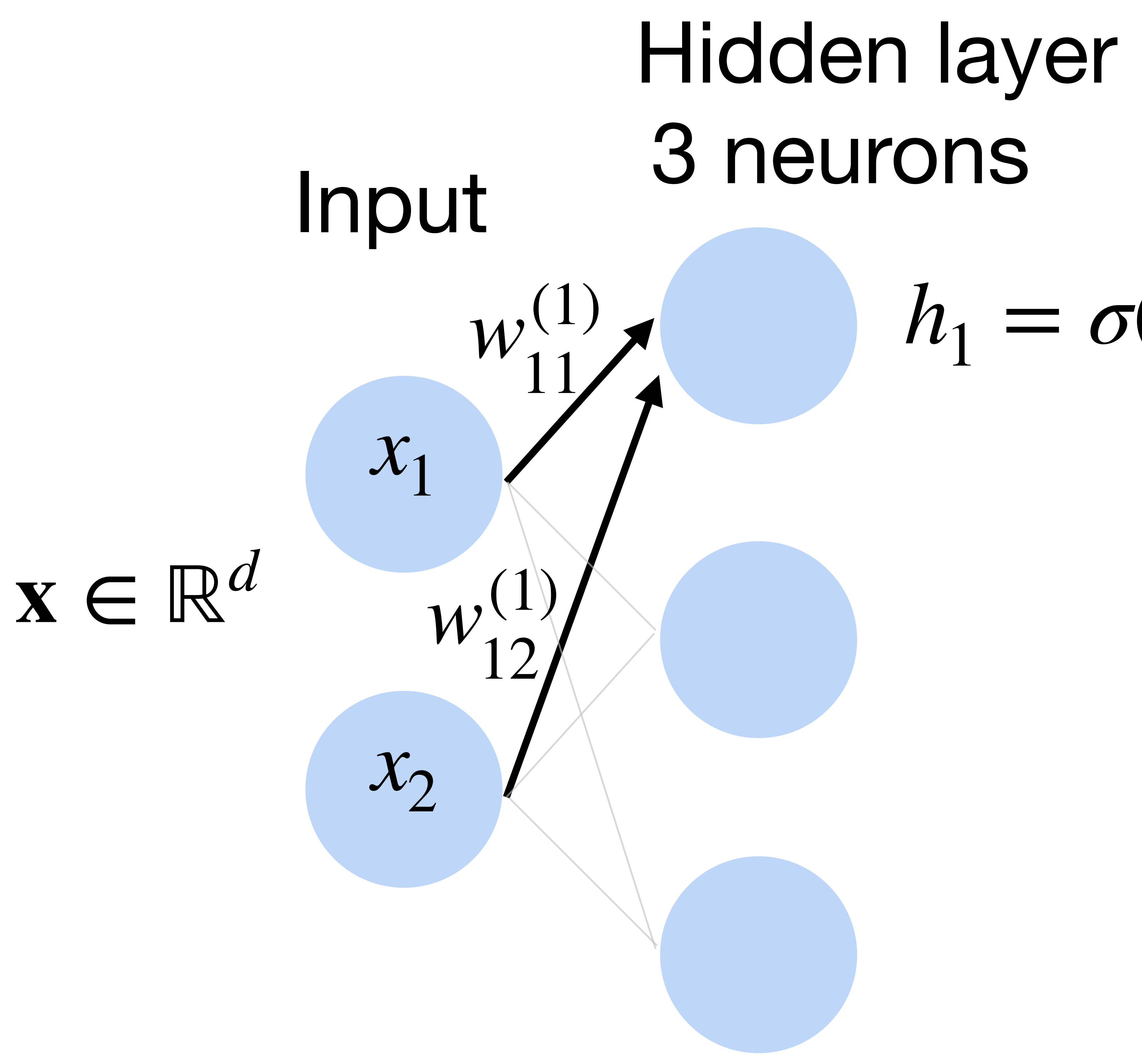


Multilayer Perceptron



Multi-layer perceptron: Example

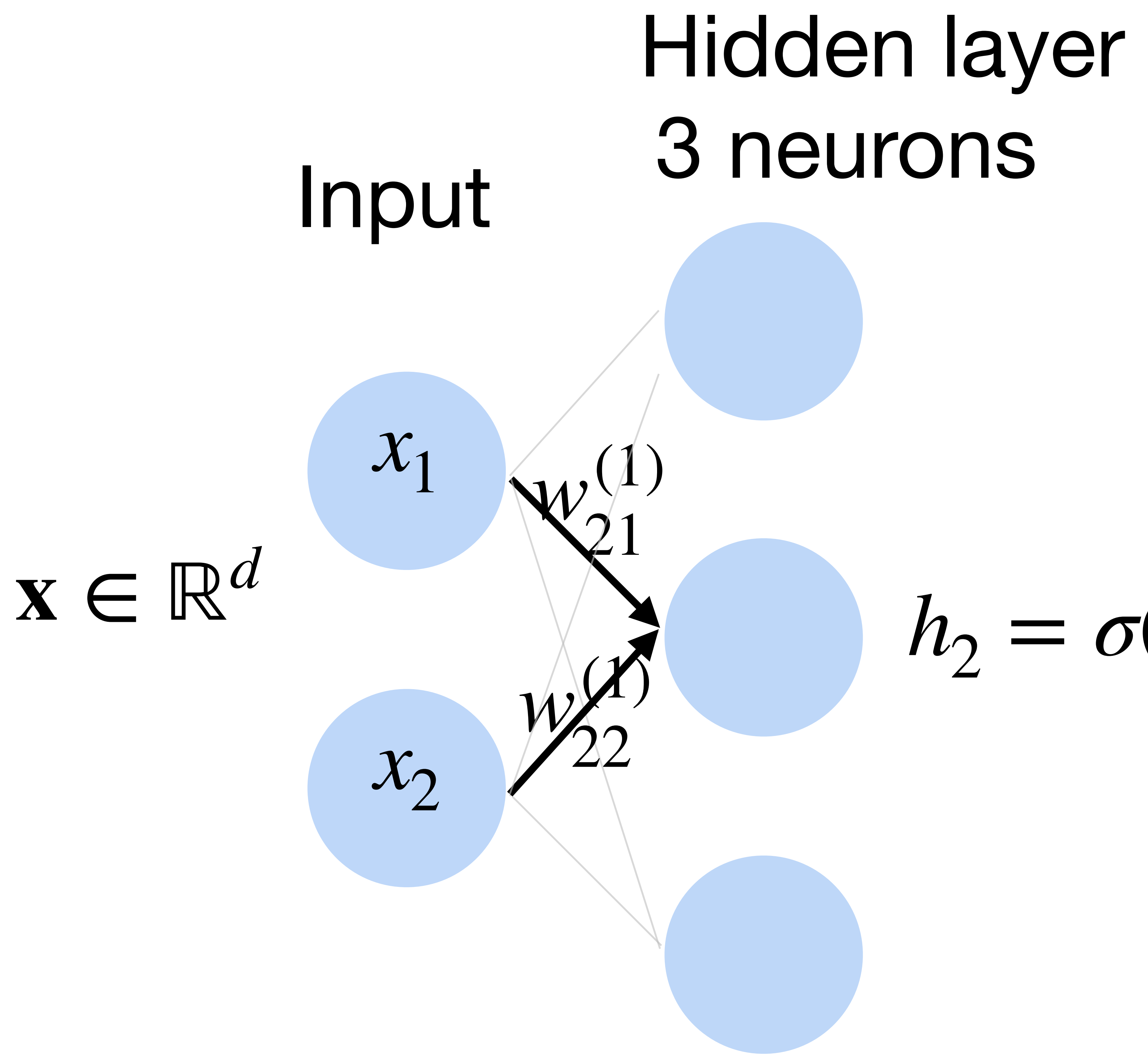
- Standard way to connect Perceptrons
- Example: 1 hidden layer, 1 output layer, depth = 2



$$h_1 = \sigma\left(\sum_{i=1}^{d} x_i w_{1i}^{(1)} + b_1\right)$$

Multi-layer perceptron: Example

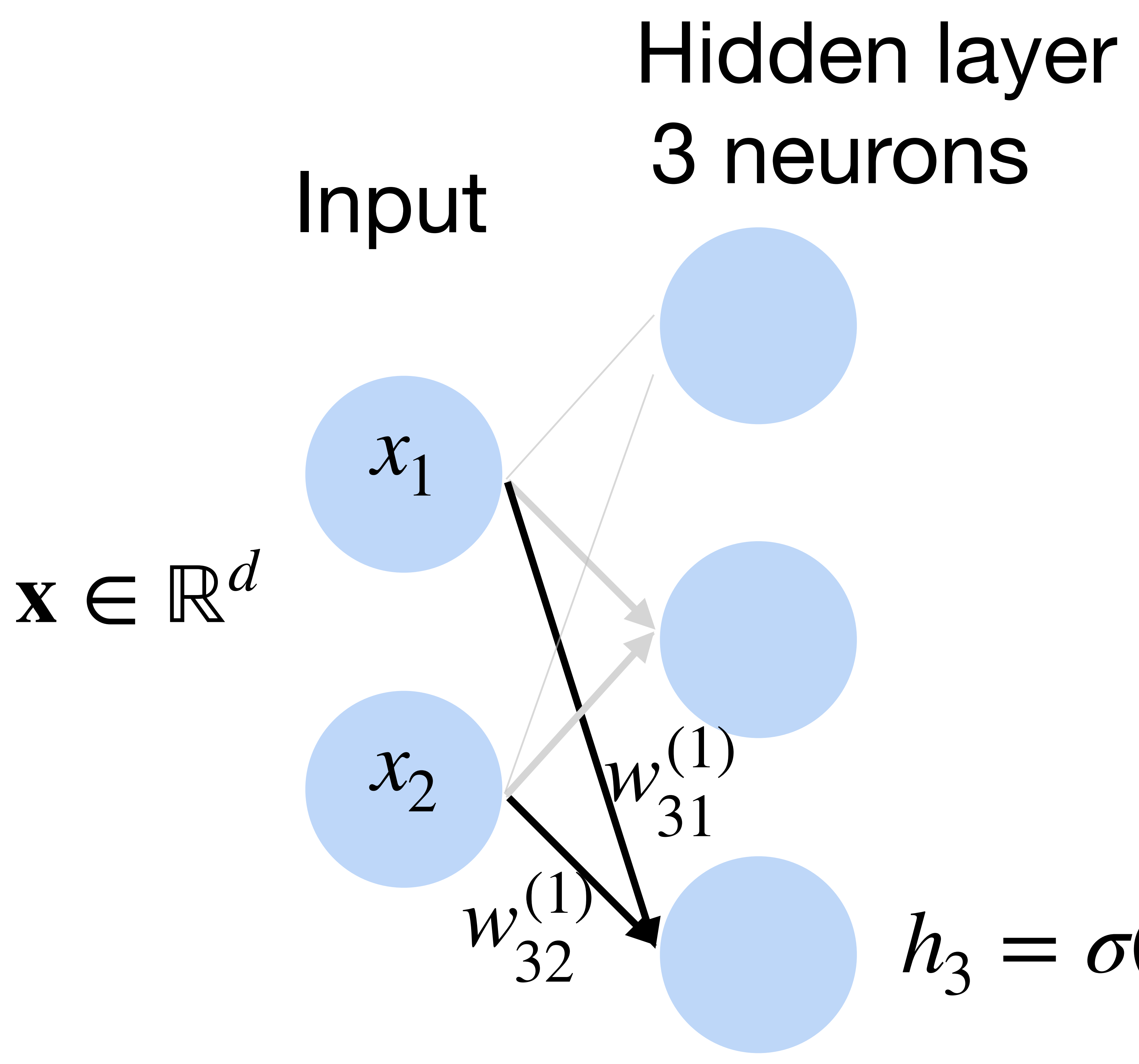
- Standard way to connect Perceptrons
- Example: 1 hidden layer, 1 output layer, depth = 2



$$h_2 = \sigma(\sum_{i=1}^{d} x_i w_{2i}^{(1)} + b_2)$$

Multi-layer perceptron: Example

- Standard way to connect Perceptrons
- Example: 1 hidden layer, 1 output layer, depth = 2



$$h_3 = \sigma(\sum_{i=1}^{d} x_i w_{3i}^{(1)} + b_3)$$

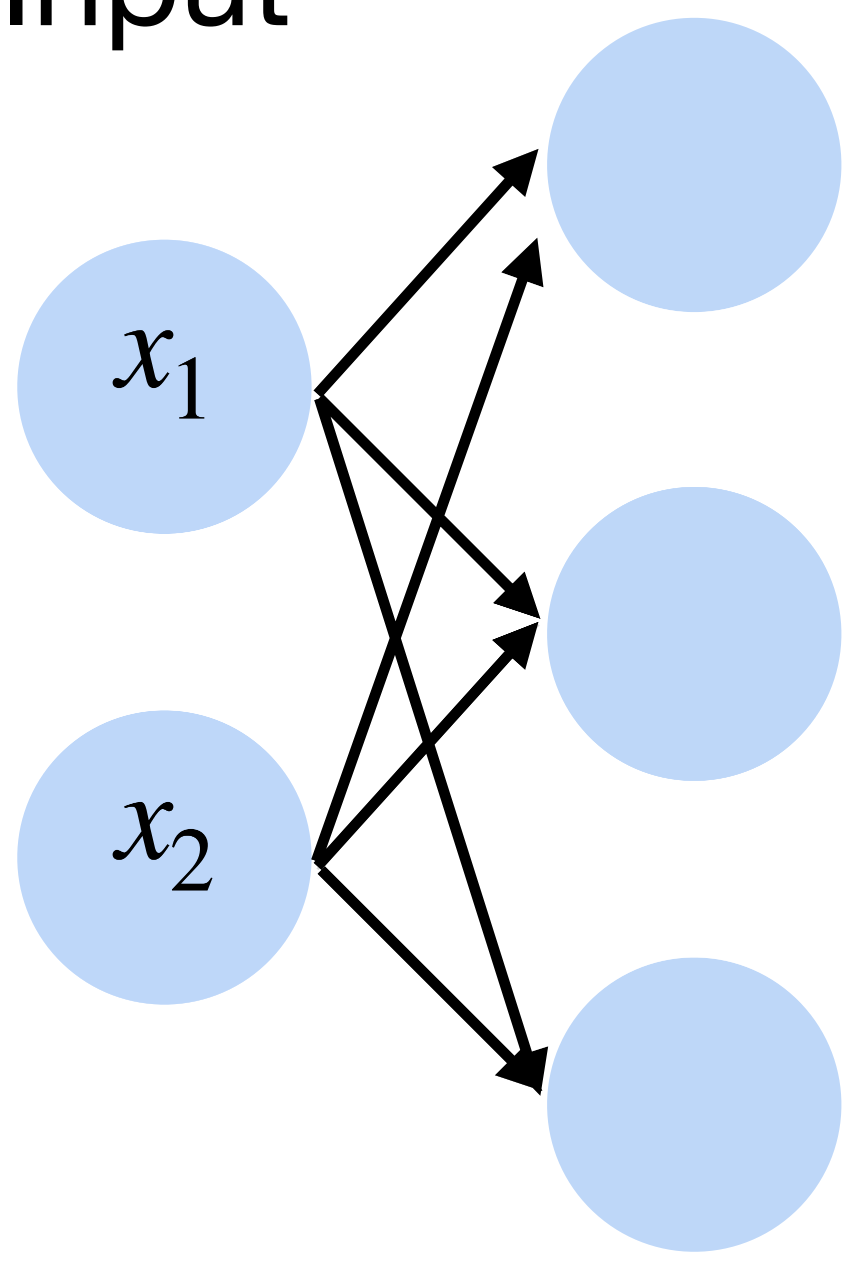
Multi-layer perceptron: Example

- Standard way to connect Perceptrons
- Example: 1 hidden layer, 1 output layer, depth = 2

Hidden layer
m=3 neurons

Input

$\mathbf{x} \in \mathbb{R}^d$



$$h_1 = \sigma(\sum_{i=1}^{d} x_i w_{1i}^{(1)} + b_1)$$

$$h_2 = \sigma(\sum_{i=1}^{d} x_i w_{2i}^{(1)} + b_2)$$

$$h_3 = \sigma(\sum_{i=1}^{d} x_i w_{3i}^{(1)} + b_3)$$

Multi-layer perceptron: Example

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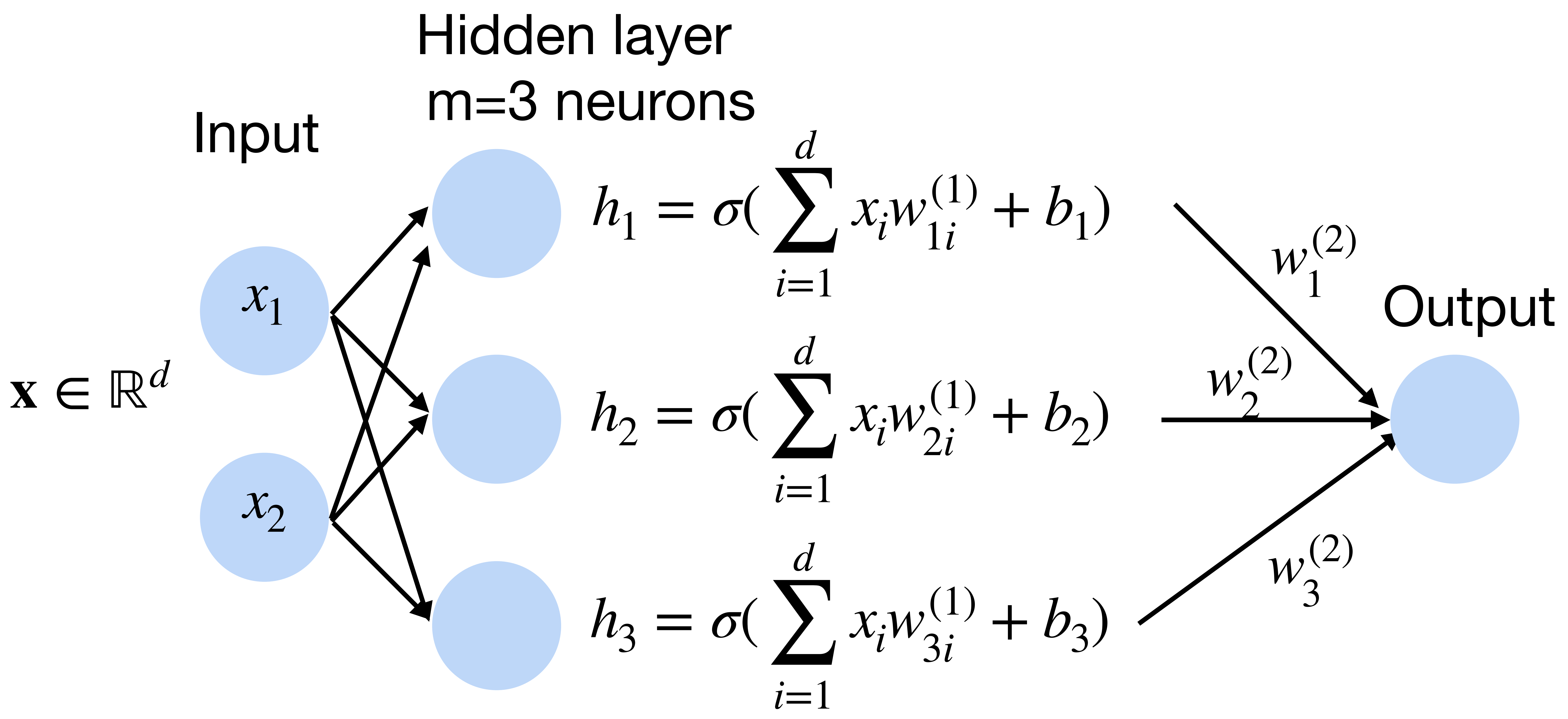

Multi-layer perceptron: Example

- Standard way to connect Perceptrons
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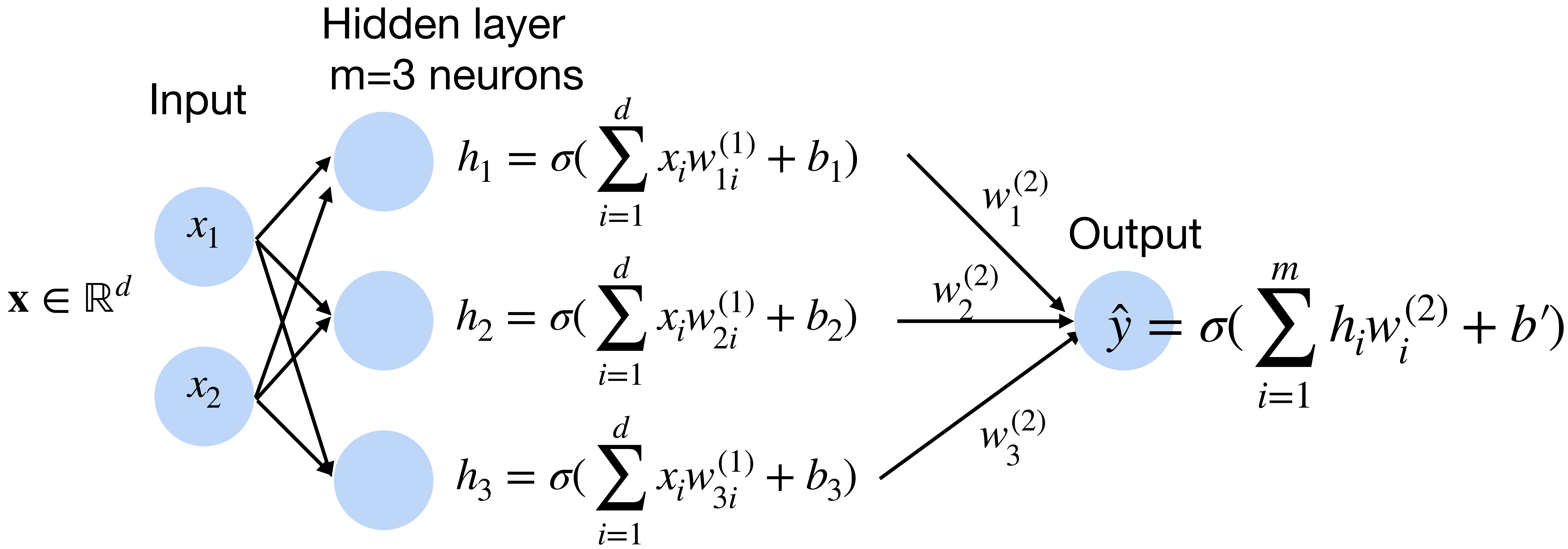

Multi-layer perceptron: Example

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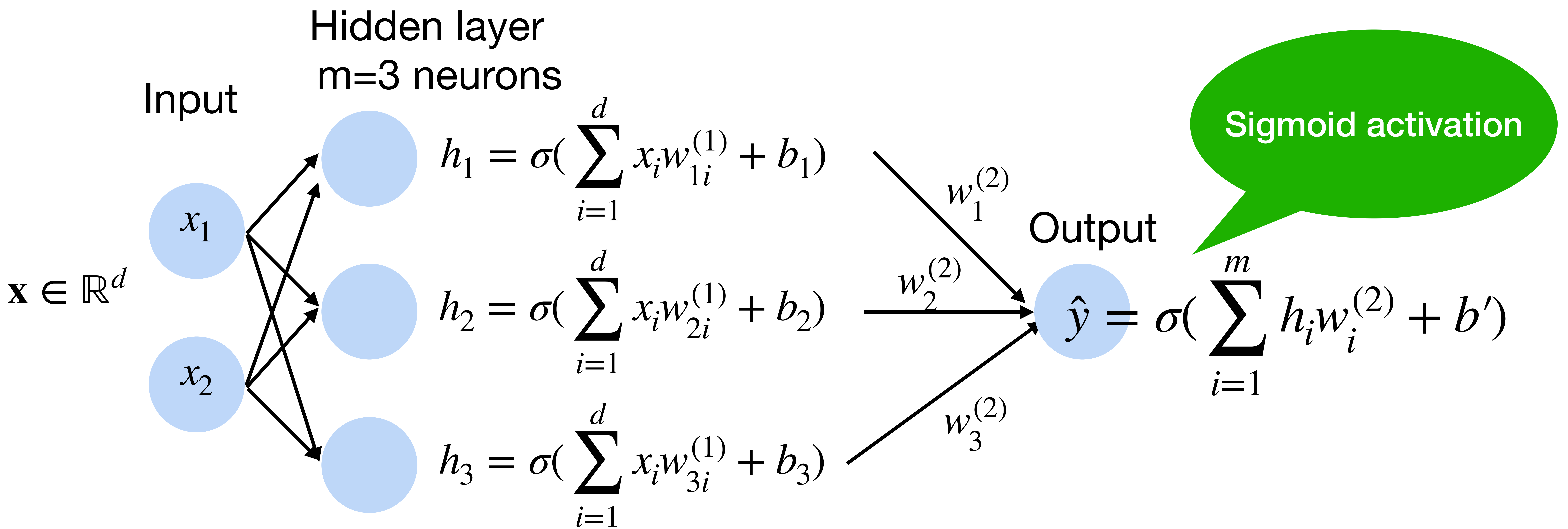
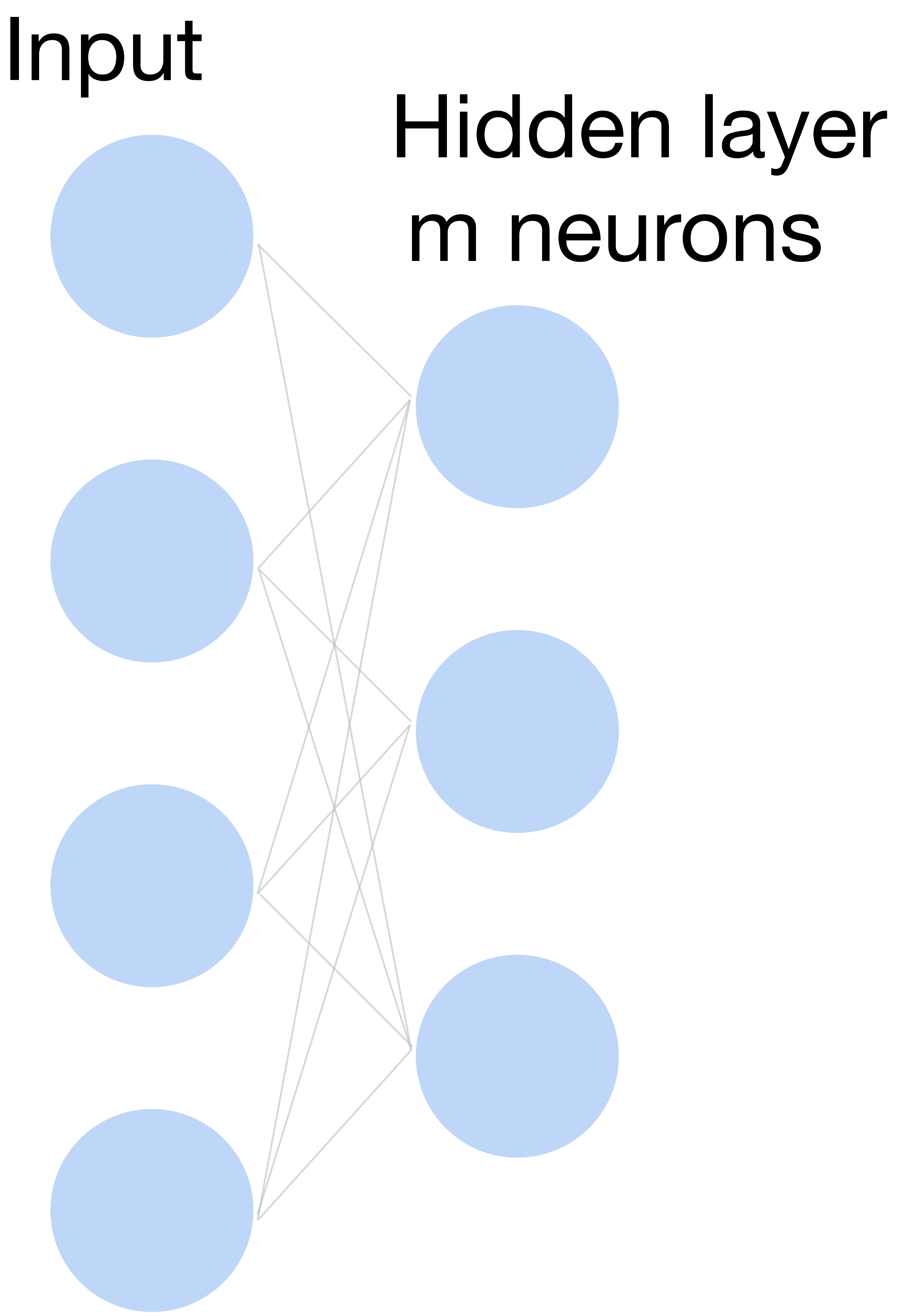


