Outline

• Uninformed vs Informed Search
  – Review of uninformed strategies, adding heuristics

• A* Search
  – Heuristic properties, stopping rules, analysis

• Extensions: Beyond A*
  – Iterative deepening, beam search
Breadth-First Search

Recall: expand **shallowest** node first

- Data structure: queue

**Properties:**
- Complete
- Optimal (if edge cost 1)
- Time $O(b^d)$
- Space $O(b^d)$
Uniform Cost Search

Like BFS, but keeps track of cost

• Expand least cost node
• Data structure: priority queue
• **Properties:**
  – Complete
  – Optimal (if weight lower bounded by $\varepsilon$)
  – Time $O(b^{C*/\varepsilon})$
  – Space $O(b^{C*/\varepsilon})$

Optimal goal path cost
Recall: expand **deepest** node first

- Data structure: stack
- **Properties:**
  - Incomplete (stuck in infinite tree...)
  - Suboptimal
  - Time $O(b^m)$
  - Space $O(bm)$
Iterative Deepening DFS

Repeated limited DFS

• Search like BFS, fringe like DFS

• **Properties:**
  – Complete
  – Optimal (if edge cost 1)
  – Time $O(b^d)$
  – Space $O(bd)$

A good option!
Uninformed vs Informed Search

Uninformed search (all of what we saw). Know:
• Path cost $g(s)$ from start to node $s$
• Successors.

Informed search. Know:
• All uninformed search properties, plus
• Heuristic $h(s)$ from $s$ to goal (recall game heuristic)
Informed Search

Informed search. Know:

• All uninformed search properties, plus
• Heuristic $h(s)$ from $s$ to goal (recall game heuristic)

• Goal: *speed up search.*
Using the Heuristic

Back to uniform-cost search
• We had the priority queue
• Expand the node with the smallest \( g(s) \)
  – \( g(s) \) “first-half-cost”
• Now let’s use the heuristic (“second-half-cost”)
  – Several possible approaches: let’s see what works
Attempt 1: Best-First Greedy

One approach: just use $h(s)$ alone

- Specifically, expand node with smallest $h(s)$
- This isn’t a good idea. Why?

- Not optimal! **Get** $A \rightarrow C \rightarrow G$. **Want**: $A \rightarrow B \rightarrow C \rightarrow G$
Attempt 2: A Search

Next approach: use both $g(s) + h(s)$ alone
- Specifically, expand node with smallest $g(s) + h(s)$
- Again, use a priority queue
- Called “A” search

- Still not optimal! (Does work for former example).
Attempt 3: A* Search

Same idea, use $g(s) + h(s)$, with one requirement

• Demand that $h(s) \leq h^*(s)$
• If heuristic has this property, “admissible”
  – Optimistic! Never over-estimates
• Still need $h(s) \geq 0$
  – Negative heuristics can lead to strange behavior
• This is A* search
Attempt 3: A* Search

**Origins:** robots and planning

Shakey the Robot, 1960’s

Credit: Wiki

**Animation:** finding a path around obstacle

Credit: Wiki
Admissible Heuristic Functions

Have to be careful to ensure admissibility (optimism!)

• Example: 8 Game

Example State

\[
\begin{array}{ccc}
1 & & 5 \\
2 & 6 & 3 \\
7 & 4 & 8 \\
\end{array}
\]

Goal State

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & \text{(shaded)} \\
\end{array}
\]

• One useful approach: relax constraints
  – \( h(s) \) = number of tiles in wrong position
    • allows tiles to fly to destination in a single step
Q 1.1: Consider finding the fastest driving route from one US city to another. Measure cost as the number of hours driven when driving at the speed limit. Let $h(s)$ be the number of hours needed to ride a bike from city $s$ to your destination. $h(s)$ is

• A. An admissible heuristic
• B. Not an admissible heuristic
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• A. An admissible heuristic
• **B. Not an admissible heuristic**
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- A. An admissible heuristic No: riding your bike takes longer.
- B. Not an admissible heuristic
Q 1.2: Which of the following are admissible heuristics?

(i) \( h(s) = h^*(s) \)
(ii) \( h(s) = \max(2, h^*(s)) \)
(iii) \( h(s) = \min(2, h^*(s)) \)
(iv) \( h(s) = h^*(s) - 2 \)
(v) \( h(s) = \sqrt{h^*(s)} \)

• A. All of the above
• B. (i), (iii), (iv)
• C. (i), (iii)
• D. (i), (iii), (v)
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Q 1.2: Which of the following are admissible heuristics?

(i) \( h(s) = h^*(s) \)
(ii) \( h(s) = \max(2, h^*(s)) \)  
   No: \( h(s) \) might be too big
(iii) \( h(s) = \min(2, h^*(s)) \)
(iv) \( h(s) = h^*(s) - 2 \)  
   No: \( h(s) \) might be negative
(v) \( h(s) = \sqrt{h^*(s)} \)  
   No: if \( h^*(s) < 1 \) then \( h(s) \) is bigger

• A. All of the above
• B. (i), (iii), (iv)
• C. (i), (iii)
• D. (i), (iii), (v)
Heuristic Function Tradeoffs

Dominance: $h_2$ dominates $h_1$ if for all states $s$,

$$h_1(s) \leq h_2(s) \leq h^*(s)$$

- **Idea**: we want to be as close to $h^*$ as possible
  - But not over!

- **Tradeoff**: being very close might require a very complex heuristic, expensive computation
  - Might be better off with cheaper heuristic & expand more nodes.
A* Termination

When should A* stop?

• One idea: as soon as we reach goal state?

