CS 540 Introduction to Artificial Intelligence

Logic

University of Wisconsin-Madison

Fall, 2022
Announcements

- **Homeworks:**
  - Good work on HW1, HW2! HW3 is next.

- **Office hours**

- **Class roadmap:**

<table>
<thead>
<tr>
<th>Date</th>
<th>Topic</th>
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<tbody>
<tr>
<td>Tuesday, Sept. 13</td>
<td>Probability</td>
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<tr>
<td>Thursday, Sept. 15</td>
<td>Linear Algebra and PCA</td>
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<tr>
<td>Tuesday, Sept. 20</td>
<td>PCA, Statistics and Math Review</td>
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<td><strong>Thursday, Sept. 22</strong></td>
<td><strong>Introduction to Logic</strong></td>
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<tr>
<td>Tuesday, Sept. 27</td>
<td>Natural Language Processing</td>
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Logic & AI

Why are we studying logic?

• **Traditional** approach to AI (’50s-’80s)
  – “Symbolic AI”
  – The Logic Theorist – 1956
    • Proved a bunch of theorems!

• Logic also the language of:
  – Knowledge rep., databases, etc.
Symbolic Techniques in AI

Lots of systems based on symbolic approach:

• Ex: expert systems, planning, more
• Playing great chess

• Less popular recently!
Symbolic vs Connectionist

Rival approach: **connectionist**
- Probabilistic models
- Neural networks
- **Extremely popular** last 20 years
Symbolic vs Connectionist

Analogy: Logic versus probability

• Which is better?

• Future: combination; best-of-both-worlds
  – Actually been worked on:
  – **Example**: Markov Logic Networks
Outline

• Introduction to logic
  – Arguments, validity, soundness

• Propositional logic
  – Sentences, semantics, inference

• First order logic (FOL)
  – Predicates, objects, formulas, quantifiers
Basic Logic

• Arguments, premises, conclusions
  – Argument: a set of sentences (premises) + a sentence (a conclusion)
  – **Validity:** argument is valid iff it’s necessary that if all premises are true, the conclusion is true
  – **Soundness:** argument is sound iff valid & premises true
  – **Entailment:** when valid arg., premises entail conclusion
Propositional Logic Basics

Logic Vocabulary:

• Sentences, symbols, connectives, parentheses
  – Symbols: P, Q, R, ... (atomic sentences)
  – Connectives:
    - and
    - or
    - implies
    - is equivalent
    - not

– Literal: P or negation ¬P
Propositional Logic Basics

Examples:

• \((P \lor Q) \implies S\)
  – “If it is cold or it is raining, then I need a jacket”

• \(Q \implies P\)
  – “If it is raining, then it is cold”

• \(\neg R\)
  – “It is not hot”
Propositional Logic Basics

Several rules in place

- Precedence: ¬, ∧, ∨, ⊃, ↔
- Use parentheses when needed
- Sentences: **well-formed** or not well-formed:
  - P ⊃ Q ⊃ S **not well-formed** (not associative!)
Sentences & Semantics

• Think of symbols as defined by user
• Sentences: built up from symbols with connectives
  - **Interpretation**: assigning True / False to symbols
  - **Semantics**: interpretations for which sentence evaluates to True
  - **Model**: (of a set of sentences) interpretation for which all sentences are True
Evaluating a Sentence

• Example:

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<tr>
<td>P</td>
<td>Q</td>
<td>¬P</td>
<td>P ∧ Q</td>
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• Note:
  – If P is false, P→Q is true regardless of Q (“5 is even implies 6 is odd” is True!)
  – Causality unneeded: “5 is odd implies the Sun is a star” is True!
Evaluating a Sentence: Truth Table

- **Ex:**

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<tr>
<th>P</th>
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<th>R</th>
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<th>Q∧R</th>
<th>¬P∨Q∧R</th>
<th>¬P∨Q∧R⇒Q</th>
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- **Satisfiable**
  - There exists some interpretation where sentence true
Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) \( \neg(\neg p \rightarrow \neg q) \land r \)
(ii) \( (\neg p \lor \neg q) \rightarrow (p \lor \neg r) \)

• A. Both
• B. Neither
• C. Just (i)
• D. Just (ii)
Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) \( \neg(\neg p \rightarrow \neg q) \land r \)

(ii) \( (\neg p \lor \neg q) \rightarrow (p \lor \neg r) \)

- A. Both
- B. Neither
- C. Just (i)
- D. Just (ii)
Q 1.2: Let A = “Aldo is Italian” and B = “Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