Multi-layer perceptron: Matrix Notation

- Input $\mathbf{x} \in \mathbb{R}^d$
- Hidden $\mathbf{W}^{(1)} \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m$
- Intermediate output

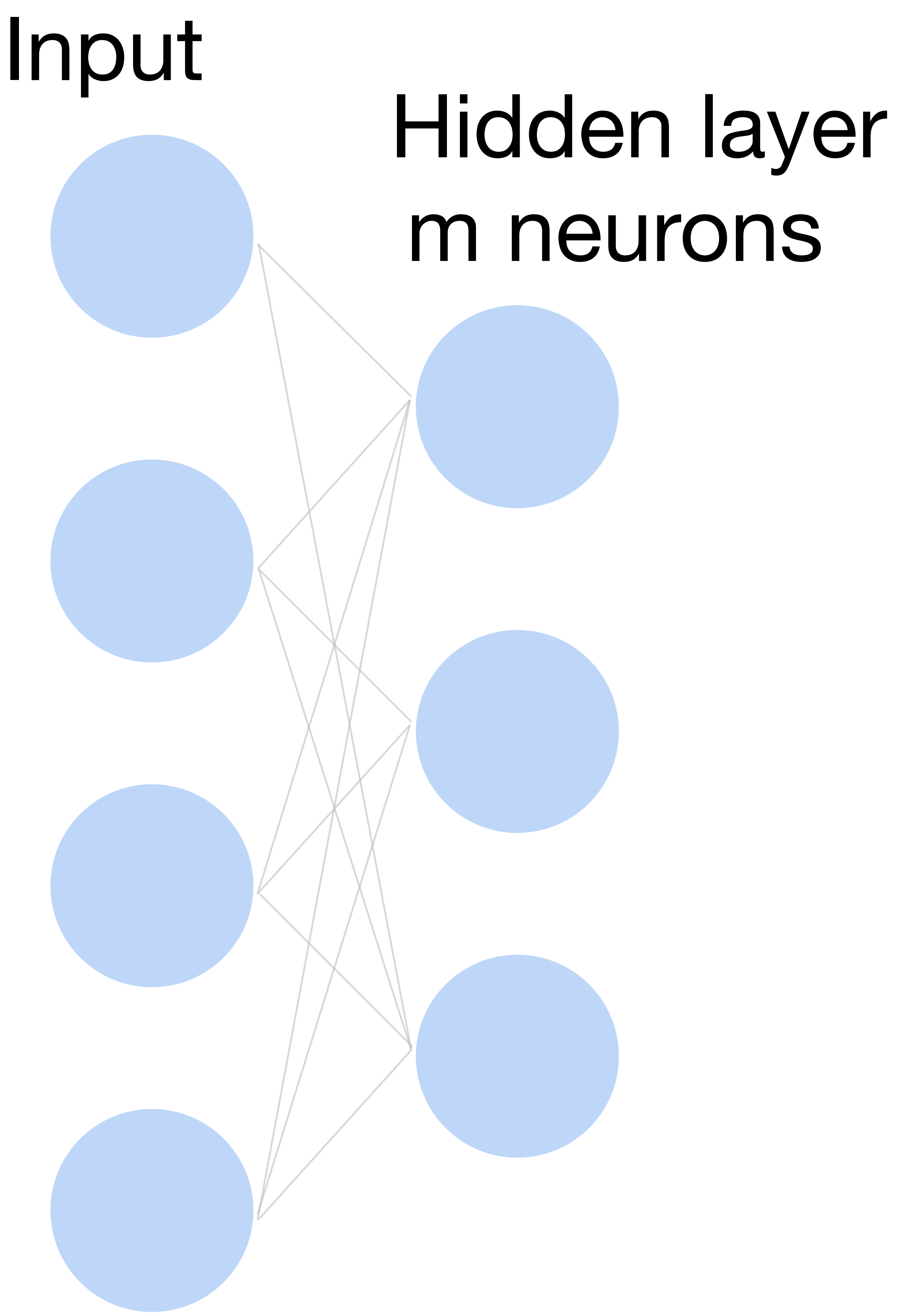


Multi-layer perceptron: Matrix Notation

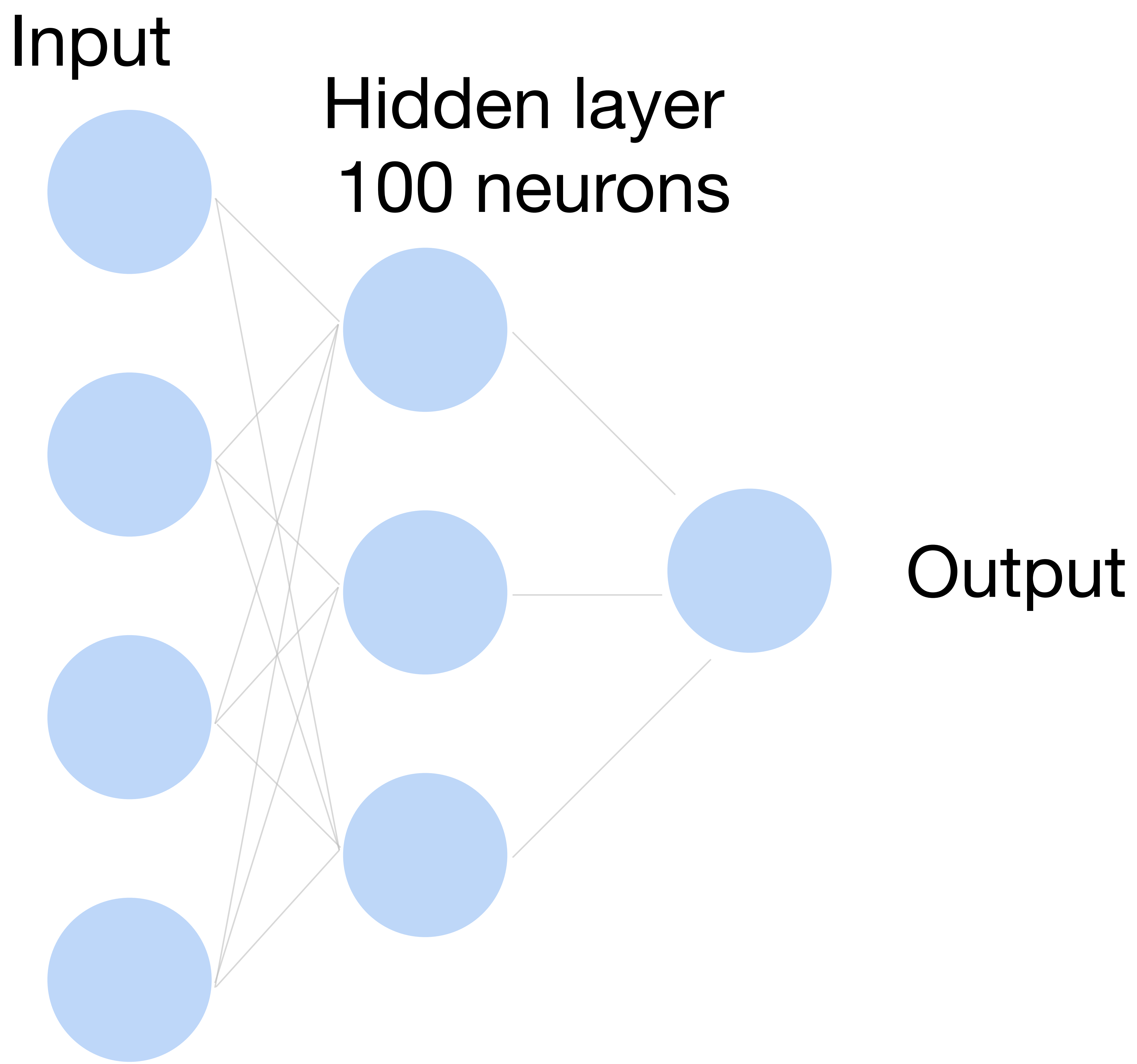
- Input $\mathbf{x} \in \mathbb{R}^d$
- Hidden $\mathbf{W}^{(1)} \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m$
- Intermediate output

$$\mathbf{h} = \sigma(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b})$$

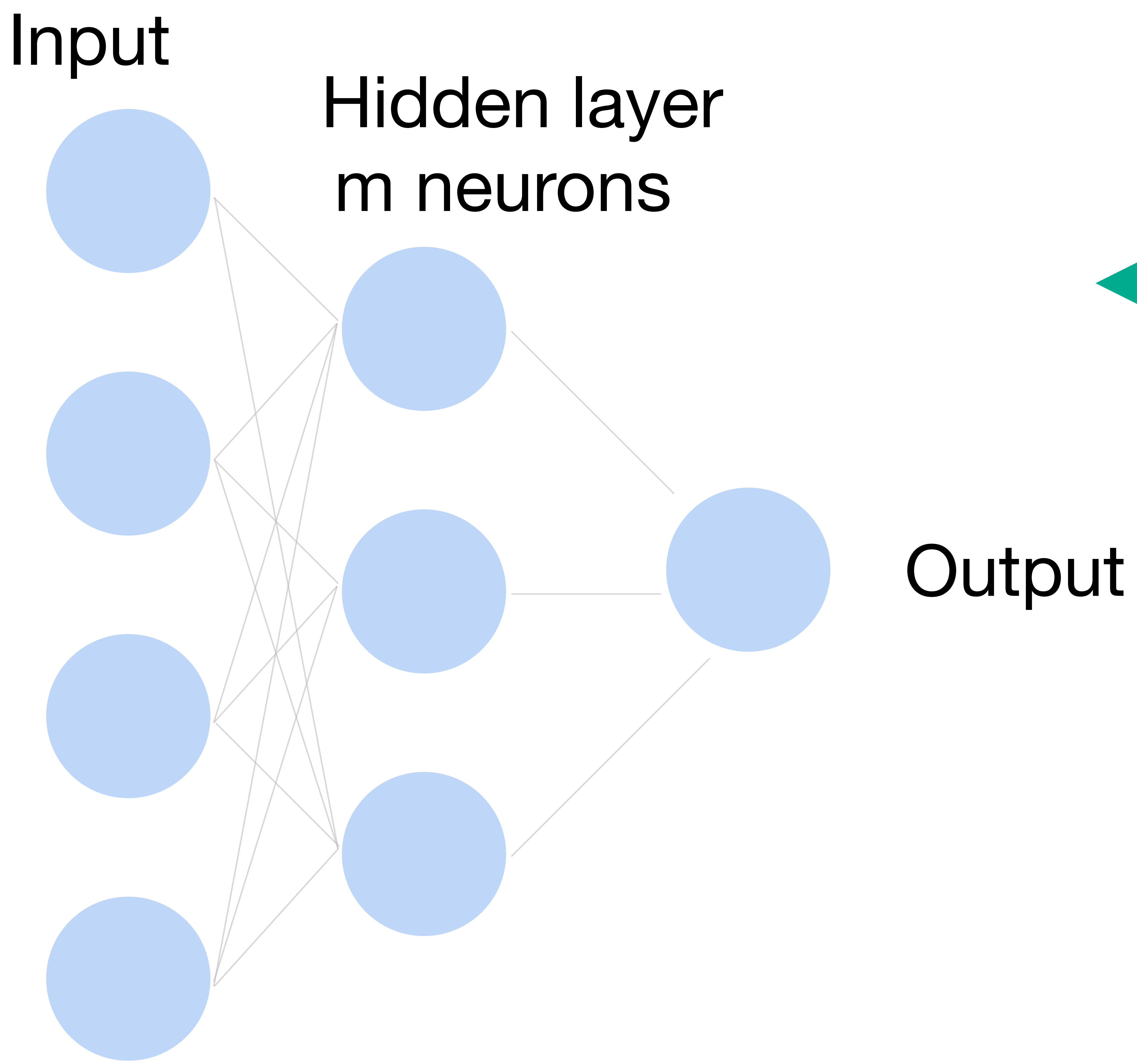
$$\mathbf{h} \in \mathbb{R}^m$$



Classify cats vs. dogs

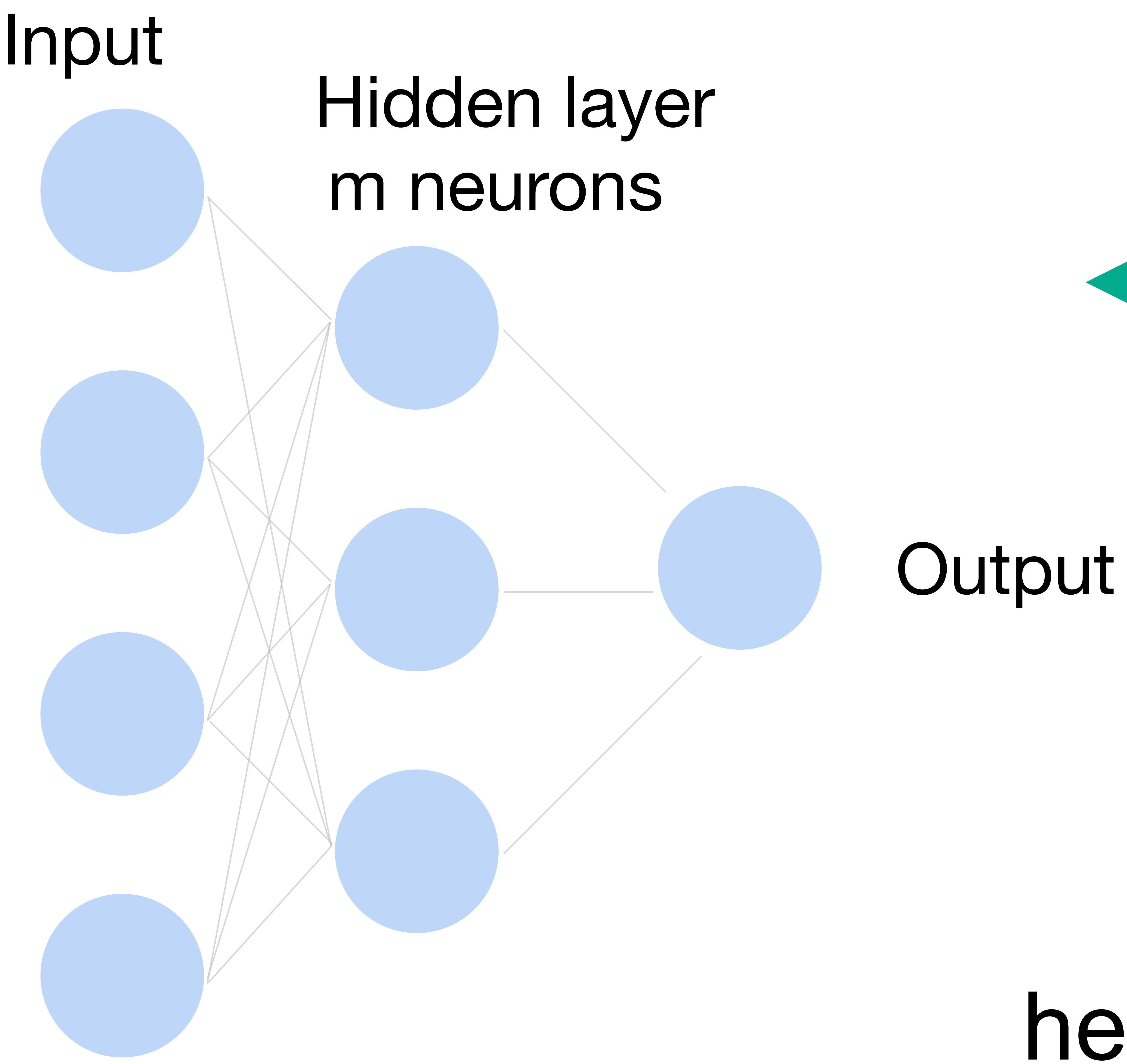


Multi-layer perceptron



Why do we need an a nonlinear activation?

Multi-layer perceptron



Why do we need an a nonlinear activation?

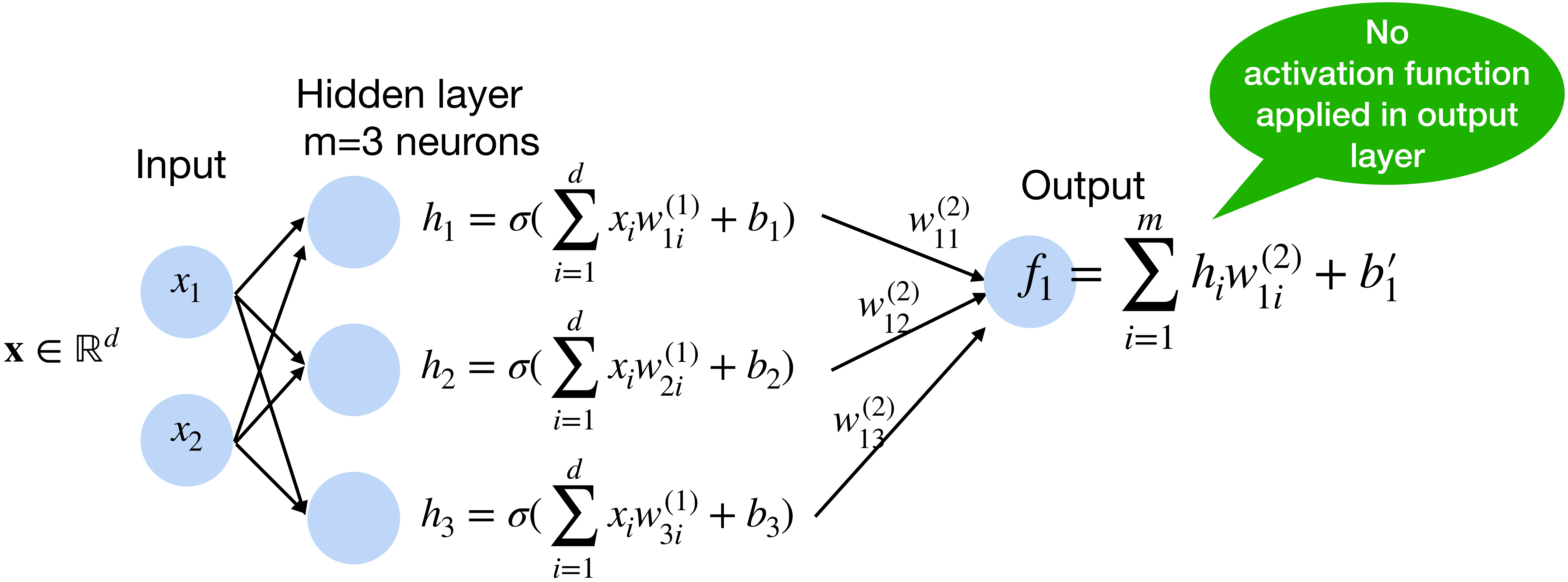
$$\mathbf{h} = \mathbf{W}\mathbf{x} + \mathbf{b}$$

$$f = \mathbf{w}_2^T \mathbf{h} + b_2$$

hence $f = \mathbf{w}_2^\top \mathbf{W}\mathbf{x} + b'$

Neural network for K-way classification

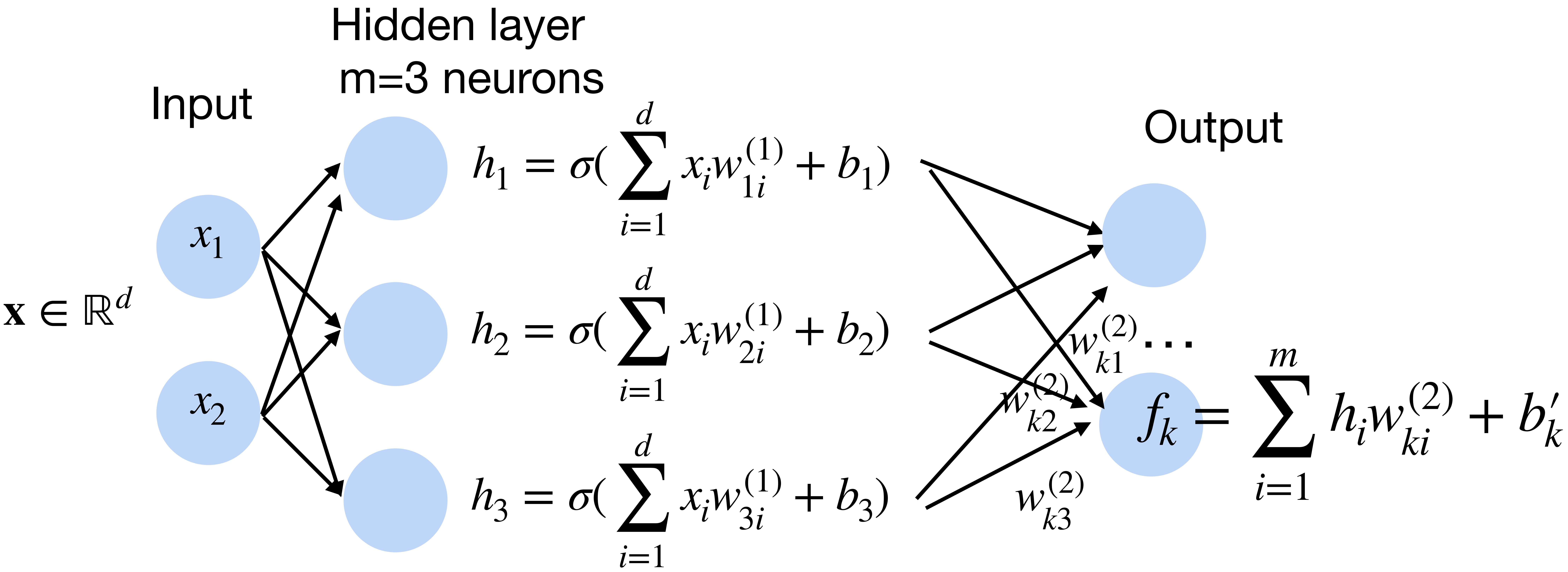
- K outputs in the final layer



Neural network for K-way classification

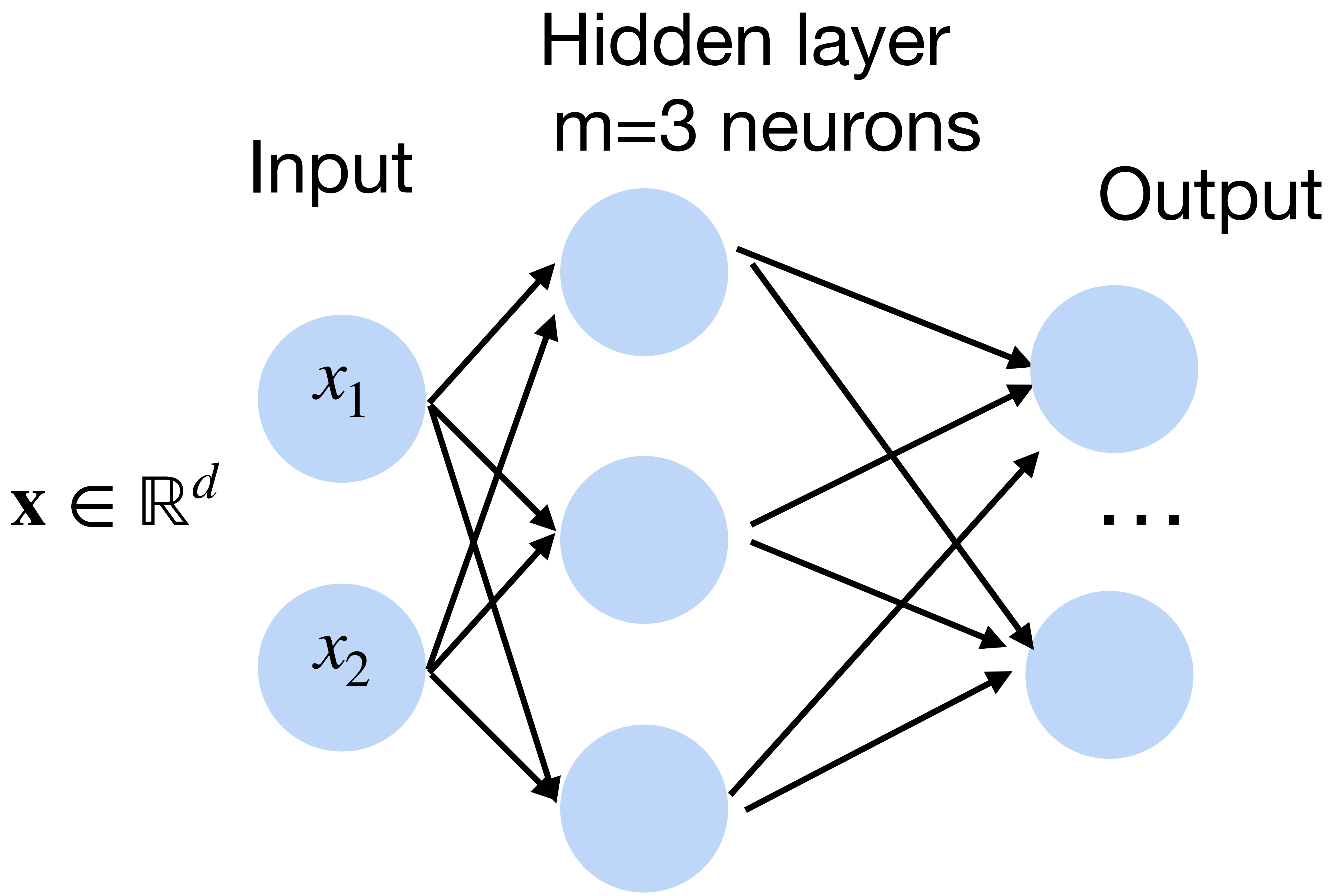
- K outputs units in the final layer

Multi-class classification (e.g., ImageNet with K=1000)



