CS639: Algorithmic Game Theory & Learning

University of Wisconsin–Madison, Spring 2026 *Instructor: Kirthevasan Kandasamy*

Homework 0. Due 01/30/2025, 11.59 pm

Homework 0 serves as a diagnostic tool to assess your background knowledge and mathematical maturity for this course. It also familiarizes you with course expectations as you advance to more complex topics. While you are not expected to know all the solutions right away, you should be able to solve most of the questions with reasonable effort, using references as needed. Future homework assignments will be *significantly* more challenging than Homework 0.

Instructions:

1. Homework is due on Canvas by 11:59 pm on the due date. Please plan to submit well before the deadline. Refer to the course website for policies on late submission.
2. Homework must be typeset using appropriate software, such as LaTeX. Handwritten and scanned submissions will **not** be accepted.
3. Your solutions will be evaluated on correctness, clarity, and conciseness.
4. Unless otherwise specified, you may use any result we have already discussed in class. Clearly state which result you are using.
5. If you use any external references, please cite them in your submission.
6. **Collaboration:** You may collaborate in groups of size up to 3 to solve problems 2 and 3. If you collaborate, please indicate your collaborators at the beginning of the homework. Even if you collaborate, *you must write the solution in your own words*. You may **not** collaborate on problem 1.

1 Mathematics background

*You may **not** collaborate on Problem 1.*

1. **[1 pts]** Let m, n be positive integers. Consider the matrices and vectors $A \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$. Which of the following is $x^\top Ay$ equal to?

   (i) $\sum_{i=1}^{\min(m,n)} \sum_{j=1}^{\min(m,n)} A_{i,j} x_i y_j$ *(ii)* $\sum_{i=1}^{\min(m,n)} \sum_{j=1}^{\min(m,n)} A_{i,j} x_j y_i$ *(iii)* $\sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} x_i y_j$ *(iv)* $\sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} x_j y_i$

   Here, $A_{i,j}$ refers to the element in the i^{th} row and j^{th} column of A, x_i refers to the i^{th} element of the vector x, and y_j refers to the j^{th} element of the vector y.

2. **[2 pts]** Let $x \in \mathbb{R}$ and define $f(x) = x^2 - 4x - 11$.
   (a) Compute $\min_{x\in\mathbb{R}} f(x)$.
   (b) Compute $\min_{x\in[0,1]} f(x)$.

3. **[2 pts]** Consider the following joint probability table where both A and B are binary random variables:

A	B	$P(A,B)$
0	0	0.3
0	1	0.1
1	0	0.2
1	1	0.4

   (a) What is $P(A = 0|B = 1)$?
   (b) What is $P(A = 1 \text{ or } B = 1)$?

4. **[2 pts]** State if the following sentences on probability are True or False.
   (a) For any events A, B, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   (b) For any events A, B, we have $P(A \cap B) = P(A|B)P(B)$.
   (c) For any events A, B, we have $P(A|B)P(A) = P(B|A)P(B)$.
   (d) If A and B are independent events, then A^c and B^c are independent.

5. Let $\mathcal{X}, \mathcal{Y}$ be sets and $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, $g : \mathcal{X} \to \mathbb{R}$, $h : \mathcal{Y} \to \mathbb{R}$ be real-valued functions.
   (a) **[1 pts]** What is the negation of the following statement?

   For all $x \in \mathcal{X}$, there exists $y \in \mathcal{Y}$ such that $f(x, y) > 0$.

   (b) **[1 pts]** What is the contrapositive of the following statement?

   If there exists $x \in \mathcal{X}$, such that $g(x) > 0$, then for all $y \in \mathcal{Y}$ we have $h(y) > 0$.

   (c) **[1 pts]** What is the converse of the following statement?

   If there exists $x \in \mathcal{X}$, such that $g(x) > 0$, then there exists $y \in \mathcal{Y}$ such that $h(y) > 0$.

6. **[2 pts]** Let $A \in \mathbb{R}^{n\times m}$ be a matrix. Prove the following statement,

$$\max_i \min_j A_{i,j} \le \min_j \max_i A_{i,j}.$$

   Here, $\max_i \min_j A_{i,j}$ is the value obtained when we first compute the minimum value of each row of A, and then take the maximum of these values. Similarly, $\min_j \max_i A_{i,j}$ is the value obtained when we first compute the maximum value of each column of A, and then take the minimum of these values.

7. **[2 pts]** Let $G = (V, E)$ be a finite directed graph in which every vertex has exactly one outgoing edge. Show that G contains at least one directed cycle.

8. Let $G = (V_1 \cup V_2, E)$ be a finite (undirected) bipartite graph. A *matching* is a set of edges $M \subseteq E$ such that no two edges in M share a common endpoint.

   (a) **[1 pts]** Prove that if M is a matching, then $|M| \leq \min\{|V_1|, |V_2|\}$.

   (b) **[2 pts]** A matching M is called *maximal* if no edge can be added to M without violating the matching property, and *maximum* if it has the largest possible cardinality among all matchings. Give an example of a bipartite graph and a matching that is maximal but not maximum.

   (c) **[1 pts]** Suppose every vertex in V_1 has degree at least 1. Does this guarantee the existence of a matching that covers all vertices in V_1? Either prove the statement or give a counterexample.

2 A resource usage game

In this question, we will analyze a simple two-player game and study its equilibrium. When doing so, it is common to consider the *utility* of an agent for a certain outcome, in order to predict agent behavior. The utility measures the degree of satisfaction for the agent, and each agent aims to maximize their utility.