• a. $A \lor (\neg A \rightarrow B)$
• b. $A \lor B$
• c. $A \lor (A \rightarrow B)$
• d. $A \rightarrow B$
Q 1.2: Let A = “Aldo is Italian” and B = “Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

• a. $A \lor (\neg A \rightarrow B)$
• b. $A \lor B$ (equivalent!)
• c. $A \lor (A \rightarrow B)$
• d. $A \rightarrow B$
Q 1.3: How many different assignments can there be to
\((x_1 \land y_1) \lor (x_2 \land y_2) \lor ... \lor (x_n \land y_n)\)

• A. 2
• B. \(2^n\)
• C. \(2^{2n}\)
• D. \(2n\)
Q 1.3: How many different assignments can there be to 

\[(x_1 \land y_1) \lor (x_2 \land y_2) \lor \ldots \lor (x_n \land y_n)\]

- A. 2
- B. \(2^n\)
- C. \(2^{2n}\)
- D. 2n
Knowledge Bases

- **Knowledge Base (KB):** A set of sentences
  - Like a long sentence, connect with conjunction

**Model of a KB:** interpretations where all sentences are True

**Goal:** inference to discover new sentences
Entailment: a sentence logically follows from others

• Like from a KB. Write $A \models B$

• $A \models B$ iff in every interpretation where $A$ is true, $B$ is also true
Inference

• Given a set of sentences (a KB), **logical inference** creates new sentences
  – Compare to prob. inference!

• **Challenges:**
  – Soundness
  – Completeness
  – Efficiency
Methods of Inference: 1. Enumeration

- Enumerate all interpretations; look at the truth table
  - “Model checking”

- Downside: $2^n$ interpretations for $n$ symbols
Methods of Inference: 2. Using Rules

• *Modus Ponens*: $(A \implies B, A) \models B$

• And-elimination

• Many other rules
  – Commutativity, associativity, de Morgan’s laws, distribution for conjunction/disjunction
Logical equivalences

\((\alpha \land \beta) \equiv (\beta \land \alpha)\)  \textit{commutativity of \land}

\((\alpha \lor \beta) \equiv (\beta \lor \alpha)\)  \textit{commutativity of \lor}

\(((\alpha \land \beta) \land \gamma) \equiv (\alpha \land (\beta \land \gamma))\)  \textit{associativity of \land}

\(((\alpha \lor \beta) \lor \gamma) \equiv (\alpha \lor (\beta \lor \gamma))\)  \textit{associativity of \lor}

\(\neg(\neg \alpha) \equiv \alpha\)  \textit{double-negation elimination}

\((\alpha \Rightarrow \beta) \equiv (\neg \beta \Rightarrow \neg \alpha)\)  \textit{contraposition}

\((\alpha \Rightarrow \beta) \equiv (\neg \alpha \lor \beta)\)  \textit{implication elimination}

\((\alpha \iff \beta) \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))\)  \textit{biconditional elimination}

\(\neg(\alpha \land \beta) \equiv (\neg \alpha \lor \neg \beta)\)  \textit{de Morgan}

\(\neg(\alpha \lor \beta) \equiv (\neg \alpha \land \neg \beta)\)  \textit{de Morgan}

\((\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma))\)  \textit{distributivity of \land over \lor}

\((\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))\)  \textit{distributivity of \lor over \land}

You can use these equivalences to modify sentences.
Methods of Inference: 3. Resolution

- Convert to special form and use a single rule
- **Conjunctive Normal Form (CNF)**

\[(\neg A \lor B \lor C) \land (\neg B \lor A) \land (\neg C \lor A)\]

Conjunction of clauses; each clause disjunction of literals

- Simple rules for converting to CNF
 Conjunctive Normal Form (CNF)

\((\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})\)

- Replace all ⇔ using biconditional elimination
- Replace all ⇒ using implication elimination
- Move all negations inward using
  - double-negation elimination
  - de Morgan's rule
- Apply distributivity of \(\lor\) over \(\land\)
Convert example sentence into CNF

\[
B_{1,1} \iff (P_{1,2} \lor P_{2,1}) \quad \text{starting sentence}
\]

\[
(B_{1,1} \Rightarrow (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \Rightarrow B_{1,1})
\]

biconditional elimination

\[
(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg (P_{1,2} \lor P_{2,1}) \lor B_{1,1})
\]

implication elimination

\[
(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1})
\]

move negations inward

\[
(-B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (-P_{1,2} \lor B_{1,1}) \land (-P_{2,1} \lor B_{1,1})
\]

distribute \lor over \land
Resolution Steps

• Given KB and $\beta$ (query)
• Add $\neg \beta$ to KB, show this leads to empty (False. Proof by contradiction)
• Everything needs to be in CNF
• Example KB:
  - $B_{1,1} \leftrightarrow (P_{1,2} \lor P_{2,1})$
  - $\neg B_{1,1}$
• Example query: $\neg P_{1,2}$
Resolution Preprocessing