Softmax

Turns outputs f into probabilities (sum up to 1 across K classes)

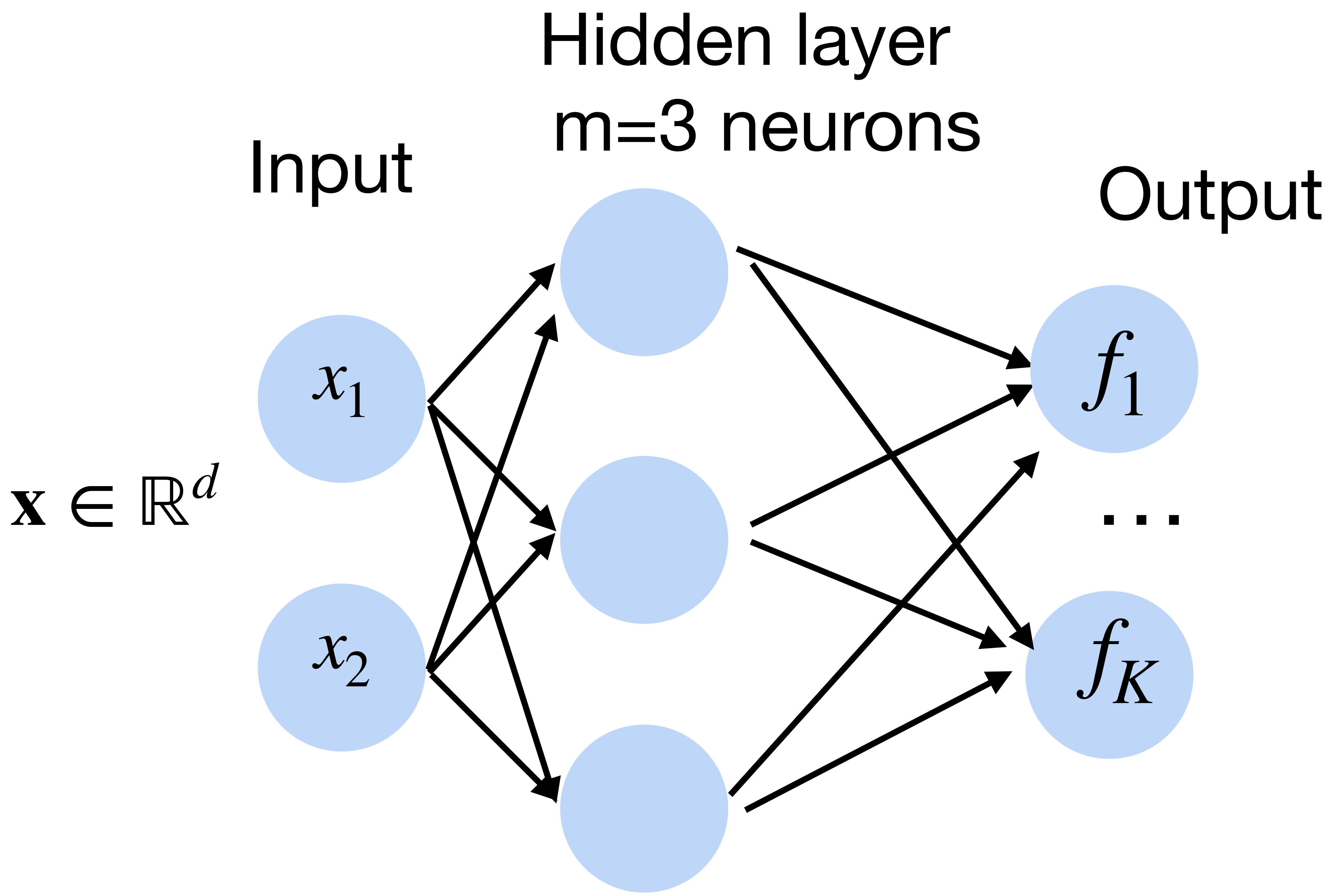


$$p(y \mid \mathbf{x}) = \text{softmax}(f)$$

$$= \frac{\exp f_y(x)}{\sum_{k=1}^{K} \exp f_k(x)}$$

Softmax

Turns outputs f into probabilities (sum up to 1 across K classes)



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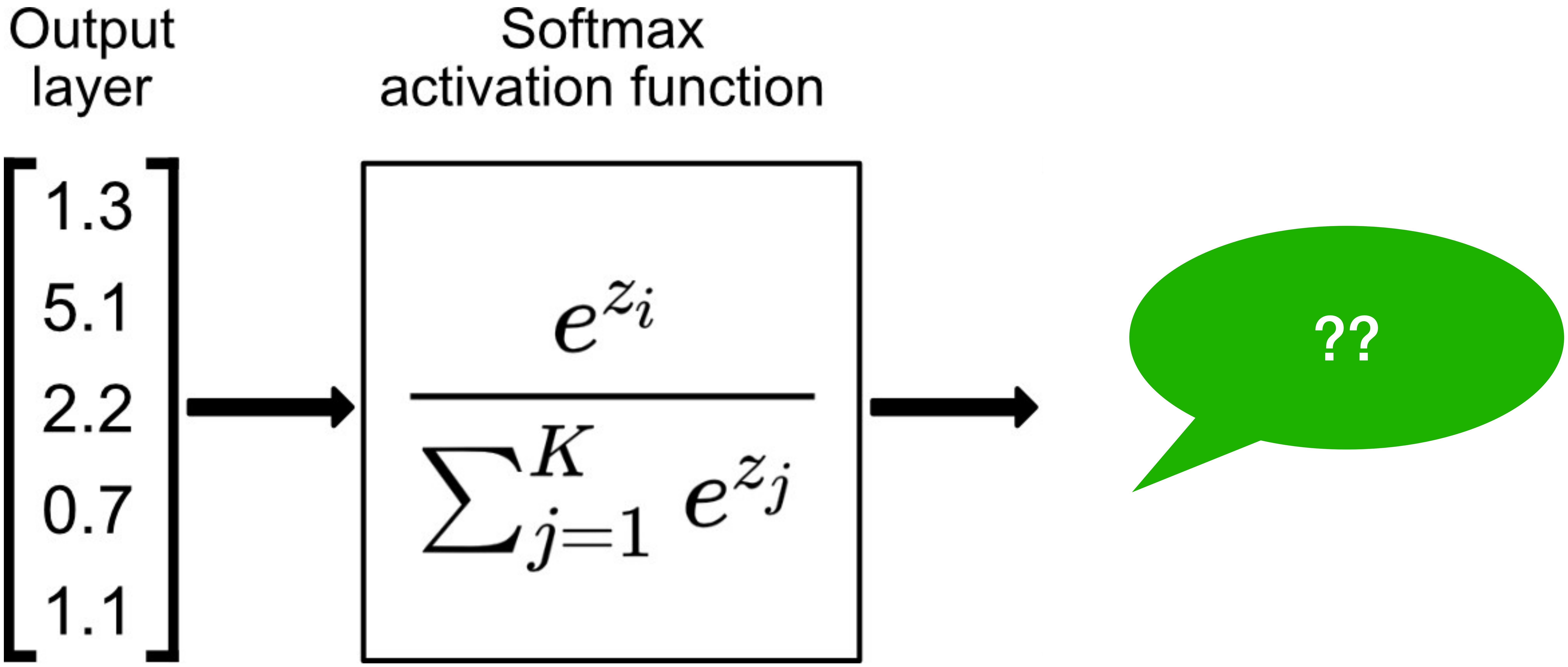

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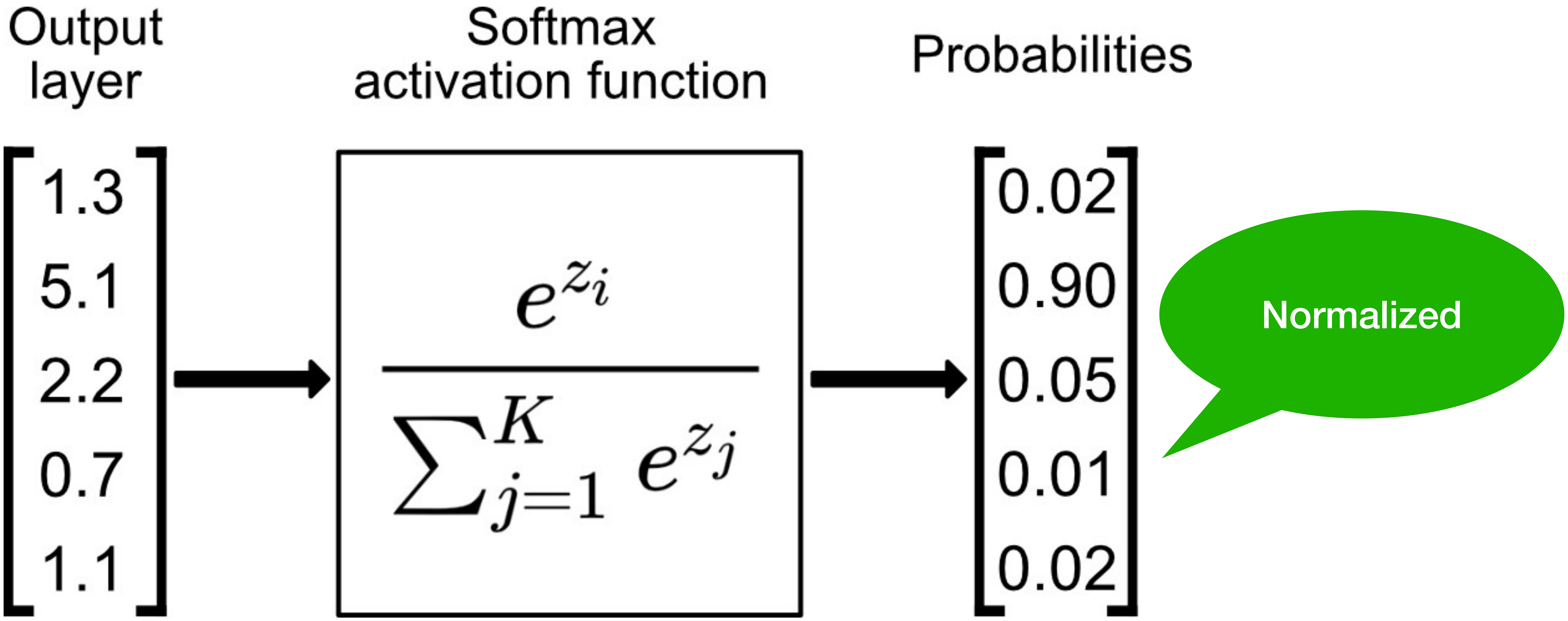
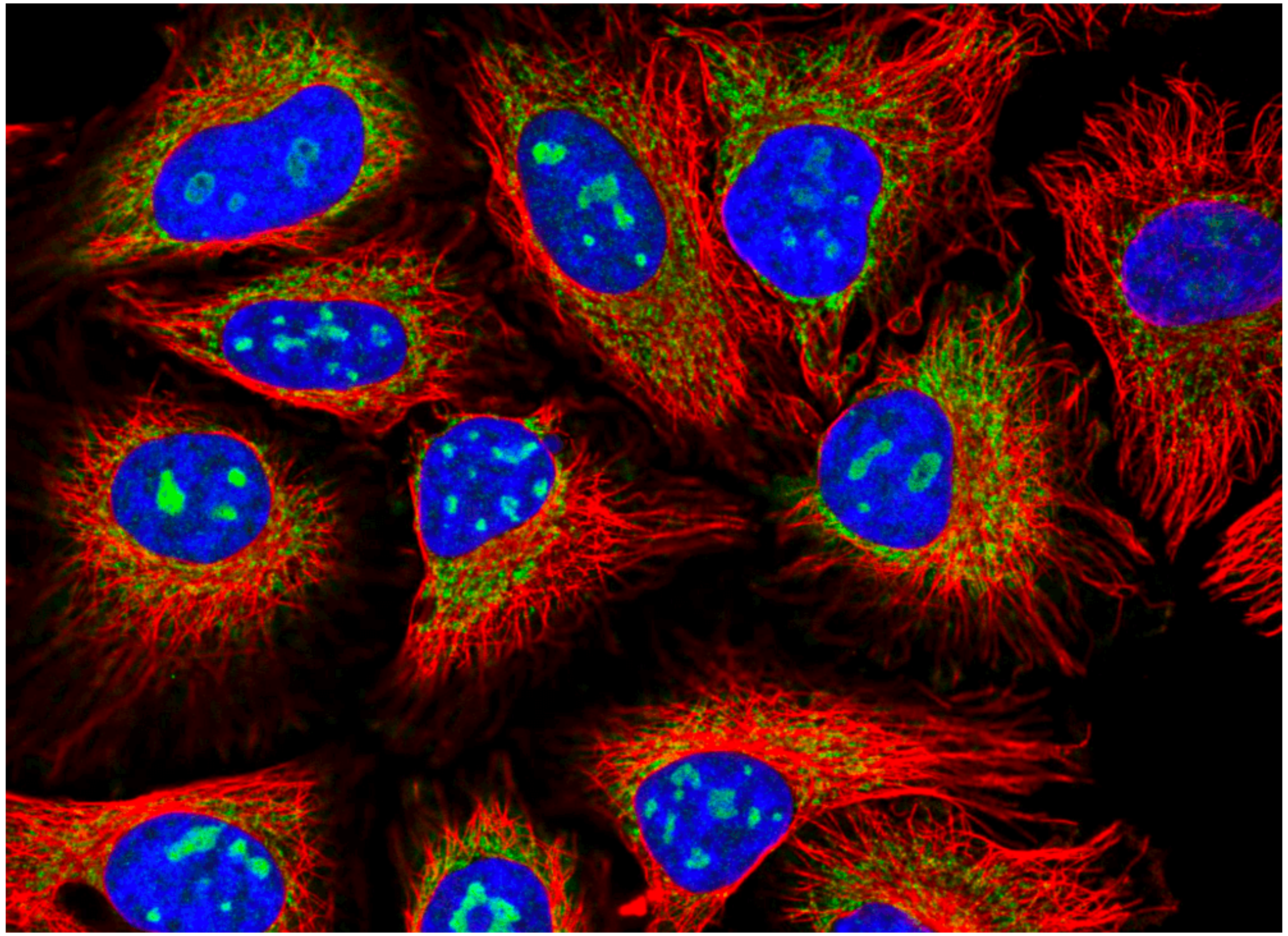


Classification Tasks at Kaggle

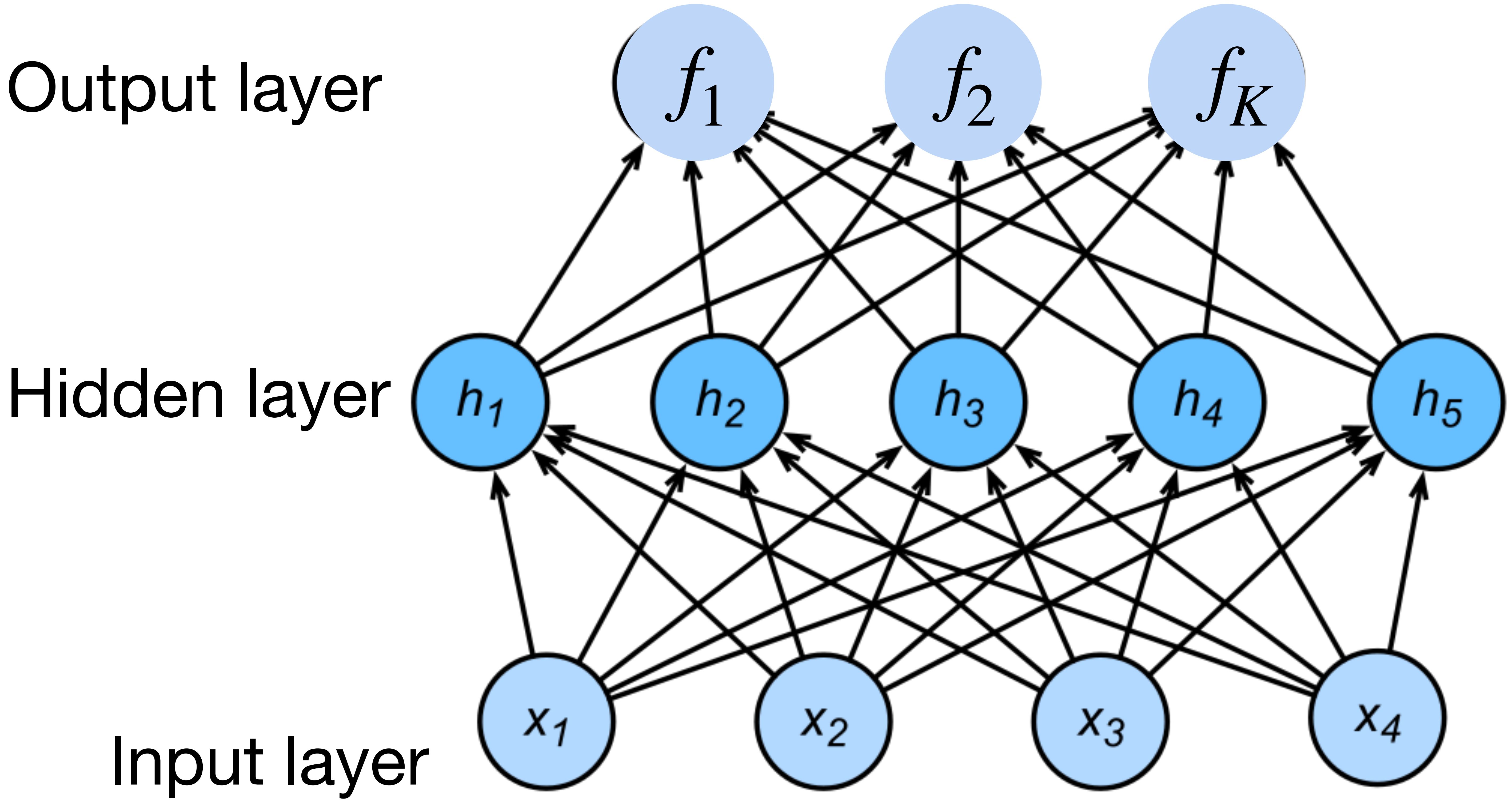
Classify human protein microscope images into 28 categories



```
0.  Nucleoplasm
1.  Nuclear membrane
2.  Nucleoli
3.  Nucleoli fibrillar
4.  Nuclear speckles
5.  Nuclear bodies
6.  Endoplasmic reticu
7.  Golgi apparatus
8.  Peroxisomes
9.  Endosomes
10.  Lysosomes
11.  Intermediate fila
12.  Actin filaments
13.  Focal adhesion si
14.  Microtubules
15.  Microtubule ends
16.  Cytokinetic bridg
```

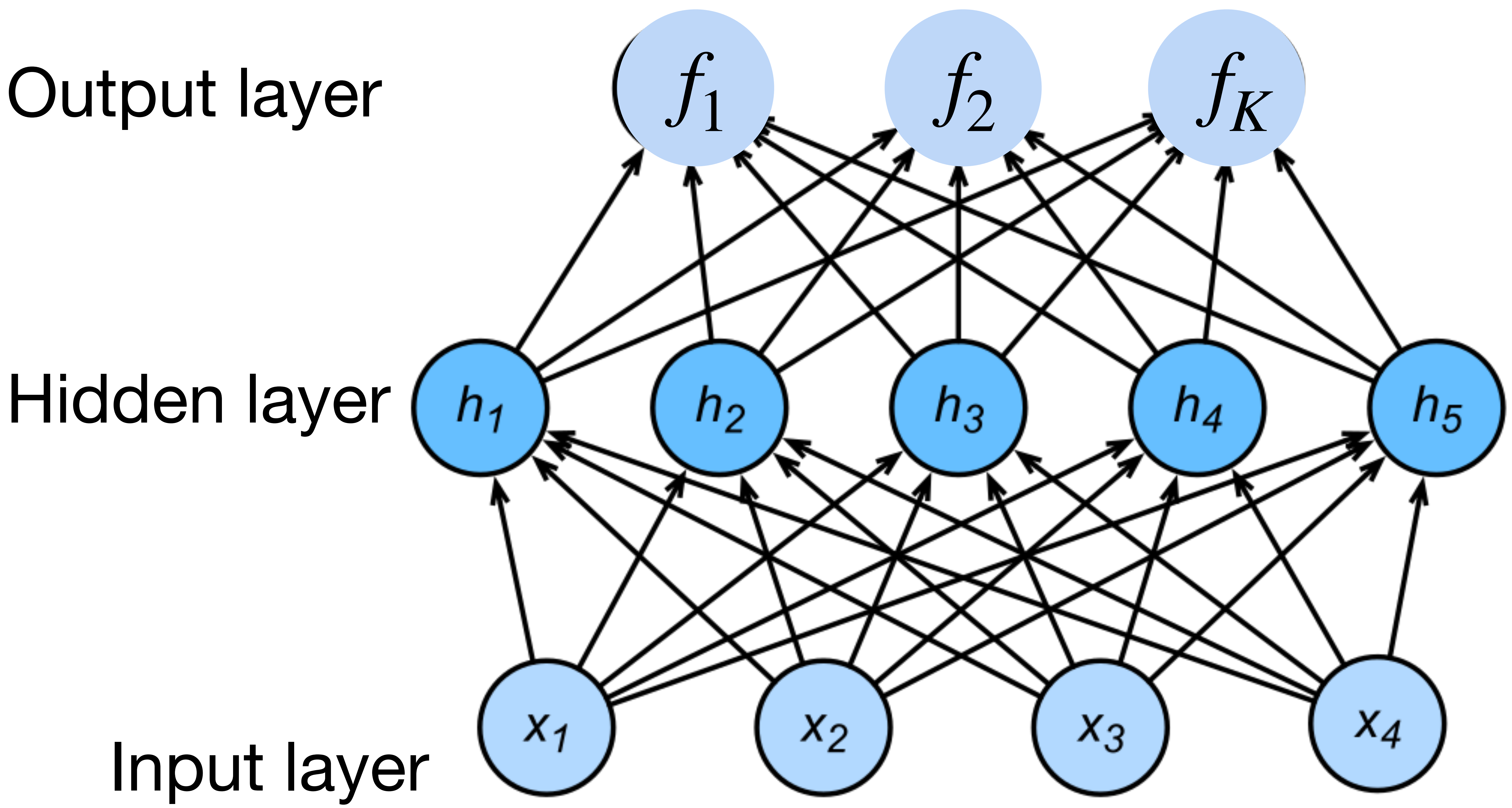
https://www.kaggle.com/c/human-protein-atlas-image-classification

More complicated neural networks



More complicated neural networks

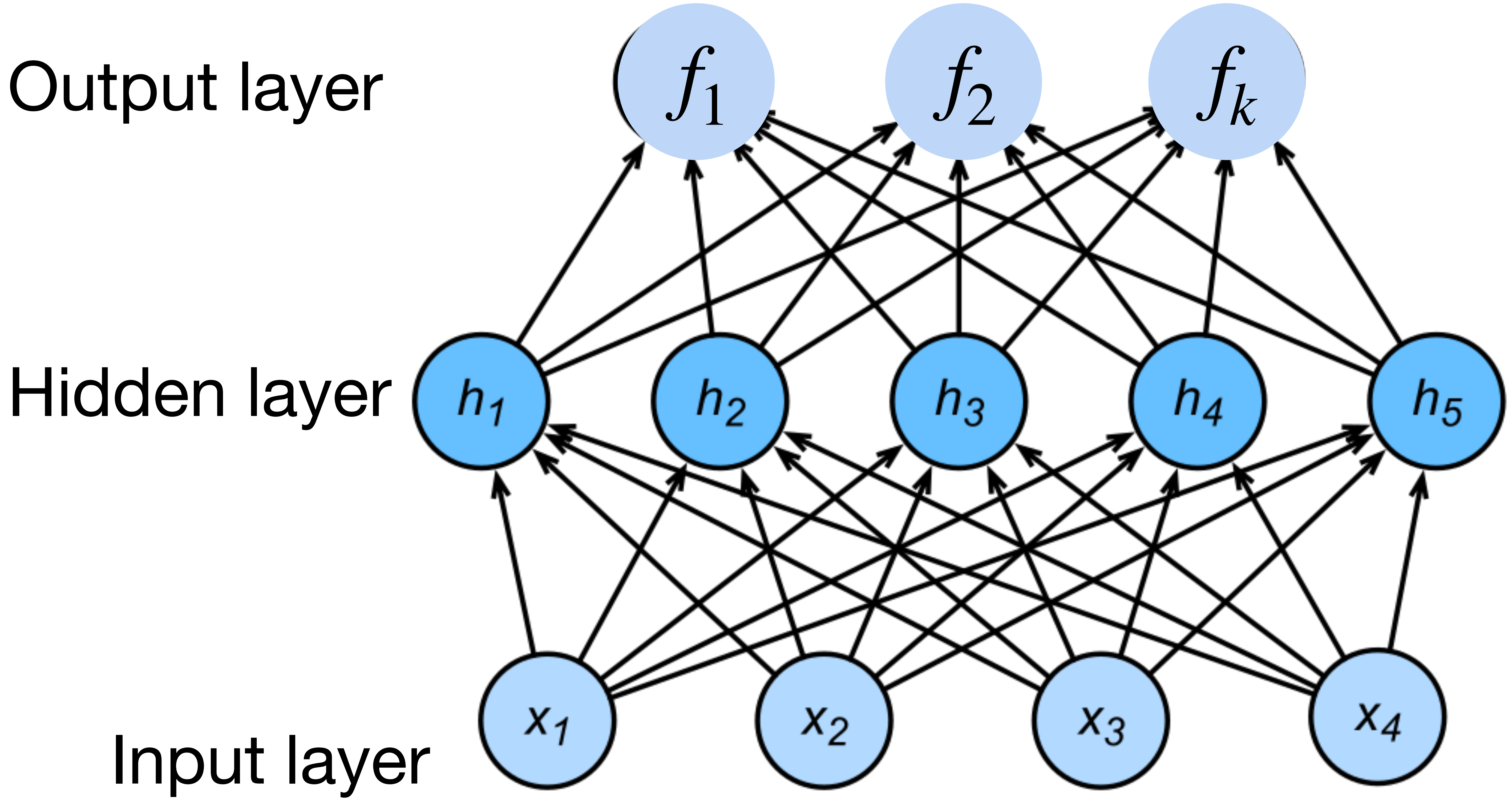
$$p_1, p_2, \ldots, p_K = \text{softmax}(f_1, f_2, \ldots, f_K)$$