Alice and Bob have access to a cluster of 100 CPUs. They can use any number of CPUs in this cluster to run their jobs. However, if they use too many CPUs, the performance of the jobs will suffer due to interference between the CPUs and contention for shared communication channels. Suppose Alice is using x_A CPUs and Bob is using x_B CPUs (we will assume that they can use fractional CPUs). If $x_A + x_B < 100$, we will assume that the utilities of Alice (u_A) and Bob (u_B), take the following forms:

$$u_A(x_A, x_B) = x_A\,(100 - x_A - x_B)\,, \qquad u_B(x_A, x_B) = x_B\,(100 - x_A - x_B)\,,$$

On the other hand, if $x_A + x_B \geq 100$, then $u_A(x_A, x_B) = u_B(x_A, x_B) = 0$.

1. **[1 pts]** Suppose Bob is using x_B resources. How many resources should Alice use to maximize her utility? (This shows that it is not always beneficial for a user to simply use a large number of CPUs.)

   Hint: For this question, you need to compute $\mathrm{argmax}_{x_A \in [0, 100 - x_B]}\, u_A(x_A, x_B)$. You may use the fact that u_A as a quadratic polynomial in x_A.

2. **[2 pts]** Suppose both Alice and Bob are using 25 CPUs each.

   (a) Compute the utilities of both users when $x_A = x_B = 25$.

   (b) Say Bob continues to use 25 CPUs. Based on your answer to part 1, is it better for Alice to continue using 25 CPUs, decrease the amount of CPUs she uses, or increase the amount of CPUs she uses? (By symmetry, you can draw a similar conclusion about Bob.)

3. **[2 pts]** Suppose both Alice and Bob are using 33.33 CPUs each.

   (a) Compute the utilities of both users when $x_A = x_B = 33.33$.

   (b) Using your answer to part 1, show that this is an equilibrium. That is, neither Alice nor Bob have incentive to increase or decrease the amount of CPUs they are using.

4. **[1 pts]** Using the results of parts 2 and 3, describe how this is an instance of the *tragedy of the commons*. You may refer to the overview slides from the first lecture for a definition.

3 Bargaining for resource allocation

Alice and Bob wish to purchase CPUs. On her own, Alice has enough funding to purchase 64 CPUs, whereas, on his own, Bob has enough to only purchase 16 CPUs. If they purchase together, they can use bulk discounts to purchase

100 CPUs. Clearly, it is sensible for them to purchase together. However, they should decide how to split the 100 CPUs between them after they purchase it. We will assume fractional allocations are possible.

In this question, the utility of Alice and Bob takes the following form, when they have been allocated x_A and x_B resources respectively[1]. (Note that this is different to the form used in problem 2.) We have:

$$u_A(x_A) = \sqrt{x_A}, \qquad u_B(x_B) = \sqrt{x_B},$$

Any method to determing the allocations should satisfy $x_A \geq 64$ and $x_B \geq 16$. Otherwise, at least one of the users will refuse to work together and purchase on their own. This requirement is called *individual rationality*.

1. **[2 pts]** Student 1 proposes that we should choose allocations to maximize the sum of utilities. Specifically, they propose choosing x_A, x_B via the following optimization problem:

$$\text{maximize} \quad u_A(x_A) + u_B(x_B); \qquad \text{subject to}\, x_A + x_B = 100.$$

   Compute the allocation according to the above scheme and verify that it is not individually rational.

   Hint: There are many ways to solve this problem, one of which is the method of Lagrange multipliers.

2. **[2 pts]** Student 2 modifies student 1's suggestion to explicitly encode the individual rationality constraints. She proposes the following optimization problem.

$$\text{maximize} \quad u_A(x_A) + u_B(x_B); \qquad \text{subject to}\, x_A + x_B = 100,\ x_A \geq 64,\ x_B \geq 16.$$

   Compute the allocation according to the above scheme.

   Hint: Instead of solving this explicitly, you can use the diminishing returns property of the square-root function to determine the optimal value. For this, suppose we have already allocated 64 to Alice and 16 to Bob to satisfy individual rationality and wish to allocate the remaining 20. If we are to allocate one more CPU to maximize $u_A(x_A) + u_B(x_B)$, who would you allocate it to?

3. **[2 pts]** While student 2's proposition satisfies individual rationality, Alice does not do significantly better than when she is working on her own (you should see that this is true when you solve part 2). Hence, Student 3 proposes that both Alice and Bob should do equally better than when working on their own. She proposes choosing the allocation to satisfy

$$u_A(x_A) - u_A(64) = u_B(x_B) - u_B(16), \quad \text{and} \quad x_A + x_B = 100.$$

   Compute the allocation according to the above scheme. It is not necessary to explicitly encode the individual rationality constraints as they will automatically be satisfied by the solution.

4. **[2 pts]** Student 4 proposes that we choose allocations via the following optimization scheme.

$$\text{maximize} \quad \log(u_A(x_A) - u_A(64)) + \log(u_B(x_B) - u_B(16)), \qquad \text{subject to}\, x_A + x_B = 100.$$

   Compute the allocation according to the above scheme. It is not necessary to explicitly encode the individual rationality constraints as this is ensured by non-negativity of the arguments to the $\log$ function.

   N.B: This solution is called Nash bargaining. Later in class we will see that it has several desirable properties.

   Hint: You may use the method of Lagrange multipliers to solve this. If necessary, you may use software such as Mathematica to find the roots of a quartic polynomial.

[1] The square-root utility captures situations where it is beneficial to users to have more resources, but there is diminishing returns to having an additional CPU when they already have many CPUs. To see this, you can compute the increase in utility due to receiving an additional CPU, when they already have only one CPU vs when they already have 10 CPUs.