• Add \( \neg \beta \) to KB, convert to CNF:
  
  a1: \((\neg B_{1,1} \lor P_{1,2} \lor P_{2,1})\)
  
  a2: \((\neg P_{1,2} \lor B_{1,1})\)
  
  a3: \((\neg P_{2,1} \lor B_{1,1})\)
  
  b: \(\neg B_{1,1}\)
  
  c: \(P_{1,2}\)

• Want to reach goal: empty
Resolution

• Take any two clauses where one contains some symbol, and the other contains its complement (negative)
  \[ PV \lor Q \lor R \lor \lnot Q \lor S \lor T \]

• Merge (resolve) them, throw away the symbol and its complement
  \[ P \lor R \lor S \lor T \]

• If two clauses resolve and there’s no symbol left, you have reached empty (False). \( KB \models \beta \)

• If no new clauses can be added, KB does not entail \( \beta \)
Resolution Example

a1: (¬B_{1,1} ∨ P_{1,2} ∨ P_{2,1})
a2: (¬P_{1,2} ∨ B_{1,1})
a3: (¬P_{2,1} ∨ B_{1,1})
b: ¬B_{1,1}
c: P_{1,2}
Resolution Example

a1: (¬B_{1,1} \lor P_{1,2} \lor P_{2,1})
a2: (¬P_{1,2} \lor B_{1,1})
a3: (¬P_{2,1} \lor B_{1,1})
b: ¬B_{1,1}
c: P_{1,2}

Step 1: resolve a2, c: \text{ } B_{1,1}

Step 2: resolve above and b: \text{ } \text{empty}
Q 2.1: Which has more rows: a truth table on $n$ symbols, or a joint distribution table on $n$ binary random variables?

• A. Truth table
• B. Distribution
• C. Same size
• D. It depends
Q 2.1: Which has more rows: a truth table on $n$ symbols, or a joint distribution table on $n$ binary random variables?

- A. Truth table
- B. Distribution
- C. Same size
- D. It depends
First Order Logic (FOL)

Propositional logic has some limitations
• Ex: how to say “all squares have four sides”
• No context, hard to generalize; express facts

FOL is a more expressive logic; works over
• Facts, Objects, Relations, Functions
First Order Logic Syntax

• **Term**: an object in the world
  – **Constant**: Alice, 2, Madison, Green, ...
  – **Variables**: x, y, a, b, c, ...
  – **Function**(term$_1$, ..., term$_n$)
    • Sqrt(9), Distance(Madison, Chicago)
    • Maps one or more objects to another object
    • Can refer to an unnamed object: LeftLeg(John)
    • Represents a user defined functional relation

• A **ground term** is a term without variables.
FOL Syntax

• **Atom**: smallest T/F expression
  – **Predicate**\((\text{term}_1, ..., \text{term}_n)\)
    • Teacher(Jerry, you), Bigger(\text{sqrt}(2), x)
    • Convention: read “Jerry (is) Teacher(of) you”
    • Maps one or more objects to a truth value
    • Represents a user defined relation
  – \text{term}_1 = \text{term}_2
    • Radius(Earth)=6400km, 1=2
    • Represents the equality relation when two terms refer to the same object
FOL Syntax

• **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\land \lor \neg \Rightarrow \Leftrightarrow$
    - Less(x,22) $\land$ Less(y,33)
  - Complex sentence using quantifiers $\forall, \exists$

• Sentences are evaluated under an interpretation
  – Which objects are referred to by constant symbols
  – Which objects are referred to by function symbols
  – What subsets defines the predicates
FOL Quantifiers

• Universal quantifier: $\forall$
• Sentence is true **for all** values of x in the domain of variable x.

• Main connective typically is $\Rightarrow$
  – Forms if-then rules
  – “all humans are mammals”
    $$\forall x \text{ human}(x) \Rightarrow \text{mammal}(x)$$
  – Means if x is a human, then x is a mammal
FOL Quantifiers

- Existential quantifier: $\exists$
- Sentence is true for some value of $x$ in the domain of variable $x$.

- Main connective typically is $\land$
  - “some humans are male”
    \[ \exists x \; \text{human}(x) \land \text{male}(x) \]
  - Means there is an $x$ who is a human and is a male