More complicated neural networks

- Input $\mathbf{x} \in \mathbb{R}^d$
- Hidden $\mathbf{W}^{(1)} \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m$

$$\mathbf{h} = \sigma(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b})$$
$$\mathbf{f} = \sigma(\mathbf{W}^{(2)}\mathbf{h} + \mathbf{b}^{(2)})$$
$$\mathbf{p} = \text{softmax}(\mathbf{f})$$



More complicated neural networks

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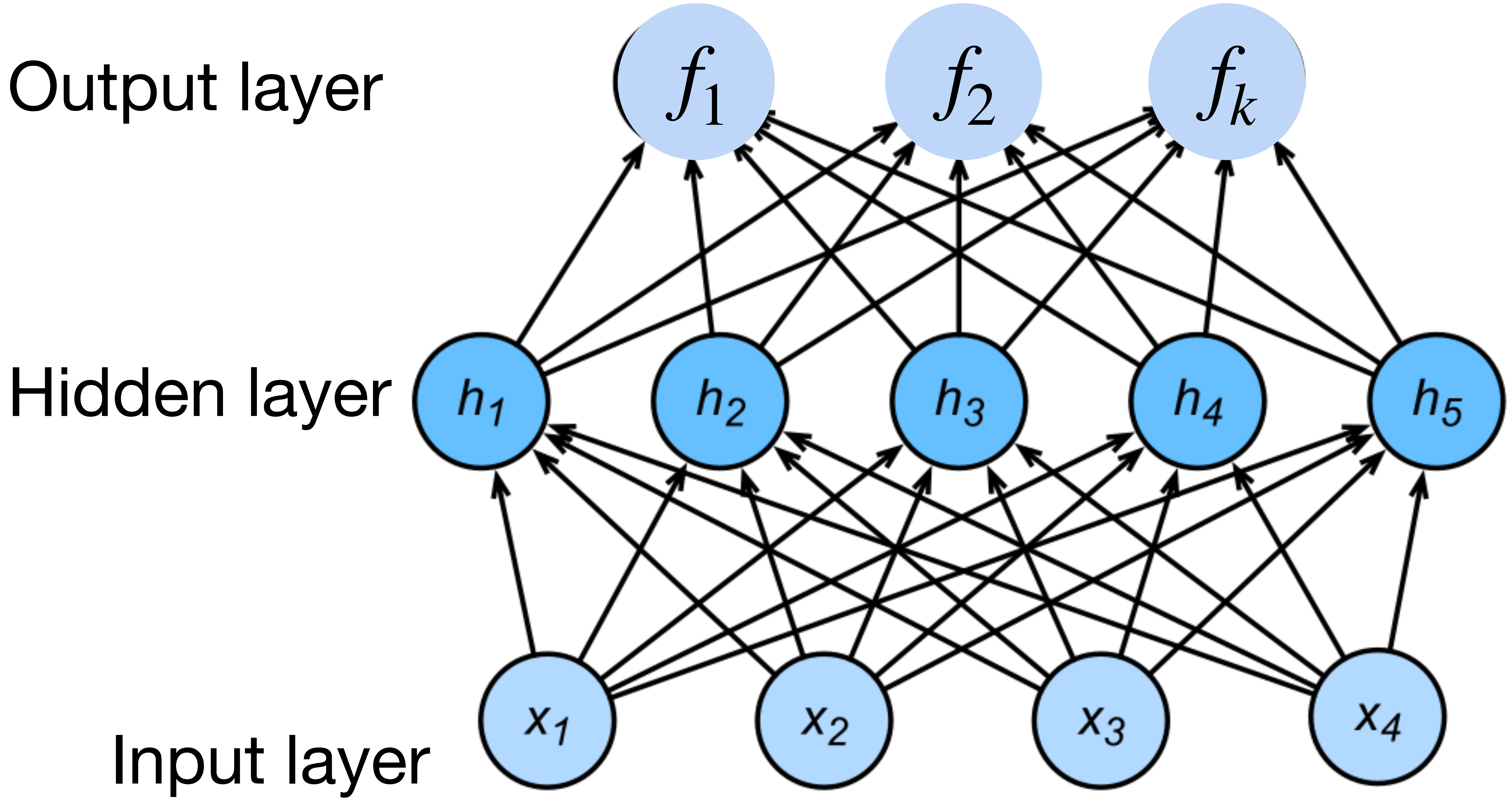
$$p_1, p_2, \ldots, p_K = \text{softmax}(f_1, f_2, \ldots, f_K)$$

$$\mathbf{h} = \sigma(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b})$$
$$\mathbf{f} = \sigma(\mathbf{W}^{(2)}\mathbf{h} + \mathbf{b}^{(2)})$$
$$\mathbf{p} = \text{softmax}(\mathbf{f})$$



More complicated neural networks: multiple hidden layers

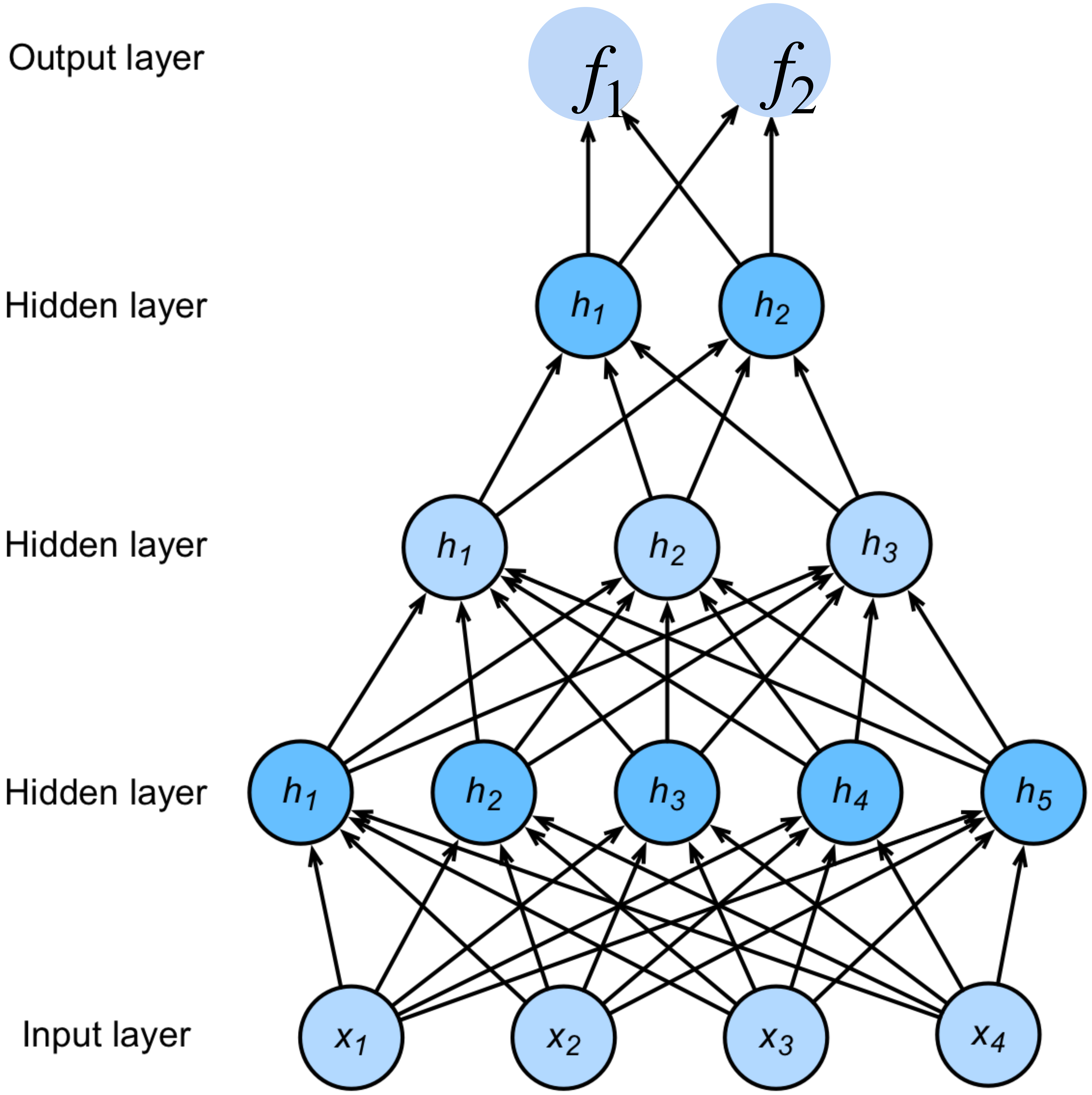
$$\mathbf{h}_1 = \sigma(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1)$$

$$\mathbf{h}_2 = \sigma(\mathbf{W}_2\mathbf{h}_1 + \mathbf{b}_2)$$

$$\mathbf{h}_3 = \sigma(\mathbf{W}_3\mathbf{h}_2 + \mathbf{b}_3)$$

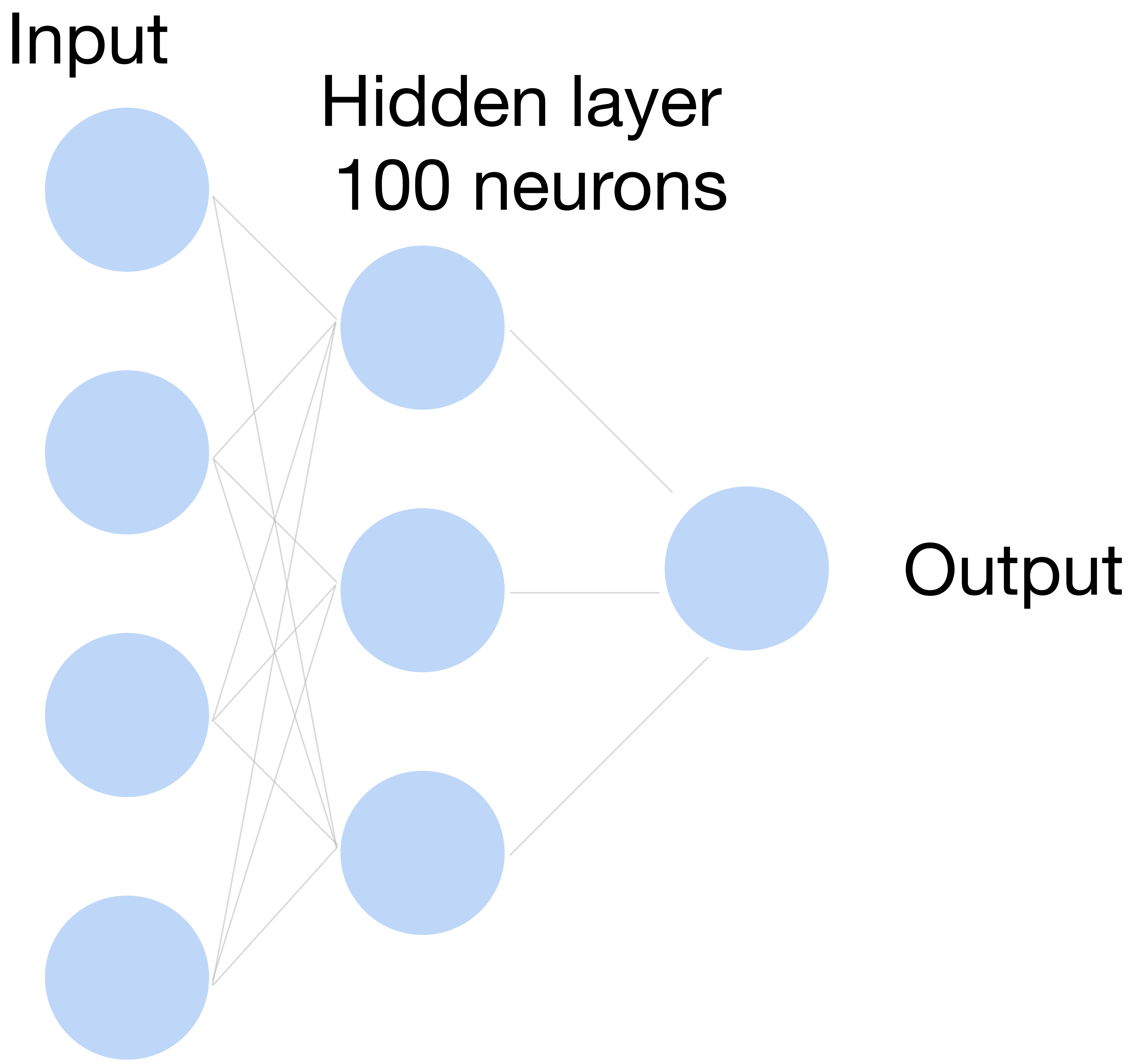
$$\mathbf{f} = \mathbf{W}_4\mathbf{h}_3 + \mathbf{b}_4$$

$$\mathbf{p} = \text{softmax}(\mathbf{f})$$



How to train a neural network?

Classify cats vs. dogs



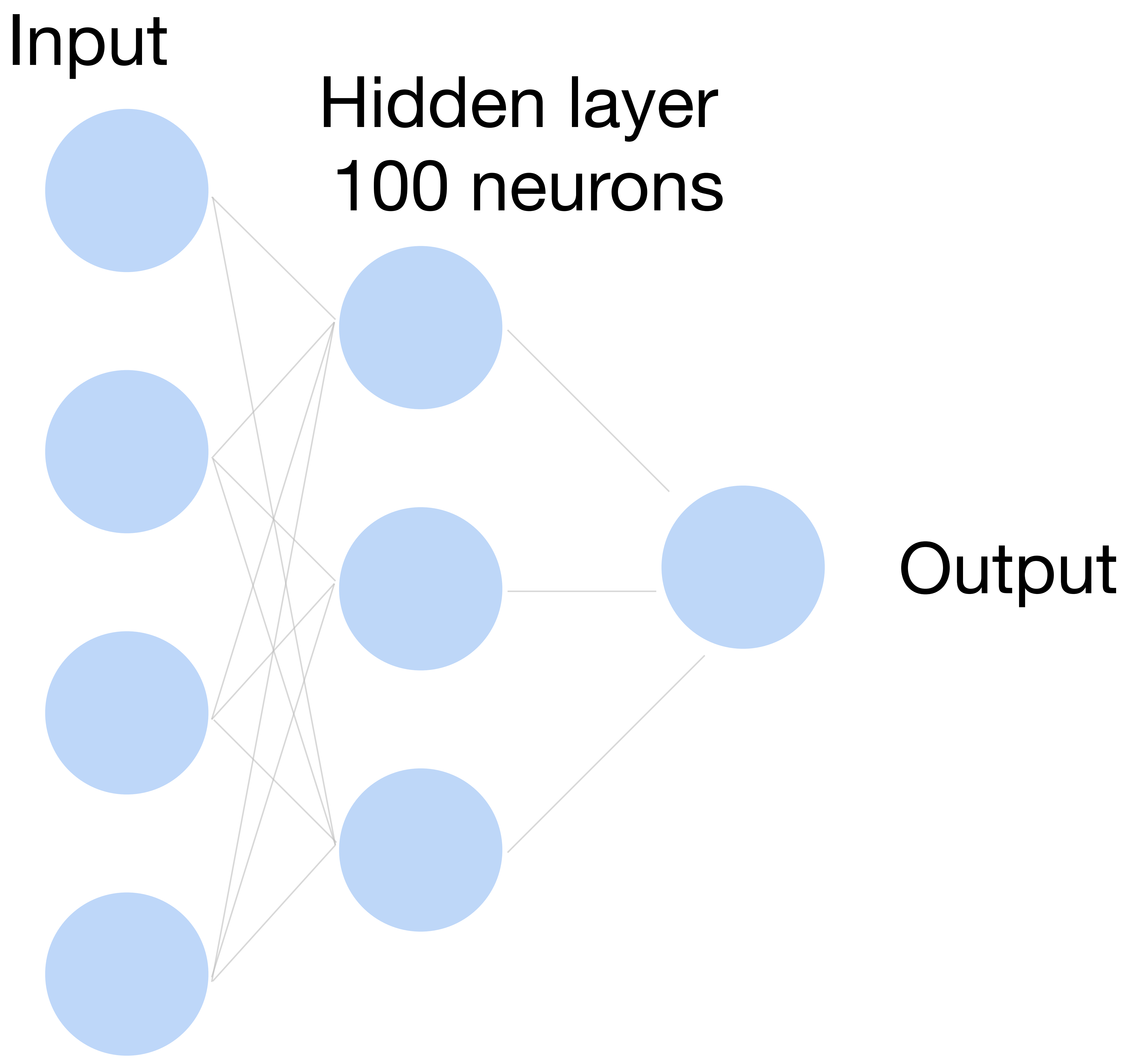
How to train a neural network? Binary classification

$\mathbf{x} \in \mathbb{R}^d$ One training data point in the training set D

$\hat{y}$ Model output for example **x**

(This is a function of all weights W)

y Ground truth label for example **x**

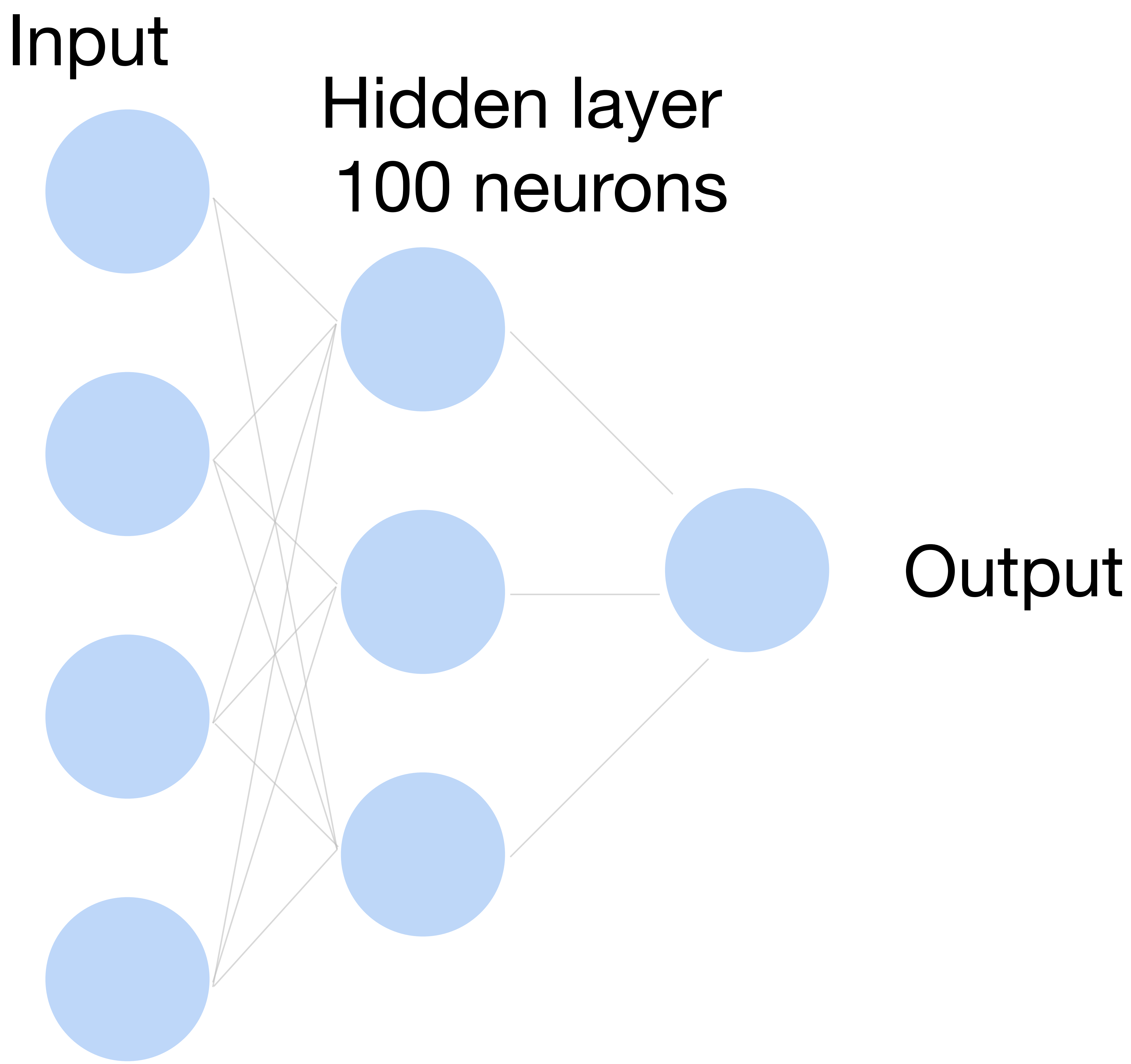


Learning by matching the output to the label

We want $\hat{y} \rightarrow 1$ **when** $y = 1$,
and $\hat{y} \rightarrow 0$ **when** $y = 0$

How to train a neural network? Binary classification

Loss function: $\frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \ell(\mathbf{x}, y)$

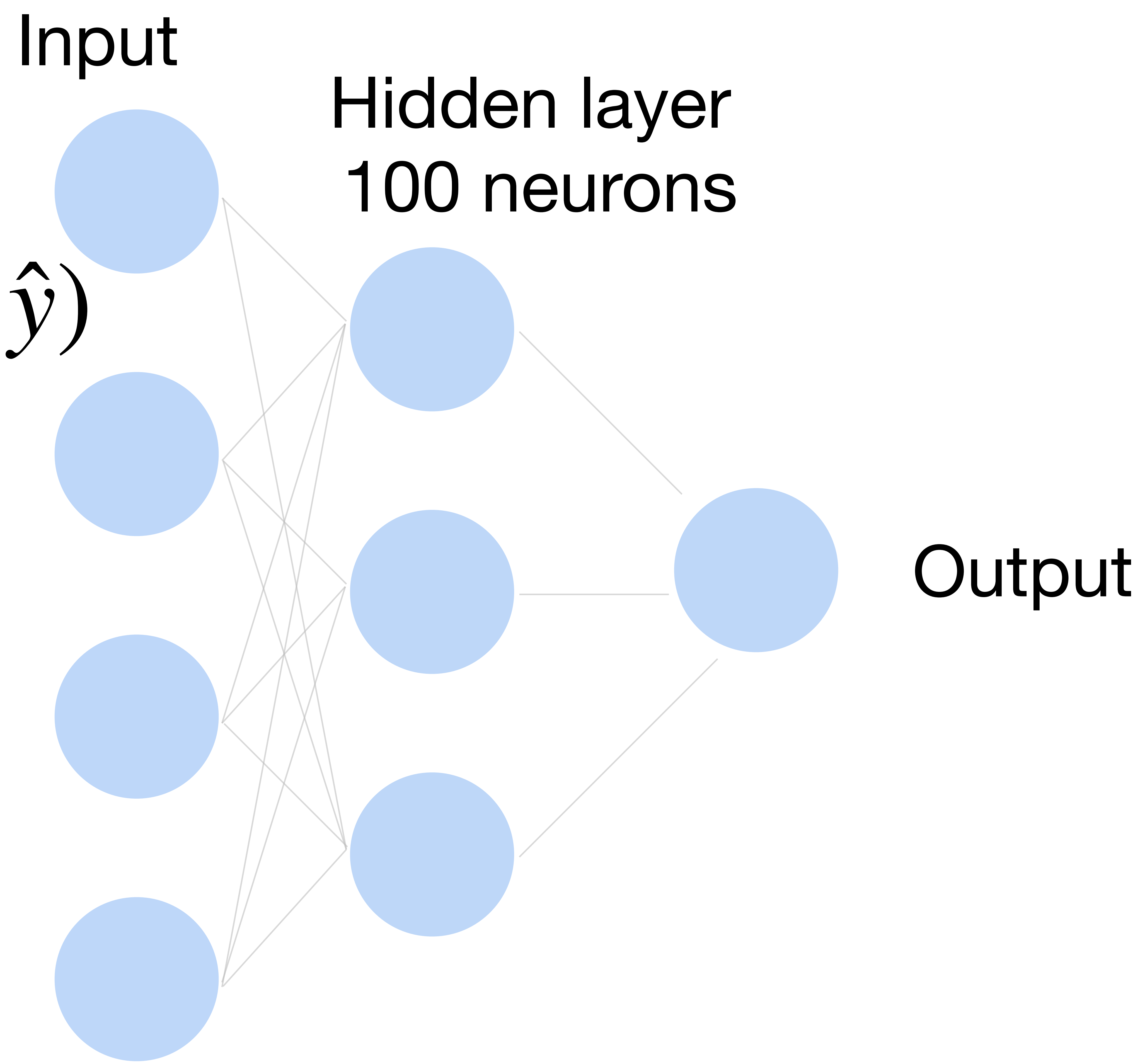


How to train a neural network? Binary classification

Loss function: $\frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \ell(\mathbf{x}, y)$