  - $h$ admissible, but note that we get $A \rightarrow B \rightarrow G$ (cost 1000)!
A* Termination

When should A* stop?

- **Rule**: terminate **when a goal is popped** from queue.

- Note: taking $h = 0$ reduces to uniform cost search rule.
**A* Revisiting Expanded States**

Possible to revisit an expanded state, get a shorter path:

- Put D back into priority queue, smaller $g+h$
A* Full Algorithm

1. Put the start node $S$ on the priority queue, called OPEN
2. If OPEN is empty, exit with failure
3. Remove from OPEN and place on CLOSED a node $n$ for which $f(n)$ is minimum (note that $f(n) = g(n) + h(n)$)
4. If $n$ is a goal node, exit (trace back pointers from $n$ to $S$)
5. Expand $n$, generating all successors and attach to pointers back to $n$. For each successor $n'$ of $n$
   1. If $n'$ is not already on OPEN or CLOSED estimate $h(n')$, $g(n') = g(n) + c(n,n')$, $f(n') = g(n') + h(n')$, and place it on OPEN.
   2. If $n'$ is already on OPEN or CLOSED, then check if $g(n')$ is lower for the new version of $n'$. If so, then:
      1. Redirect pointers backward from $n'$ along path yielding lower $g(n')$.
      2. Put $n'$ on OPEN.
   3. If $g(n')$ is not lower for the new version, do nothing.
A* Analysis

Some properties:

• Terminates!
• A* can use **lots of memory**: $O(\# \text{ states})$.
• Will run out on large problems.
• Next, we will consider some alternatives to deal with this.
Q 2.1: Consider two heuristics for the 8 puzzle problem. \( h_1 \) is the number of tiles in wrong position. \( h_2 \) is the \( l_1 \)/Manhattan distance between the tiles and the goal location. How do \( h_1 \) and \( h_2 \) relate?

- A. \( h_2 \) dominates \( h_1 \)
- B. \( h_1 \) dominates \( h_2 \)
- C. Neither dominates the other
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- A. $h_2$ dominates $h_1$
- B. $h_1$ dominates $h_2$ (No: $h_1$ is a distance where each entry is at most 1, $h_2$ can be greater)
- C. Neither dominates the other
Q 2.2: Consider the state space graph below. Goal states have bold borders. $h(s)$ is show next to each node. What node will be expanded by A* after the initial state I?

- A. A
- B. B
- C. C
Q 2.2: Consider the state space graph below. Goal states have bold borders. $h(s)$ is shown next to each node. What node will be expanded by A* after the initial state I?

- A. A
- B. B
- C. C
IDA*: Iterative Deepening A*

Similar idea to our earlier iterative deepening.

- Bound the memory in search.
- At each phase, don’t expand any node with $g(s) + h(s) > k$,
  - Assuming integer costs, do this for $k=0$, then $k=1$, then $k=2$, and so on

- Complete + optimal, might be costly time-wise
  - Revisit many nodes
- Lower memory use than A*
IDA*: Properties

How many restarts do we expect?
• With integer costs, optimal solution $C^*$, at most $C^*$

What about non-integer costs?
• Initial threshold $k$. Use the same rule for non-expansion
• Set new $k$ to be the min $g(s) + h(s)$ for non-expanded nodes
• Worst case: restarted for each state
Beam Search

General approach (beyond A* too)
• Priority queue with fixed size $k$; beyond $k$ nodes, discard!
• **Upside**: good memory efficiency
• **Downside**: not complete or optimal

Variation:
• Priority queue with nodes that are at most $\varepsilon$ worse than best node.
Recap and Examples

Example for A*:

Initial state

Goal state

h=0

h=inf

h=inf

h=inf
Example for A*: 

- OPEN
  - S(0+8)
  - A(1+7) B(5+4) C(8+3)
- CLOSED
  - S(0+8)
  - B(5+4) C(8+3) D(4+inf) E(8+inf) G(10+0) S(0+8) A(1+7)
- h=8
- h=7
- h=4
- h=3
- G → B → S

Goal state: G

Initial state: S
Recap and Examples

Example for IDA*:
Threshold = 8

PREFIX
- S
S A H F
S A D

OPEN
- S(0+8)
S A H F D(4+4)
S A D F D(4+4)

Example:
Threshold = 8

Initial state

Goal state

h=0

h=8

h=7

h=4

h=3

h=2

h=inf

h=inf

h=inf

h=2

h=4

h=1

h=1

h=1
Recap and Examples

Example for IDA*: Threshold = 9

- OPEN
  - S(0+8)
  - A(1+7) B(5+4)
  - B(5+4) H(2+2) D(4+4)
  - B(5+4) D(4+4) F(6+1)
  - B(5+4) D(4+4)
  - B(5+4)
  - G(9+0)
  - S B G

PREFIX

Initial state

Goal state
Recap and Examples

Example for Beam Search: $k=2$

- CURRENT
- OPEN
  - S(0+8)
  - A(1+7) B(5+4)
  - H(2+2) D(4+4)
  - D(4+4) F(6+1)
  - D(4+4) G(10+0)
- Goal state
- Initial state

h=8     h=7     h=4     h=3
S       A       B       C

h=1
F

h=0
G

h=2
H

h=inf
I

h=inf
J

h=inf
K

h=inf
L

h=inf
D

h=4
E

h=8
S

1  7  5  3  2  3
H  I  J  K  L  D

4

4

5

8
Summary

• Informed search: introduce heuristics
  – Not all approaches work: best-first greedy is bad

• A* algorithm
  – Properties of A*, idea of admissible heuristics

• Beyond A*
  – IDA*, beam search. Ways to deal with space requirements.
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