Per-sample loss:

$$\ell(\mathbf{x}, y) = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$$



How to train a neural network? Binary classification

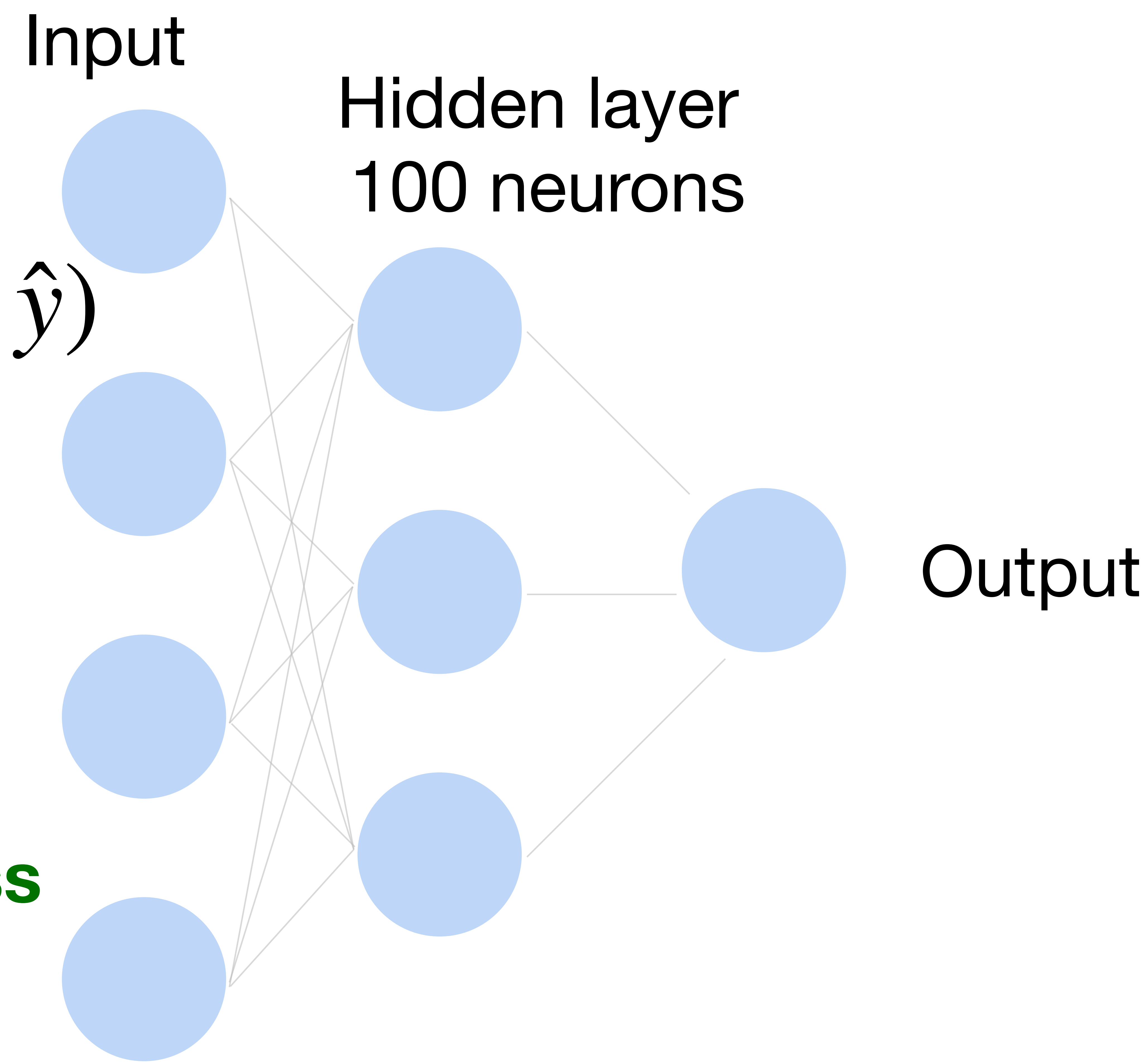
Loss function: $\frac{1}{|D|}\sum_{(\mathbf{x},y)\in D}\ell(\mathbf{x},y)$

Per-sample loss:

$$\ell(\mathbf{x},y) = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$$

Negative log likelihood

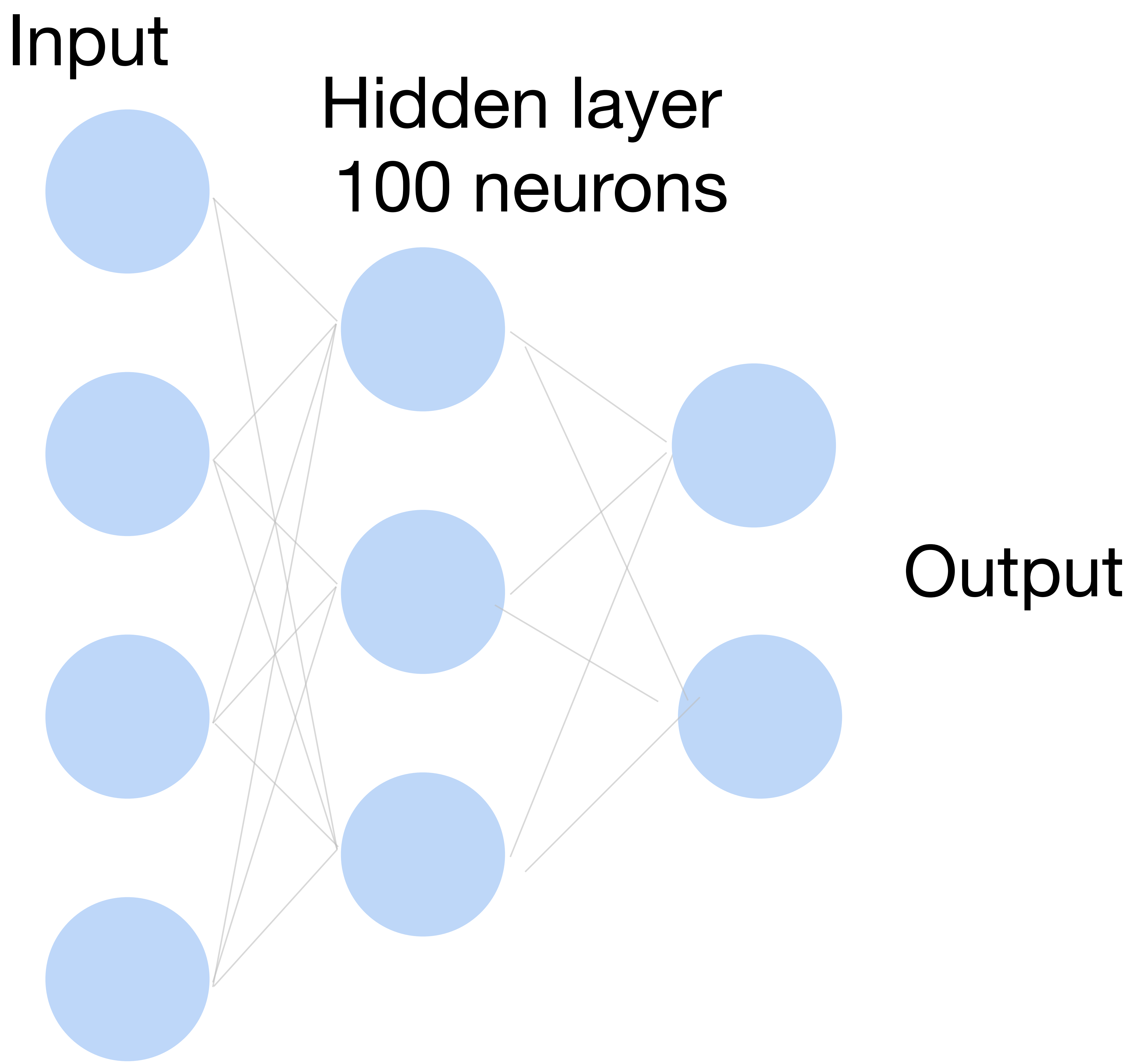
Also known as binary cross-entropy loss



How to train a neural network? Multiclass

Loss function: $$\frac{1}{|D|}\sum_{(\mathbf{x},y)\in D}\ell(\mathbf{x},y)$$

Y y



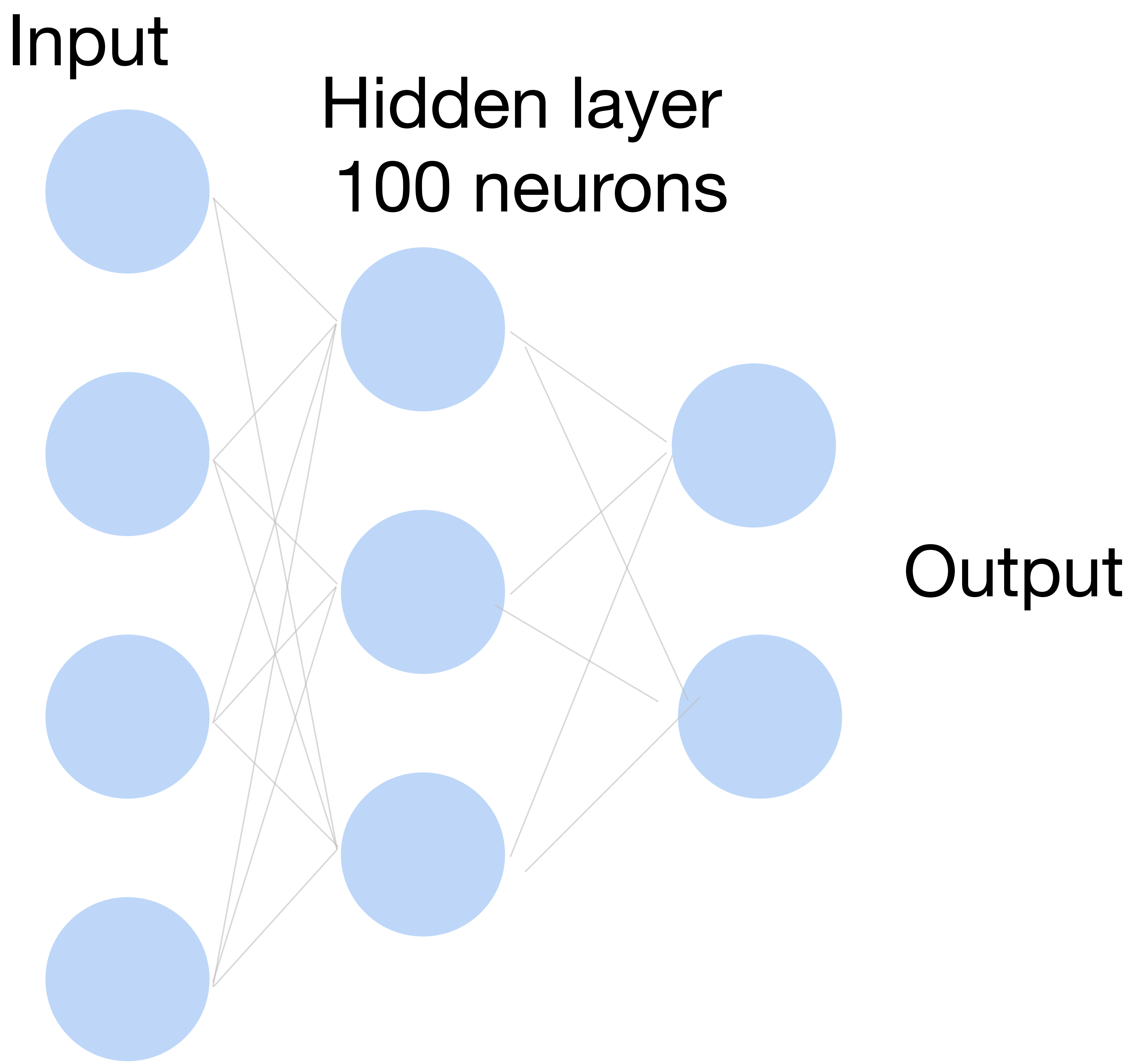
How to train a neural network? Multiclass

Loss function: $$\frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \ell(\mathbf{x}, y)$$

Per-sample loss:

$$\ell(\mathbf{x}, y) = \sum_{k=1}^{K} - Y_k \log p_k = - \log p_y$$

where Y is one-hot encoding of y



How to train a neural network? Multiclass

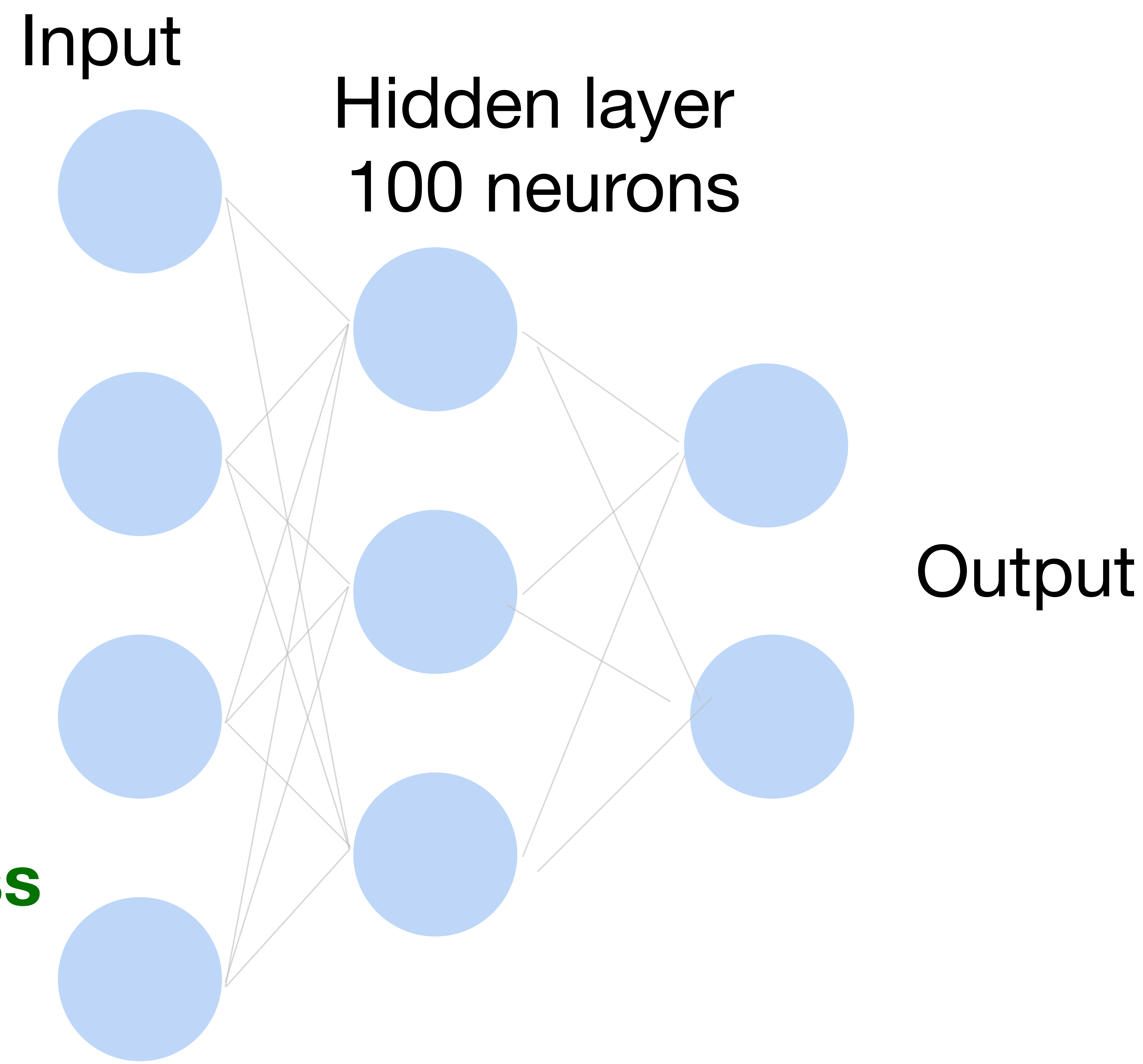
Loss function: $$\frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \ell(\mathbf{x}, y)$$

Per-sample loss:

$$\ell(\mathbf{x}, y) = \sum_{k=1}^{K} -Y_k \log p_k = -\log p_y$$

where Y is one-hot encoding of y

Also known as cross-entropy loss or softmax loss

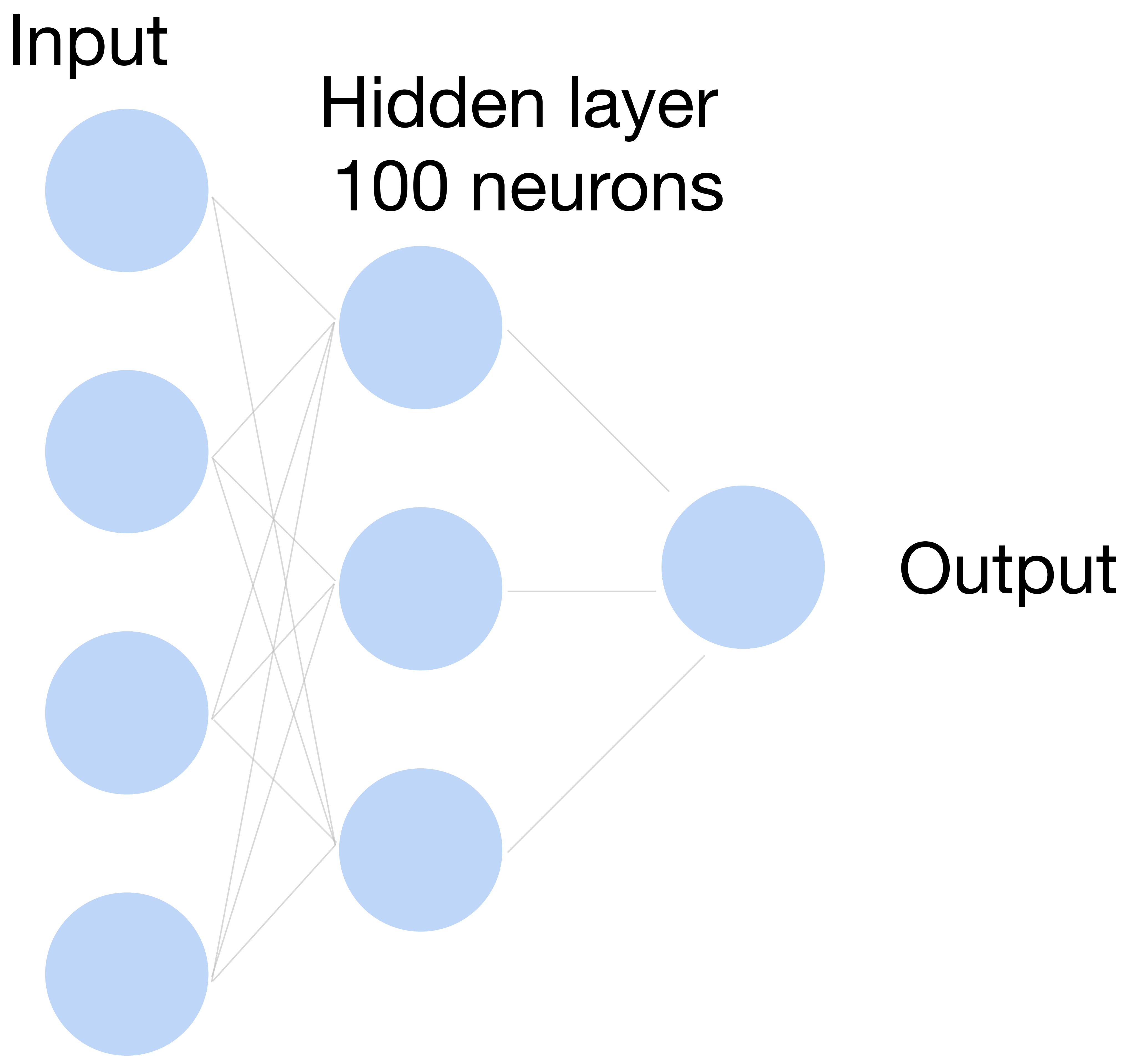


How to train a neural network?

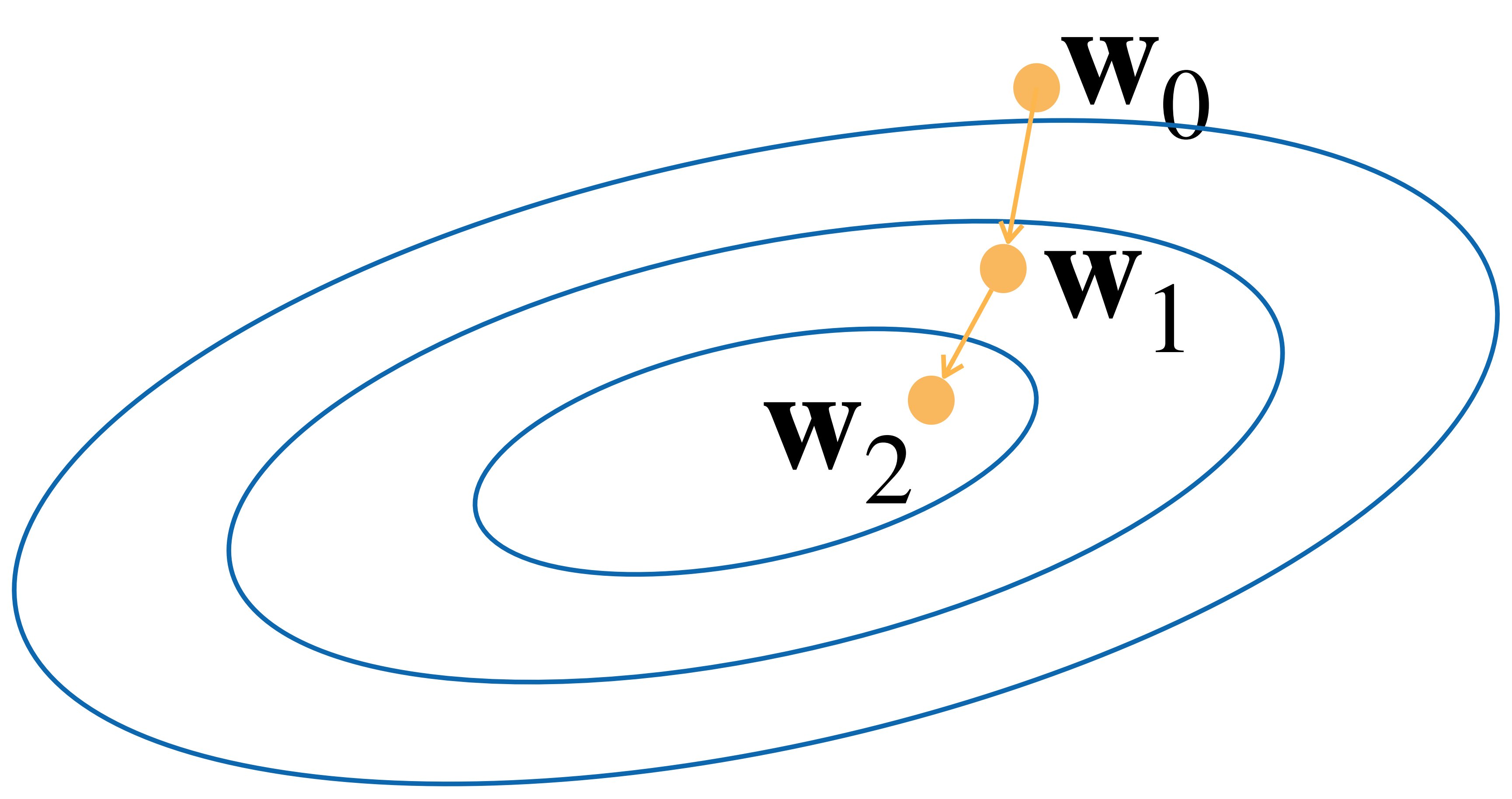
Update the weights W to minimize the loss function

$$L = \frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \ell(\mathbf{x}, y)$$

Use gradient descent!



Gradient Descent



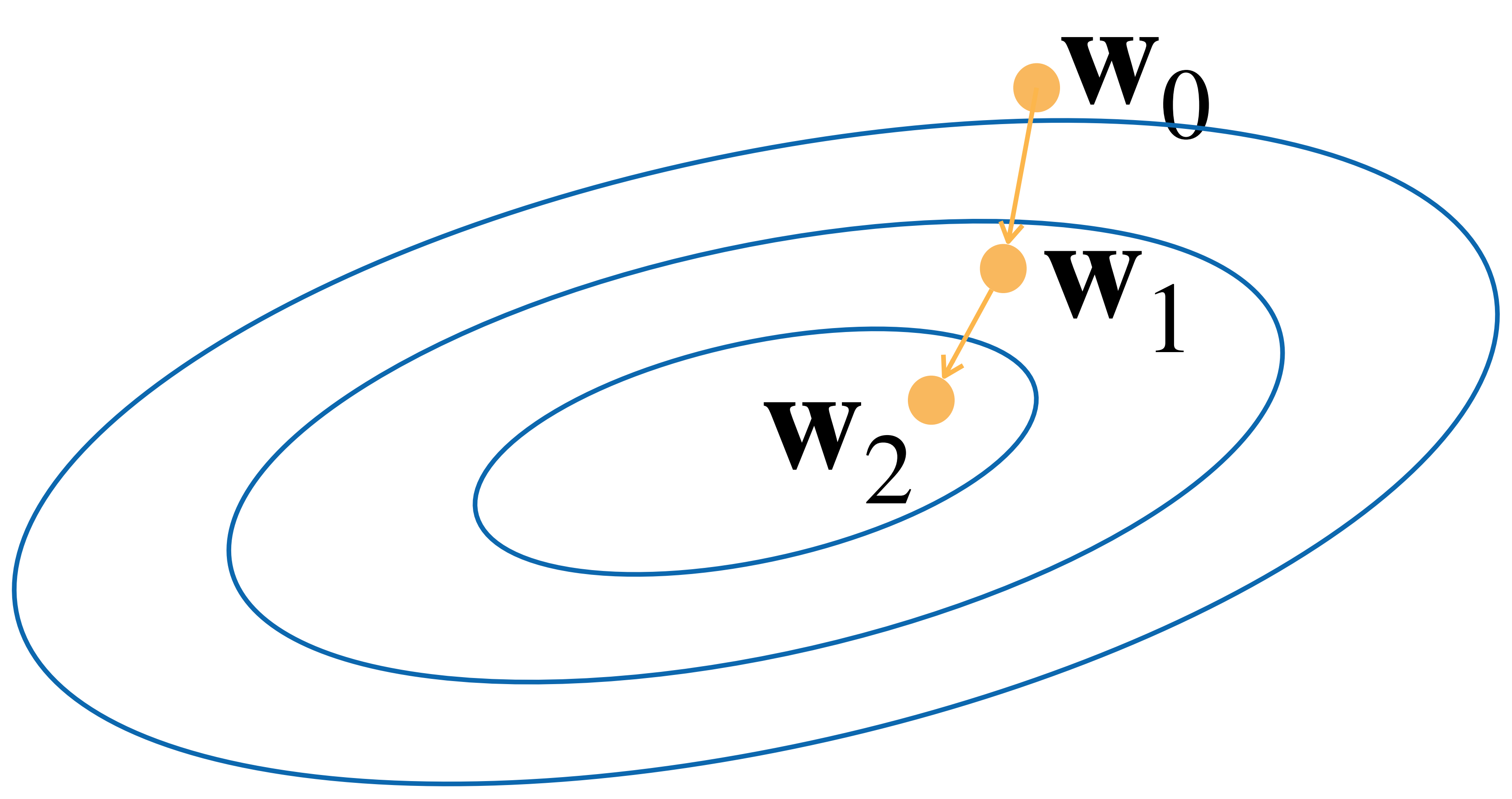
- Choose a learning rate $\alpha > 0$
- Initialize the model parameters w_0
- For t =1,2,…
    - Update parameters:

$$\mathbf{w}_t = \mathbf{w}_{t-1} - \alpha \frac{\partial L}{\partial \mathbf{w}_{t-1}}$$

$$= \mathbf{w}_{t-1} - \alpha \frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \frac{\partial \ell(\mathbf{x}, y)}{\partial \mathbf{w}_{t-1}}$$

- Repeat until converges

Gradient Descent



- Choose a learning rate $\alpha > 0$
- Initialize the model parameters w_0
- For t =1,2,…
  - Update parameters:

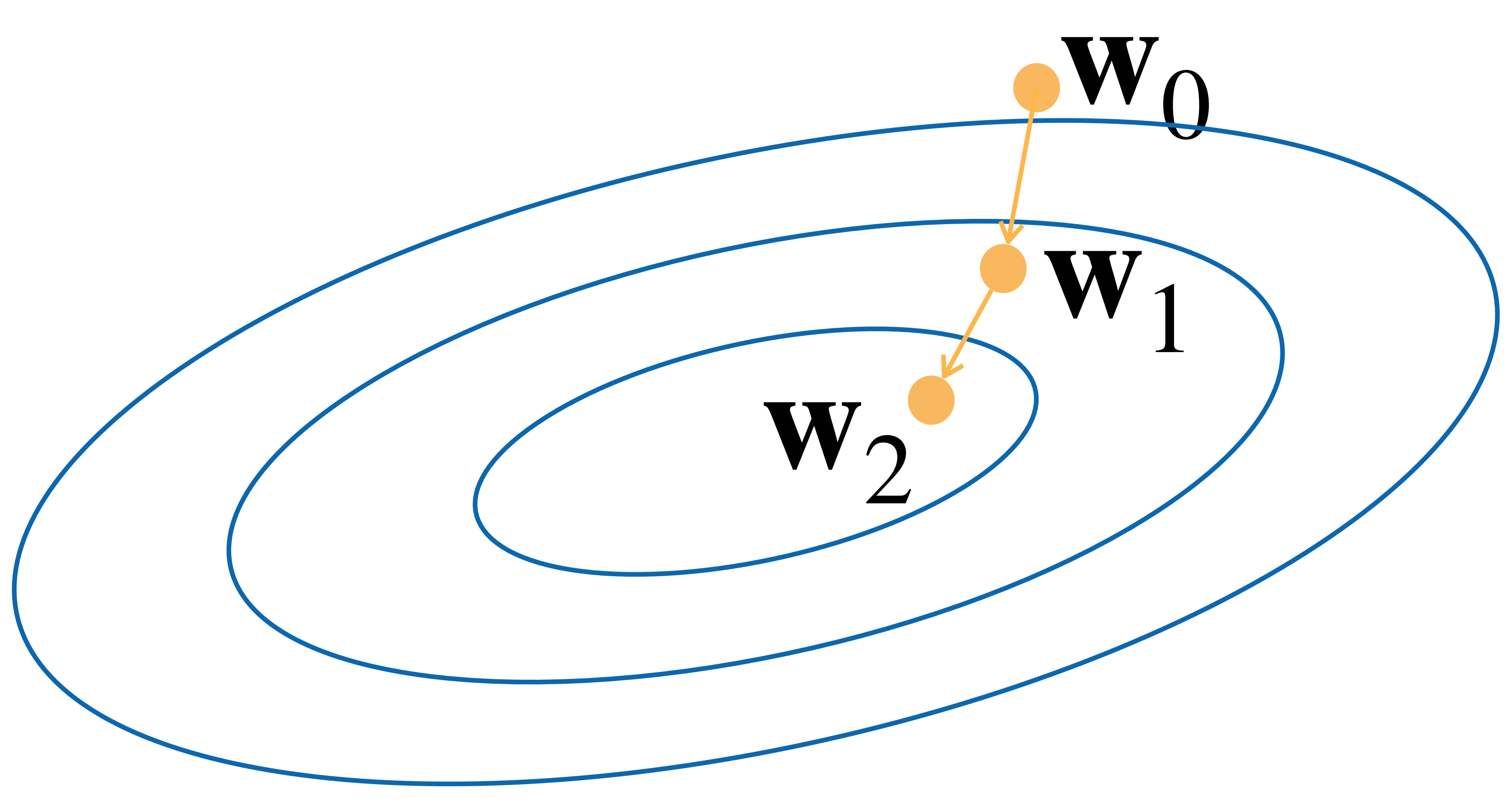
$$\mathbf{w}_t = \mathbf{w}_{t-1} - \alpha \frac{\partial L}{\partial \mathbf{w}_{t-1}}$$

$$= \mathbf{w}_{t-1} - \alpha \frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \frac{\partial \ell(\mathbf{x}, y)}{\partial \mathbf{w}_{t-1}}$$

D can
be very large.
Expensive

- Repeat until converges

Gradient Descent



- Choose a learning rate $\alpha > 0$
- Initialize the model parameters w_0
- For t =1,2,…
    - Update parameters:

$$\mathbf{w}_t = \mathbf{w}_{t-1} - \alpha \frac{\partial L}{\partial \mathbf{w}_{t-1}}$$

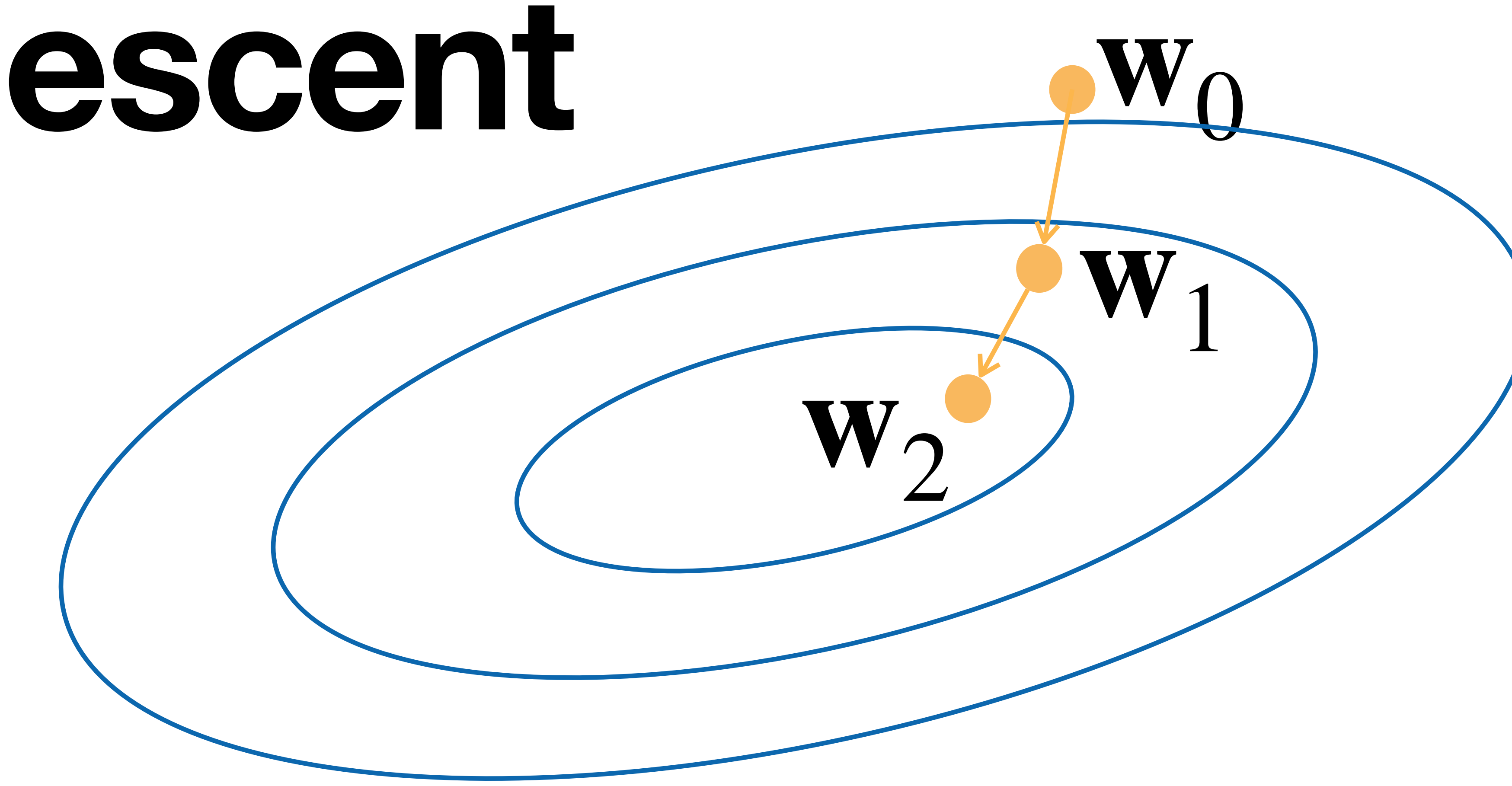
$$= \mathbf{w}_{t-1} - \alpha \frac{1}{|D|} \sum_{(\mathbf{x},y)\in D} \frac{\partial \ell(\mathbf{x}, y)}{\partial \mathbf{w}_{t-1}}$$

D can be very large. Expensive

The gradient w.r.t. all parameters is obtained by concatenating the partial derivatives w.r.t. each parameter

- Repeat until converges

Minibatch Stochastic Gradient Descent

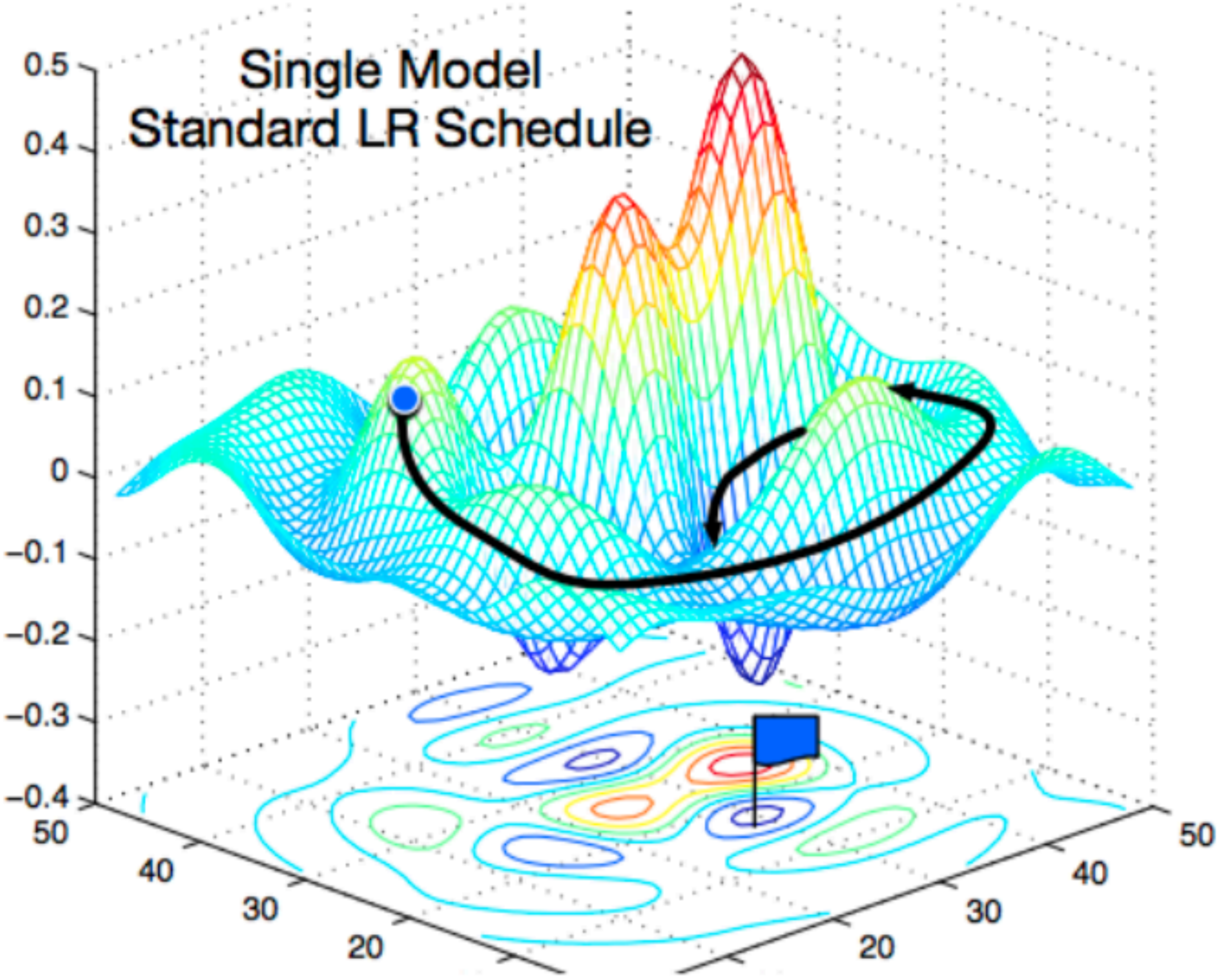


- Choose a learning rate $\alpha > 0$
- Initialize the model parameters w_0
- For t =1,2,…
    - **Randomly sample a subset (mini-batch)** $B \subset D$
      Update parameters:

$$\mathbf{w}_t = \mathbf{w}_{t-1} - \alpha \frac{1}{|B|} \sum_{(\mathbf{x},y) \in B} \frac{\partial \ell(\mathbf{x}, y)}{\partial \mathbf{w}_{t-1}}$$

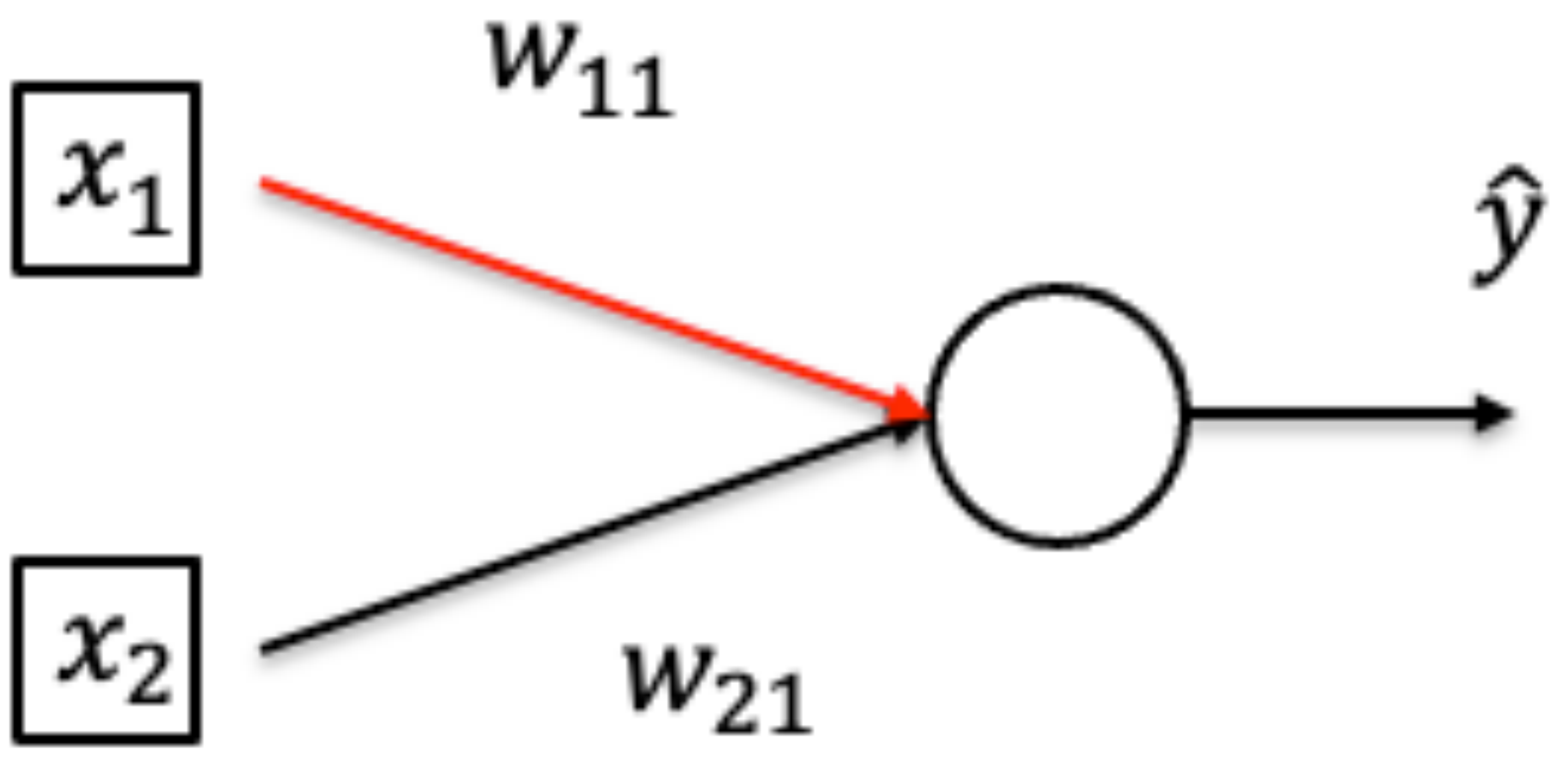
- Repeat until converges

Non-convex Optimization



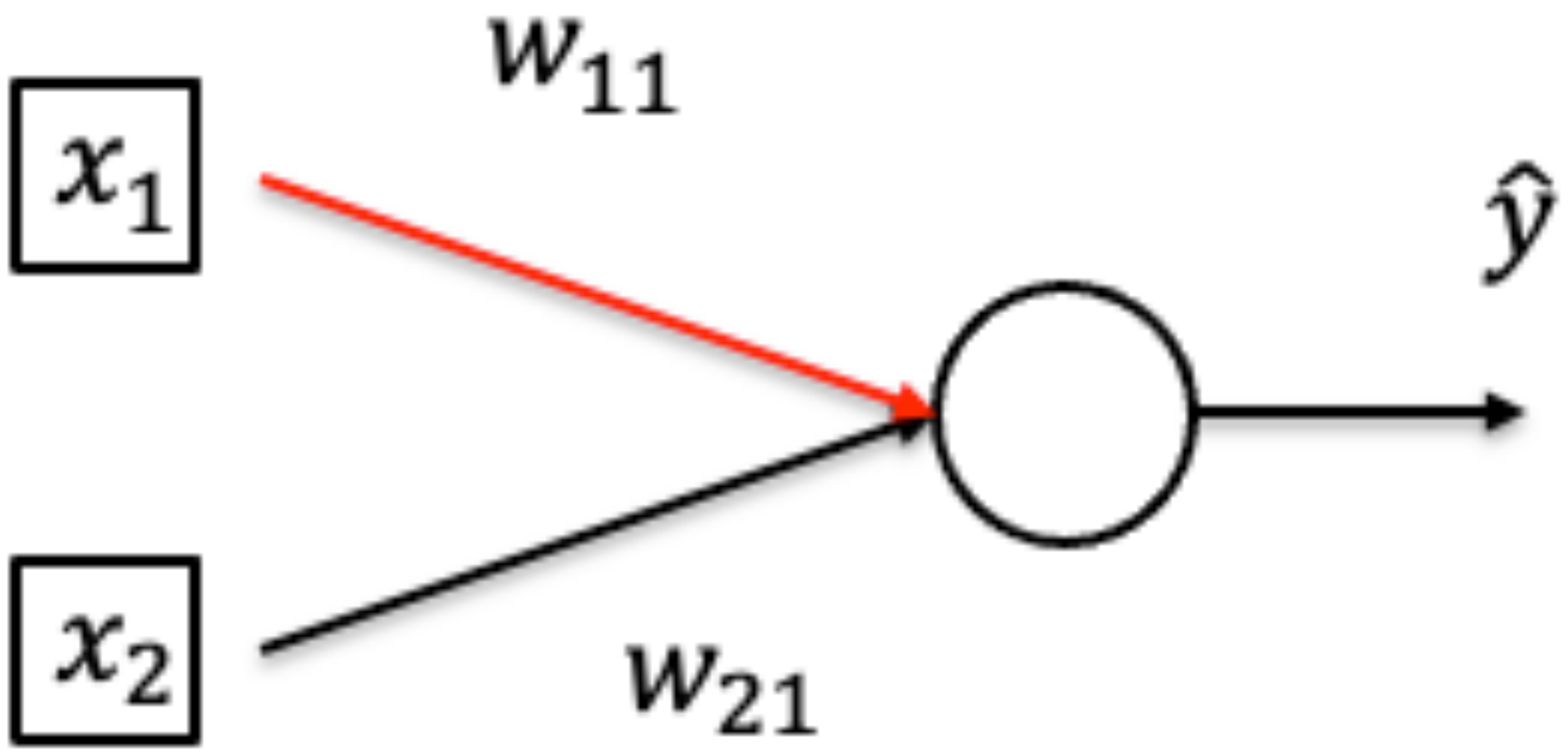
[Gao and Li et al., 2018]

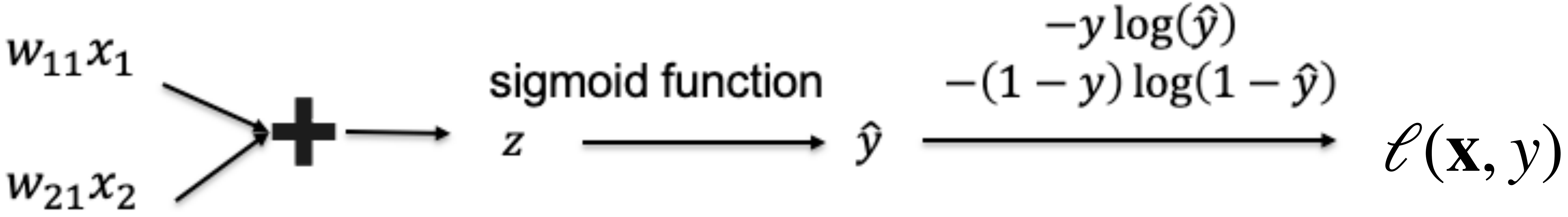
Calculate Gradient (on one data point)



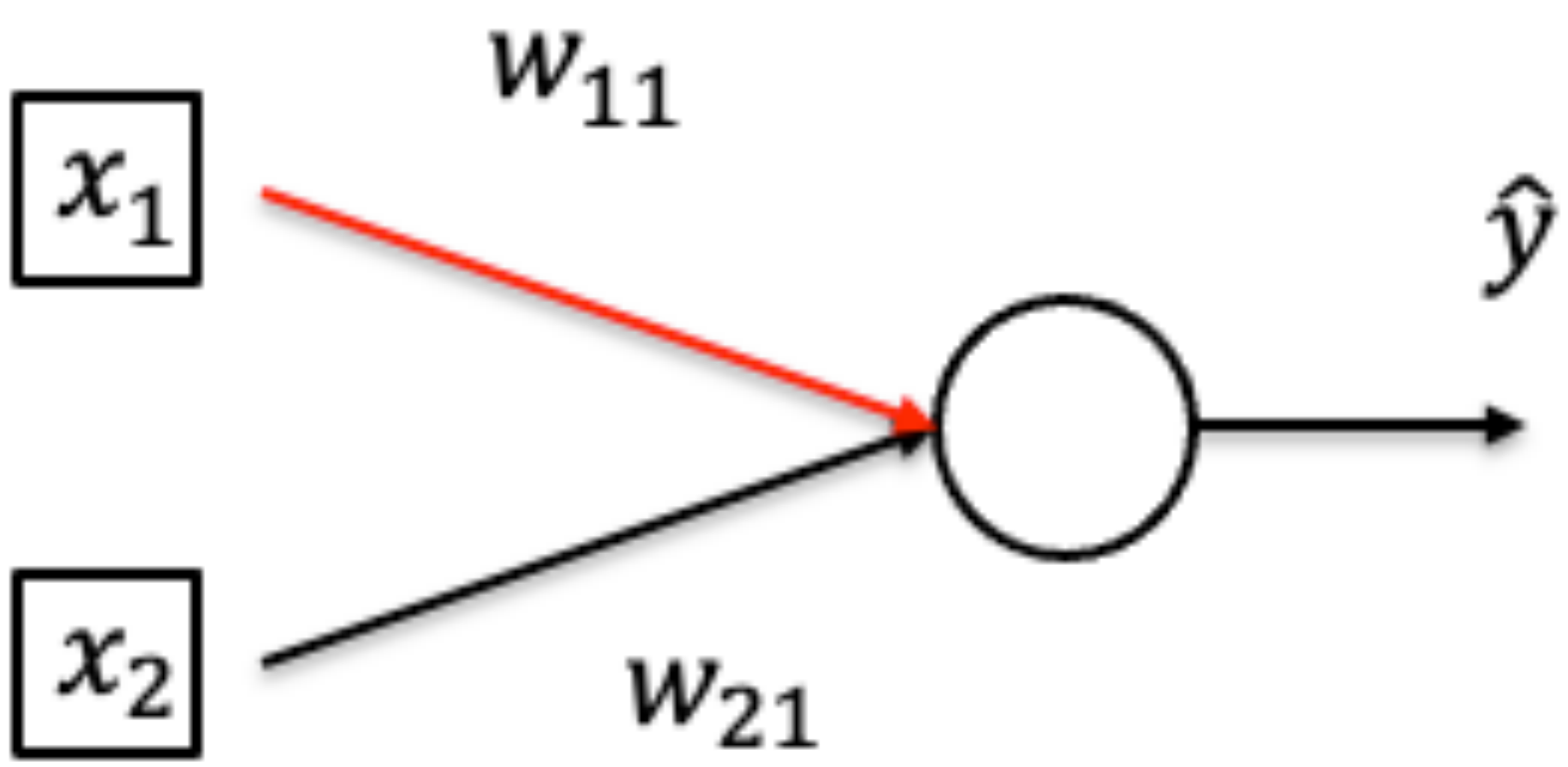
- Want to compute $\frac{\partial \ell(\mathbf{x}, y)}{\partial w_{11}}$

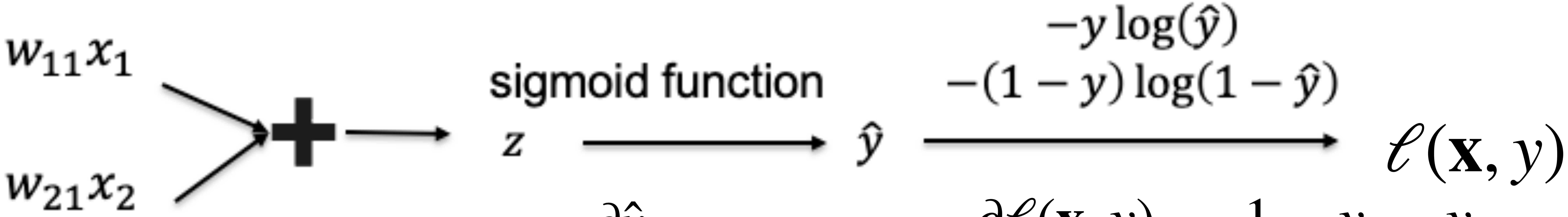
Calculate Gradient (on one data point)





Calculate Gradient (on one data point)

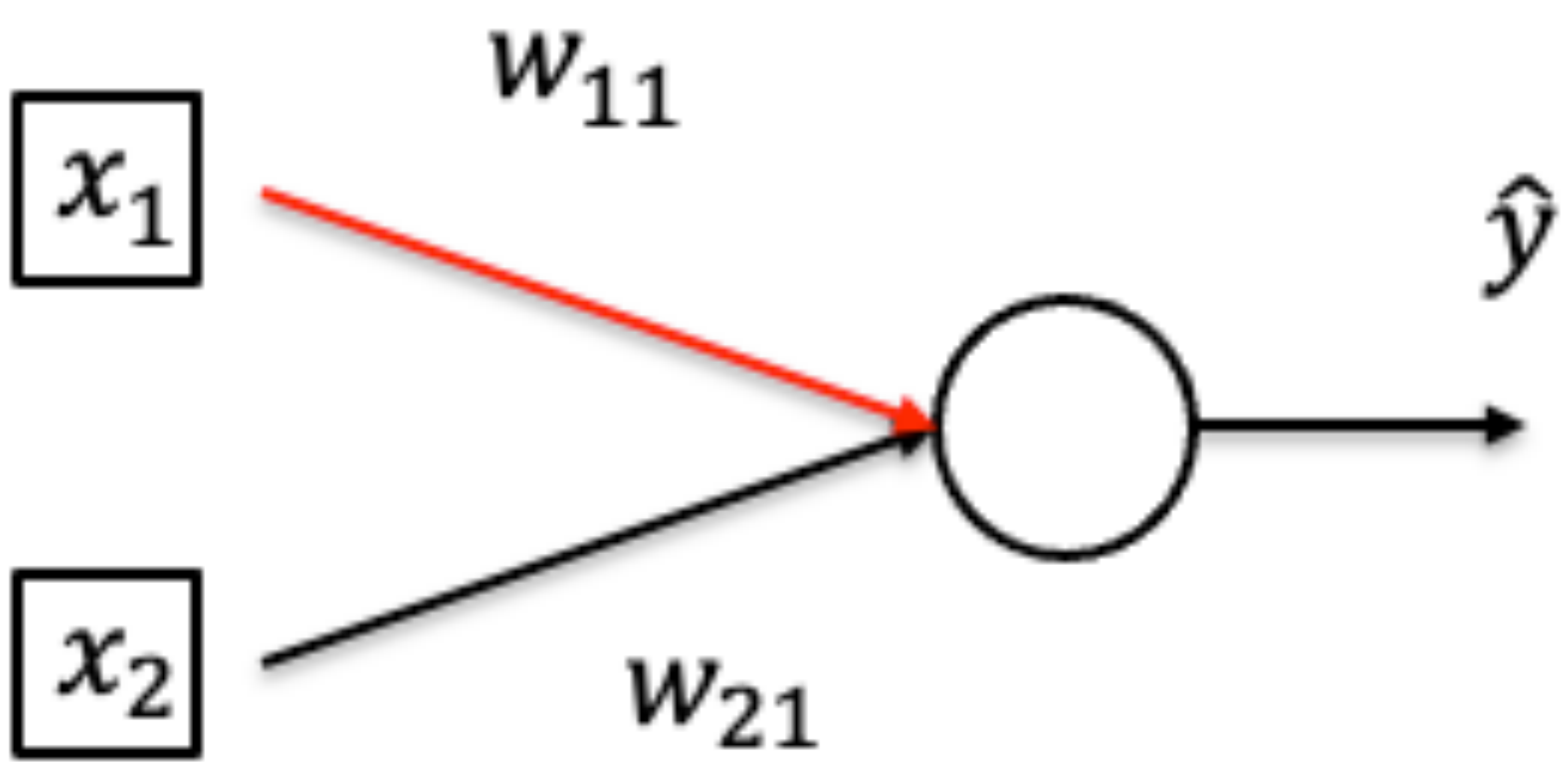


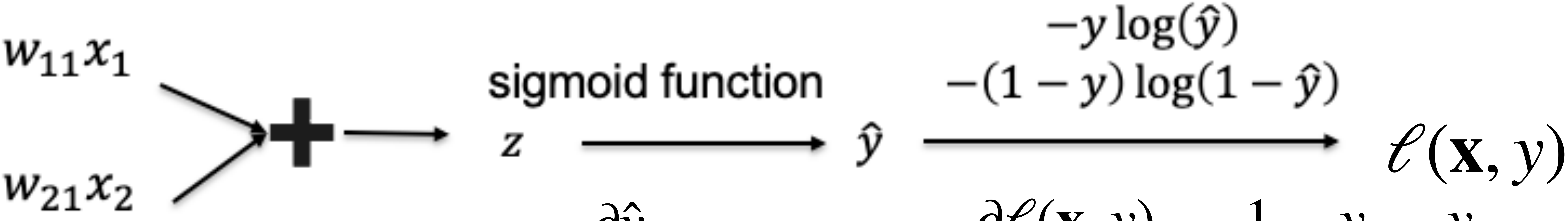


$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z)$$

$$\frac{\partial \ell(\mathbf{x}, y)}{\partial \hat{y}} = \frac{1-y}{1-\hat{y}} - \frac{y}{\hat{y}}$$

Calculate Gradient (on one data point)



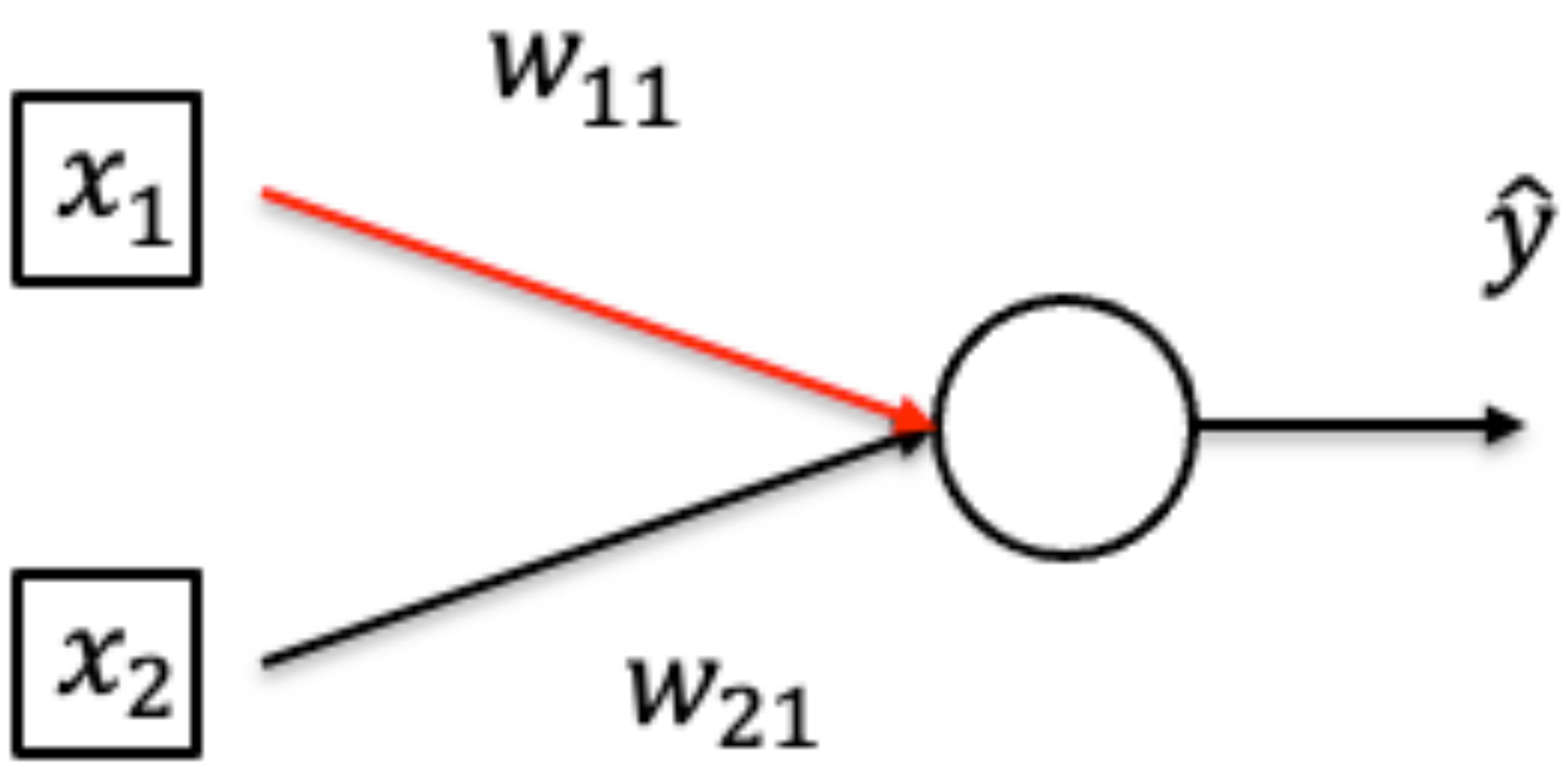


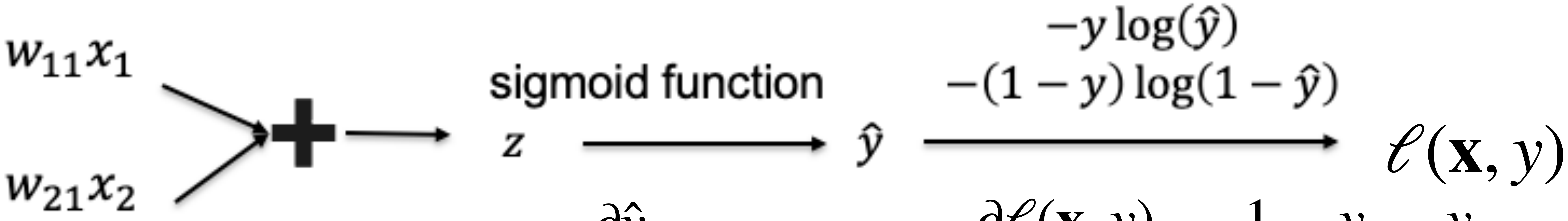
$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) \qquad \frac{\partial \ell(\mathbf{x}, y)}{\partial \hat{y}} = \frac{1-y}{1-\hat{y}} - \frac{y}{\hat{y}}$$

- By chain rule:

$$\frac{\partial l}{\partial w_{11}} = \frac{\partial l}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z} \frac{\partial z}{\partial w_{11}}$$

Calculate Gradient (on one data point)

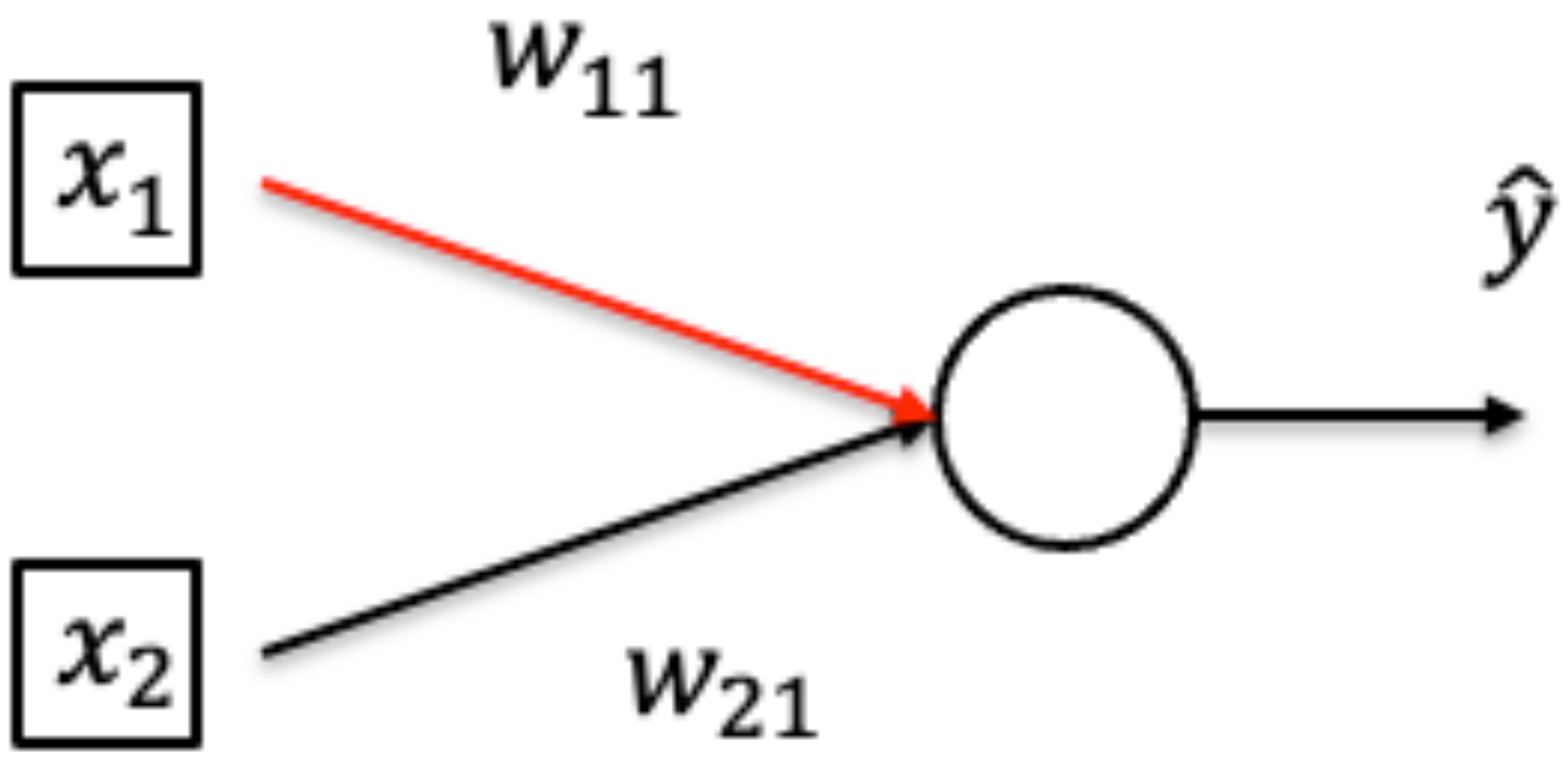


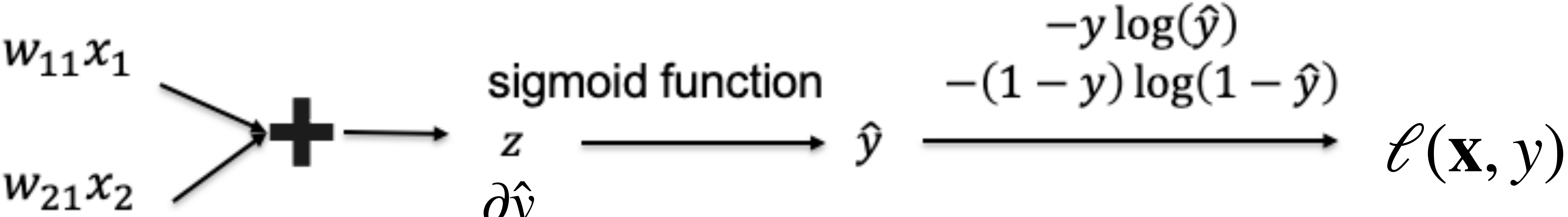


$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) \qquad \frac{\partial \ell(\mathbf{x}, y)}{\partial \hat{y}} = \frac{1-y}{1-\hat{y}} - \frac{y}{\hat{y}}$$

- By chain rule: $$\frac{\partial l}{\partial w_{11}} = \frac{\partial l}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z} x_1$$

Calculate Gradient (on one data point)

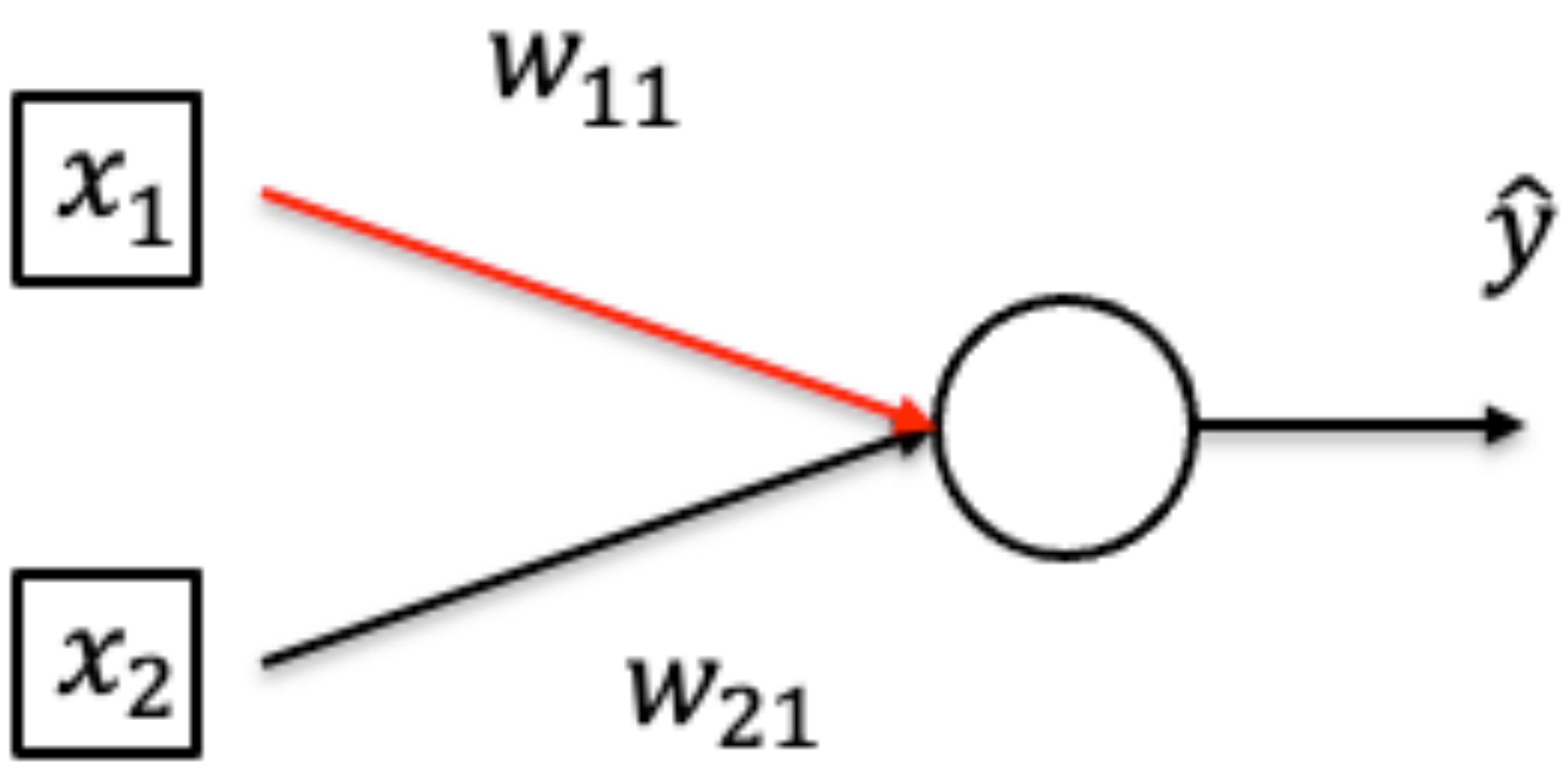


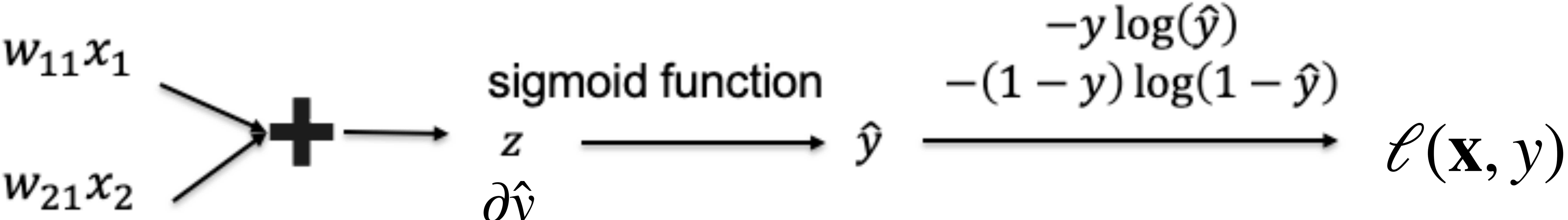


$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) = \sigma(z)(1-\sigma(z))$$

- By chain rule: $$\frac{\partial l}{\partial w_{11}} = \frac{\partial l}{\partial \hat{y}} \hat{y}(1-\hat{y})x_1$$

Calculate Gradient (on one data point)

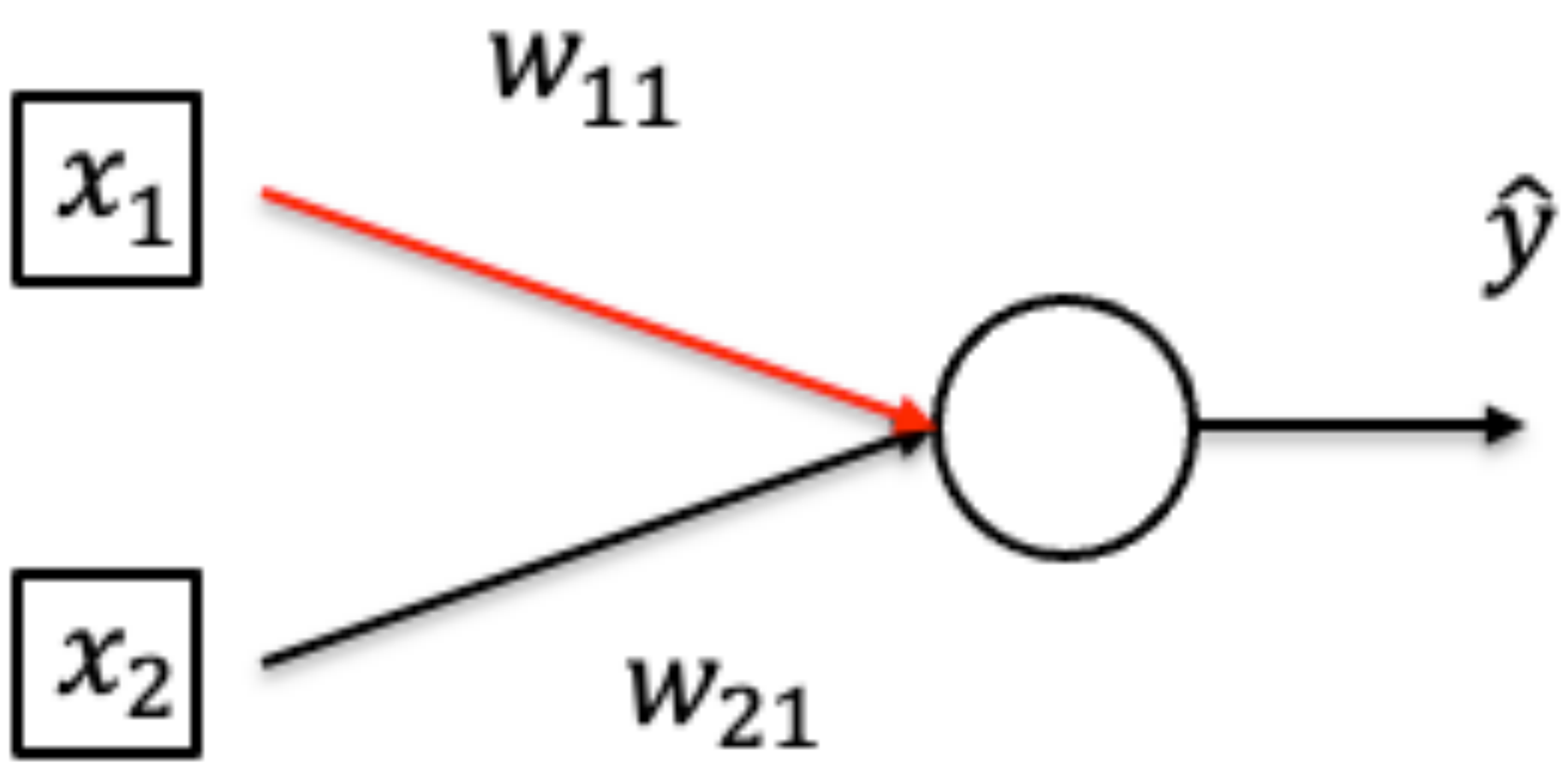


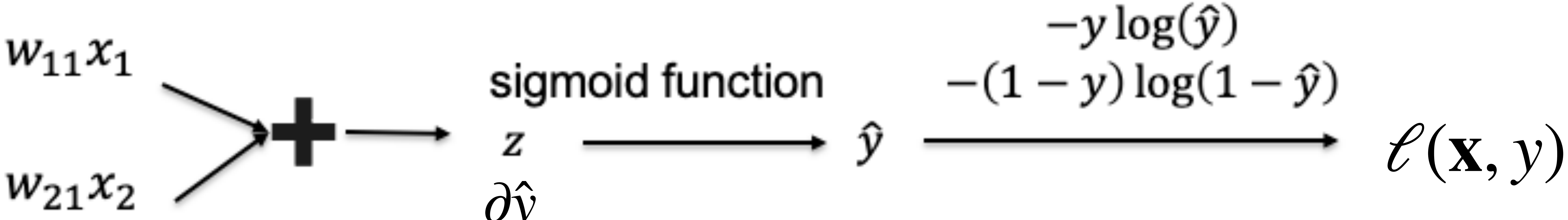


$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) = \sigma(z)(1-\sigma(z))$$

- By chain rule: $$\frac{\partial l}{\partial w_{11}} = \left(\frac{1-y}{1-\hat{y}} - \frac{y}{\hat{y}}\right)\hat{y}(1-\hat{y})x_1$$

Calculate Gradient (on one data point)

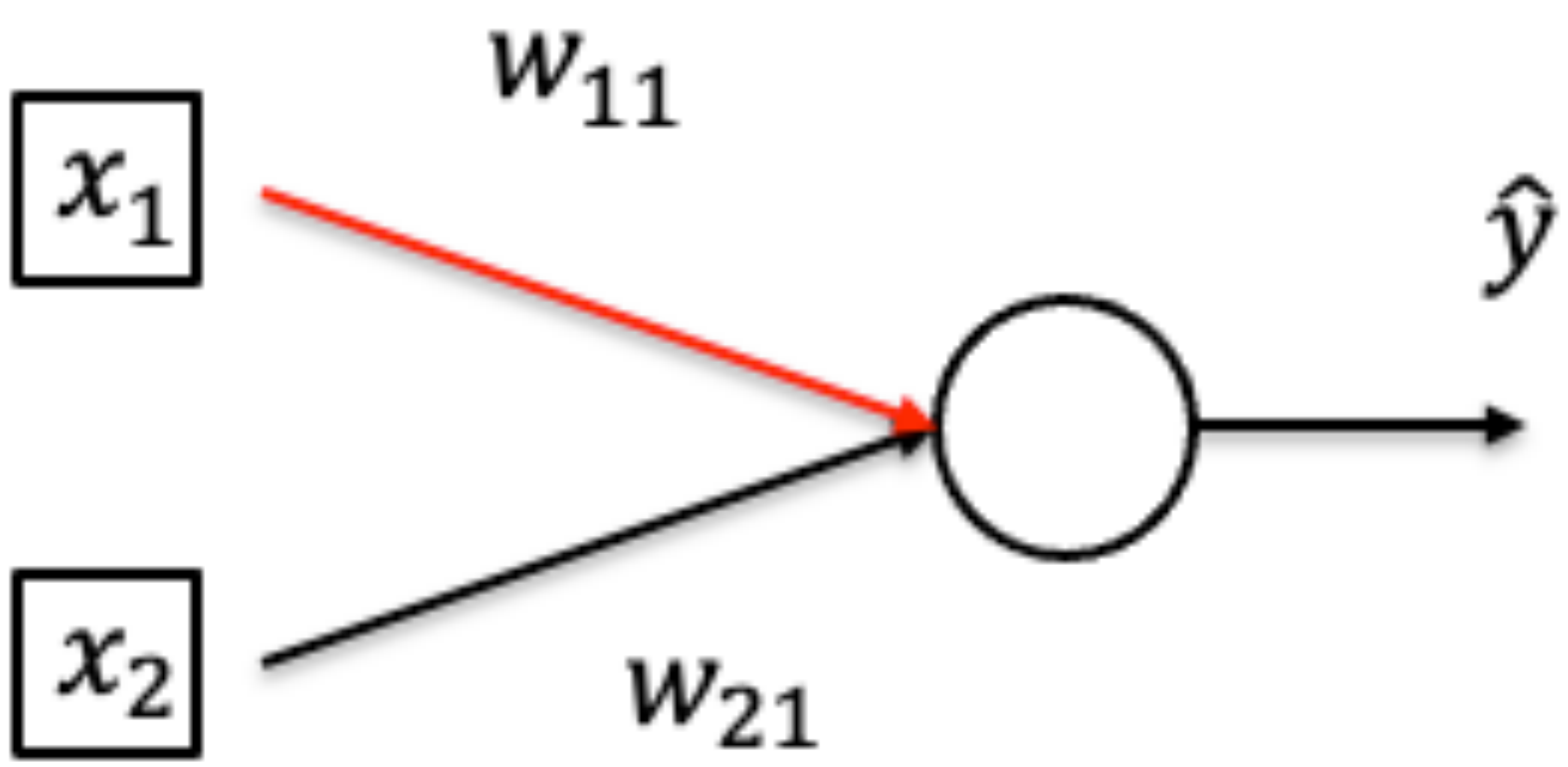


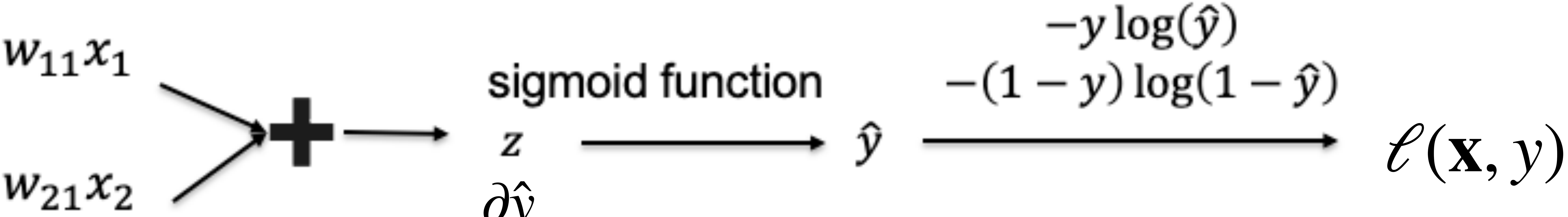


$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) = \sigma(z)(1 - \sigma(z))$$

- By chain rule: $\dfrac{\partial l}{\partial w_{11}} = (\hat{y} - y)x_1$

Calculate Gradient (on one data point)

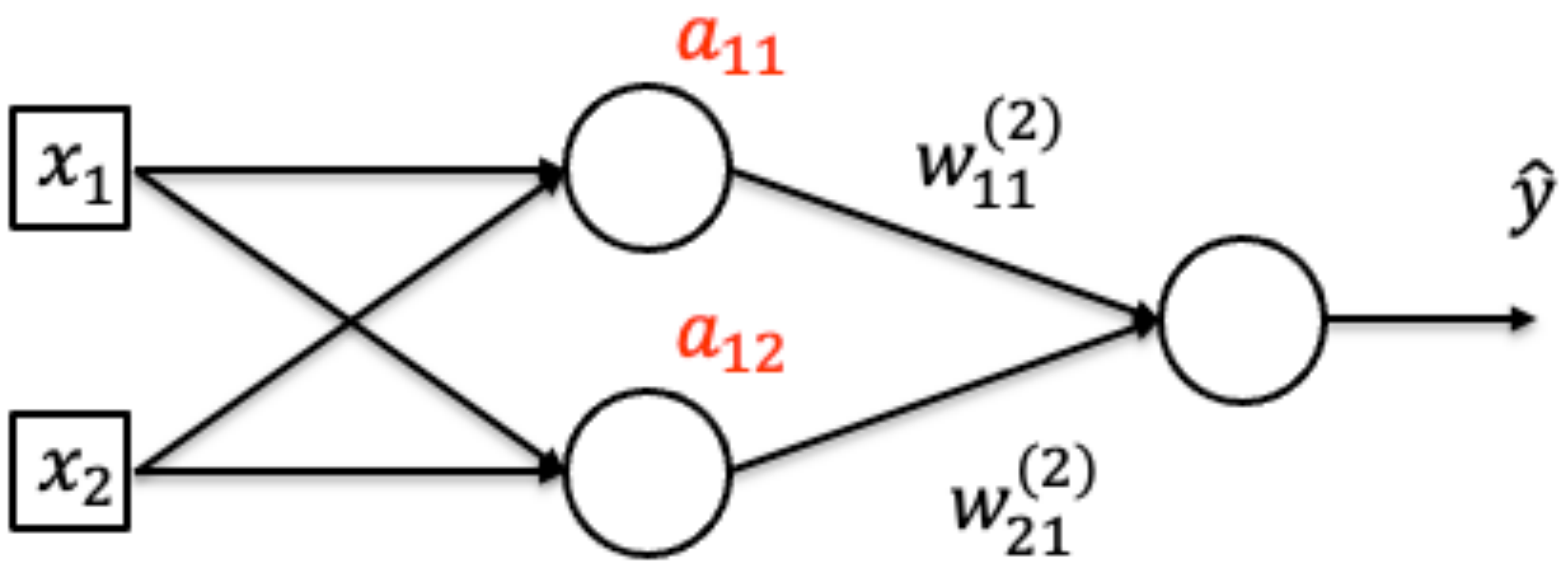




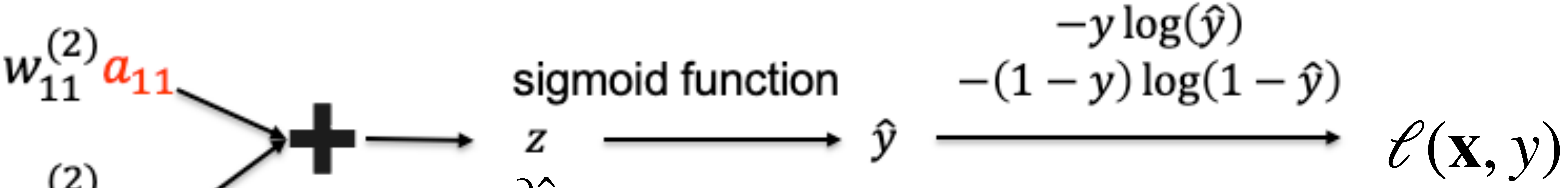
$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) = \sigma(z)(1 - \sigma(z))$$

- By chain rule: $$\frac{\partial l}{\partial x_1} = \frac{\partial l}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z} w_{11} = (\hat{y} - y) w_{11}$$

Calculate Gradient (on one data point)



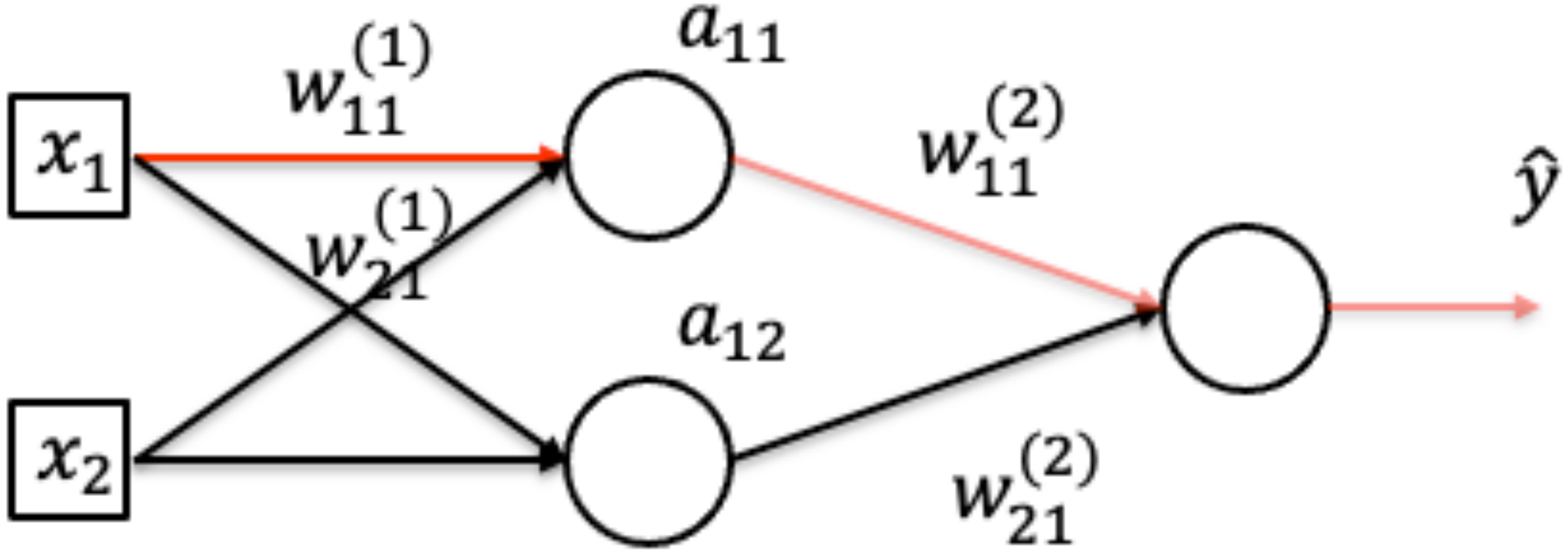
Make it deeper

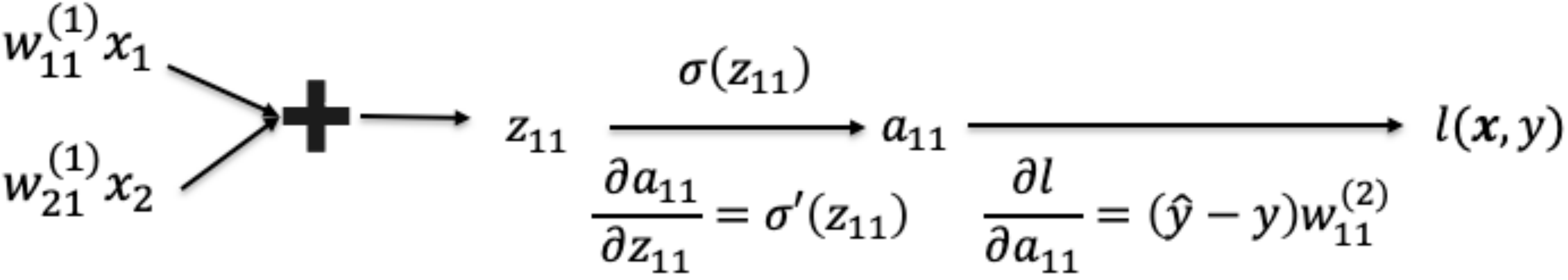


$$\frac{\partial \hat{y}}{\partial z} = \sigma'(z) = \sigma(z)(1-\sigma(z))$$

- By chain rule: $\frac{\partial l}{\partial a_{11}} = (\hat{y} - y) w_{11}^{(2)}, \ \frac{\partial l}{\partial a_{12}} = (\hat{y} - y) w_{21}^{(2)}$

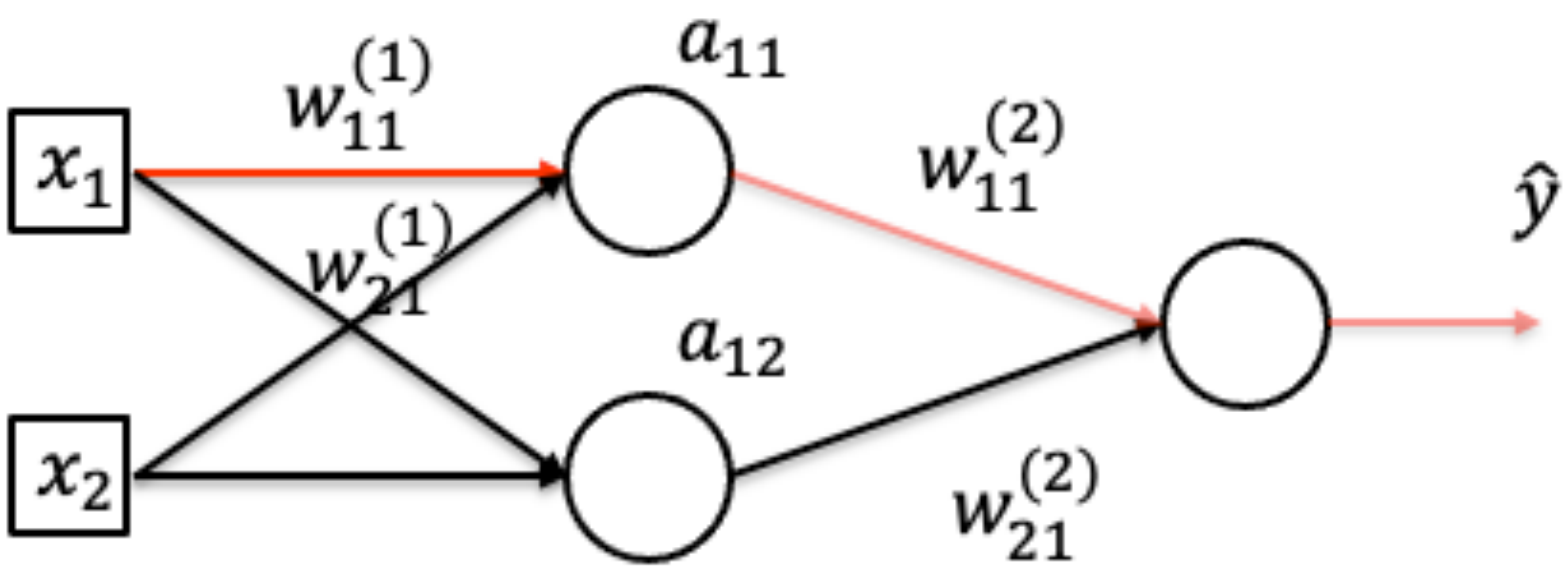
Calculate Gradient (on one data point)

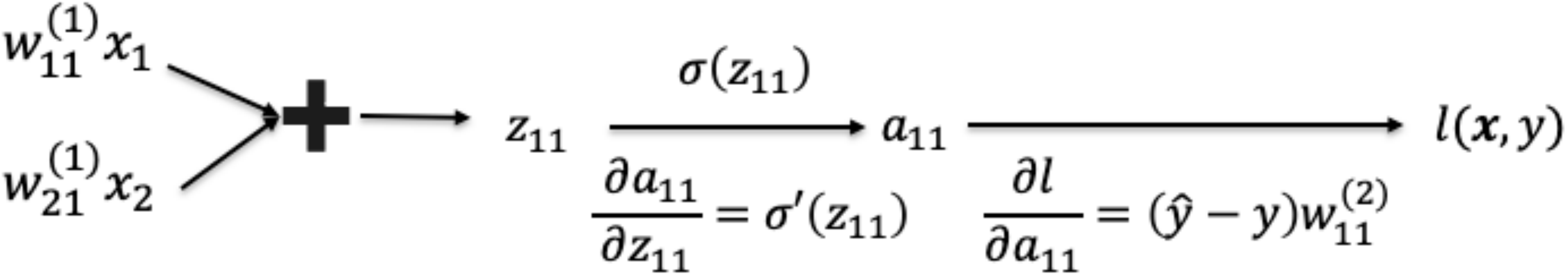




- By chain rule: $\frac{\partial l}{\partial w_{11}^{(1)}} = \frac{\partial l}{\partial a_{11}} \frac{\partial a_{11}}{\partial w_{11}^{(1)}} = (\hat{y} - y)w_{11}^{(2)} \frac{\partial a_{11}}{\partial w_{11}^{(1)}}$

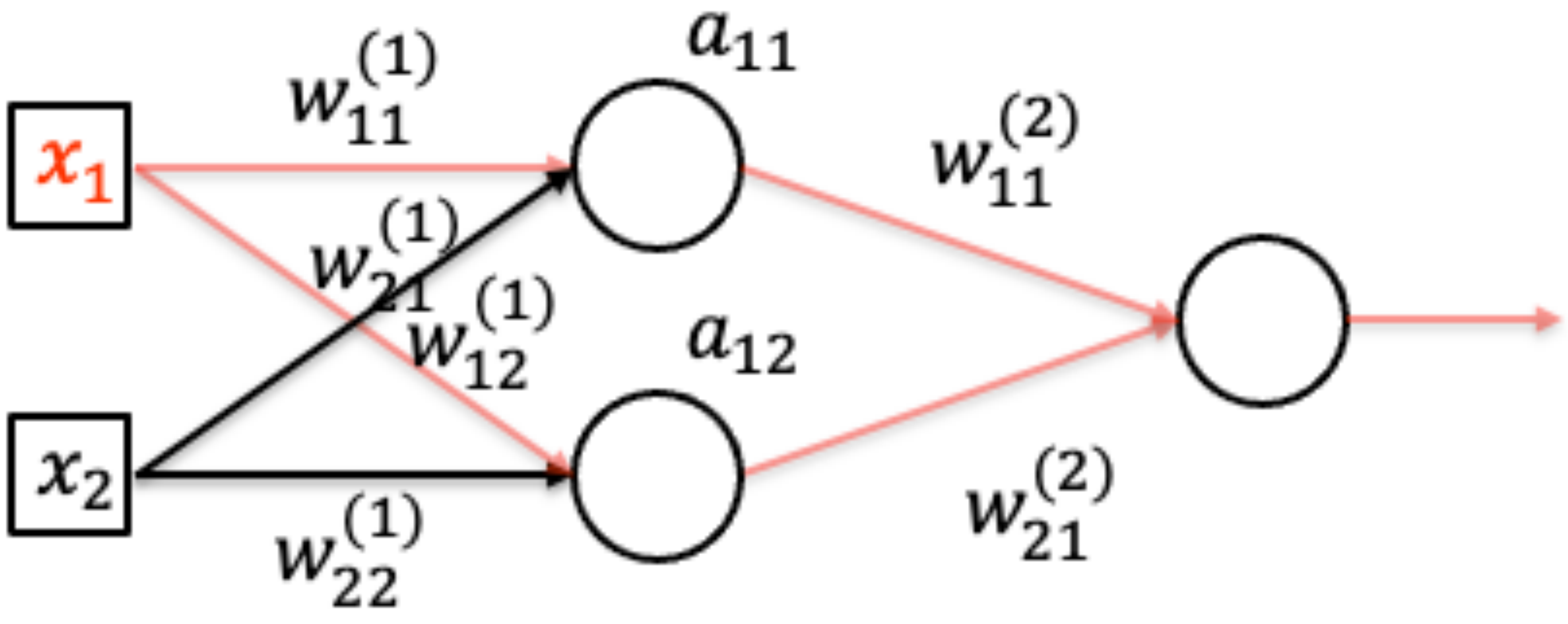
Calculate Gradient (on one data point)





- By chain rule: $\frac{\partial l}{\partial w_{11}^{(1)}} = \frac{\partial l}{\partial a_{11}} \frac{\partial a_{11}}{\partial w_{11}^{(1)}} = (\hat{y} - y) w_{11}^{(2)} a_{11}(1 - a_{11}) x_1$

Calculate Gradient (on one data point)

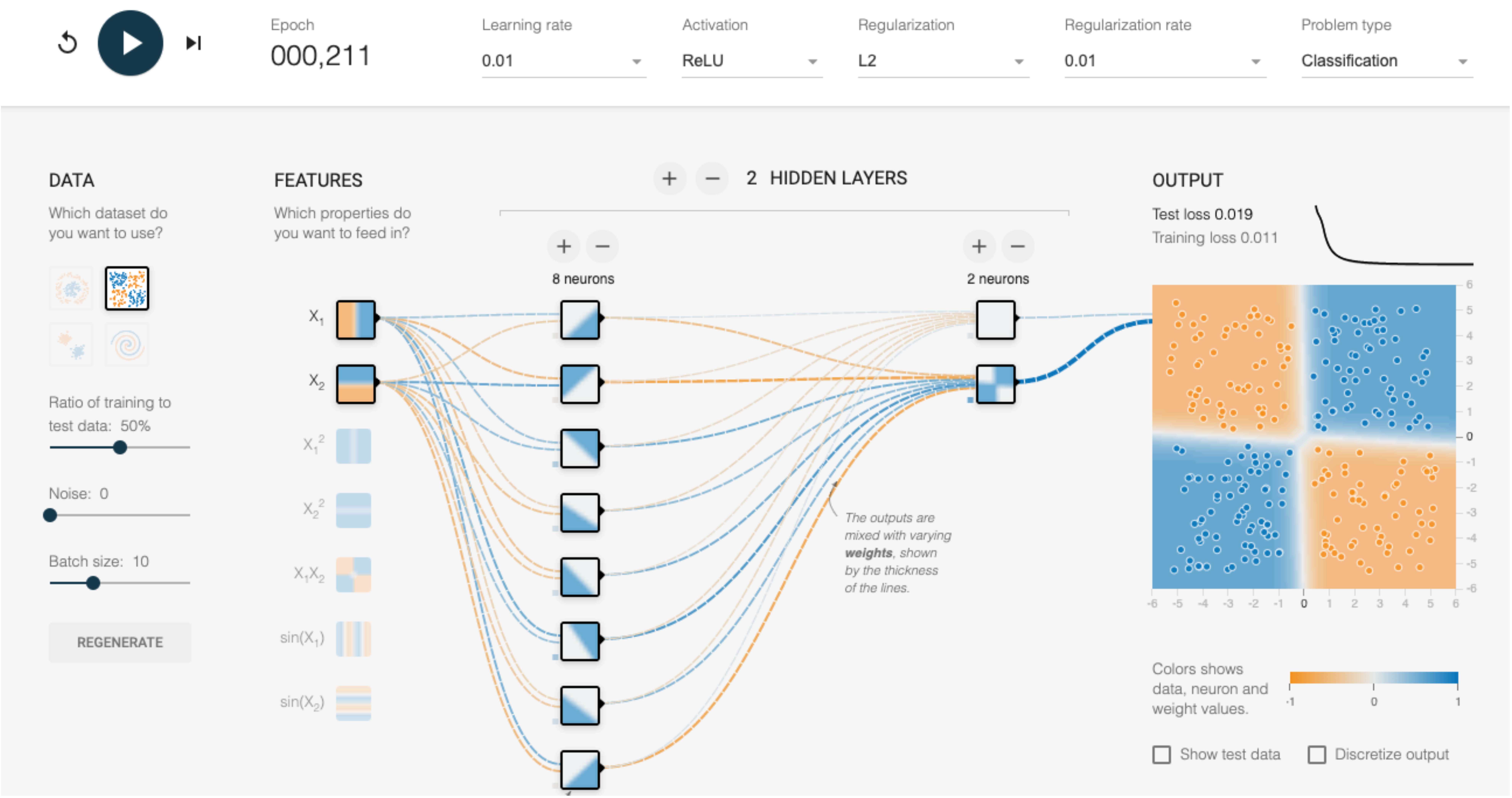


$$w_{11}^{(1)} x_1 + w_{21}^{(1)} x_2 \longrightarrow z_{11} \xrightarrow{\sigma(z_{11})} a_{11} \longrightarrow l(\boldsymbol{x}, y)$$

$$\frac{\partial a_{11}}{\partial z_{11}} = \sigma'(z_{11}) \qquad \frac{\partial l}{\partial a_{11}} = (\hat{y} - y) w_{11}^{(2)}$$

- By chain rule: $$\frac{\partial l}{\partial x_1} = \frac{\partial l}{\partial a_{11}} \frac{\partial a_{11}}{\partial x_1} + \frac{\partial l}{\partial a_{12}} \frac{\partial a_{12}}{\partial x_1}$$

Demo: Learning XOR using neural net



- https://playground.tensorflow.org/

What we've learned today…

- Single-layer Perceptron Review
- Multi-layer Perceptron
    - Single output
    - Multiple output
- How to train neural networks
    - Gradient descent