

CS639: Algorithmic Game Theory & Learning
University of Wisconsin–Madison, Spring 2026

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Homework 1.

Due 02/13/2026, 11.59 pm

Instructions:

1. Homework is due on Canvas by 11:59 pm on the due date. Please plan to submit well before the deadline. Refer to the course website for policies on late submission.
2. Homework must be typeset using appropriate software, such as \LaTeX . Handwritten and scanned submissions will **not** be accepted.
3. Your solutions will be evaluated on correctness, clarity, and conciseness.
4. Unless otherwise specified, you may use any result we have already discussed in class. Clearly state which result you are using.
5. If you use any external references, please cite them in your submission.
6. **Collaboration:** You may collaborate in groups of size up to 3. If you collaborate, please indicate your collaborators at the beginning of the homework. Even if you collaborate, *you must write the solution in your own words.*

1 On dominant strategy and Nash equilibria

- State if the following statements are true (**T**) or false (**F**) in a finite normal form game. If you indicate **T**, then prove the statement. If you indicate **F**, then provide a counter-example.
 - [2 pts]** Every dominant strategy equilibrium is a Nash equilibrium.
 - [2 pts]** Every Nash equilibrium is a dominant strategy equilibrium.
 - [2 pts]** If there exists a dominant strategy for an agent, it is necessarily unique.
- [4 pts]** (*Alternative definition for dominant strategies.*) Consider a normal form game where agent i has an action space \mathcal{A}_i and a strategy space $\mathcal{S}_i = \Delta^{\mathcal{A}_i}$. In class, we defined a dominant strategy as a strategy $s_i \in \mathcal{S}_i$ which maximizes agent i 's utility regardless of the other player's strategies:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \text{for all } s'_i \in \mathcal{S}_i, \text{ for all } s_{-i} \in \mathcal{S}_{-i}. \quad (1)$$

Alternatively, we can define a dominant strategy as a strategy $s_i \in \mathcal{S}_i$ which maximizes agent i 's utility regardless of the other player's *actions*:

$$u_i(s_i, a_{-i}) \geq u_i(s'_i, a_{-i}) \quad \text{for all } s'_i \in \mathcal{S}_i, \text{ for all } a_{-i} \in \mathcal{A}_{-i}. \quad (2)$$

Show that (1) and (2) are equivalent.

- [6 pts]** (*Existence of Nash equilibria in infinite action spaces.*) In class, we discussed John Nash's seminal result, which showed the existence of a Nash equilibrium for finite games. In this question, you will show that Nash equilibria may or may not exist in infinite games, (i.e., where $|\mathcal{A}_i|$ is infinite). In particular, we will consider two-player games where the action spaces are the natural numbers, i.e $\mathcal{A}_1 = \mathcal{A}_2 = \mathbb{N} = \{1, 2, \dots\}$. Player i 's strategy $s_i : \mathbb{N} \rightarrow [0, 1]$ is a sequence satisfying $\sum_{a_i \in \mathbb{N}} s_i(a_i) = 1$.

Design *bounded* utility functions u_1, u_2 , where $u_i : \mathbb{N} \times \mathbb{N} \rightarrow [0, 1]$, so that,

- there exists at least one Nash equilibrium in the game.
- no Nash equilibria exist in the game.

In both cases, you should give your answer with a proof.

2 Computing Nash equilibria in simple games

- (*Coordination Game, a.k.a Battle of the Sexes*) Two friends Alice and Bob are planning to attend an event together. Alice wishes to attend a ballet (B) over a lecture (L), whereas Bob wishes to attend the lecture. However, they would both like to spend time with each other, and have no value to attending the event on their own. As they cannot communicate with each other, they have to make this decision independently. The utility function for this game is given by the following matrix.

		Bob	
		Ballet (B)	Lecture (L)
Alice	Ballet (B)	(10, 7)	(0, 0)
	Lecture (L)	(0, 0)	(7, 10)

- [5 pts]** (*Nash equilibria*) Apply the indifference principle to find all Nash equilibria in this game. State the expected utility for each player at each Nash equilibrium.
- [2 pts]** (*PoA/PoS*) Compute the price of anarchy (PoA) and price of stability (PoS) defined as shown below. Here $W(s) = u_1(s) + u_2(s)$ denotes the welfare.

$$\text{PoA} = \frac{\max_s W(s)}{\min_{s; s \text{ is a NE}} W(s)}, \quad \text{PoS} = \frac{\max_s W(s)}{\max_{s; s \text{ is a NE}} W(s)}.$$

2. (*Odd vs Even*) Alice and Bob play the following game. Both players simultaneously call out one of the numbers $\{1, 2\}$. If the sum is even, Bob pays Alice the sum, whereas if the sum is odd, Alice pays Bob the sum. For example, if Alice calls “1”, and Bob calls “2”, then Alice pays Bob \$3.

- (a) [1 pts] (*Payoff matrix*) Write the payoff matrix for this game.
- (b) [5 pts] (*Nash equilibria*) Apply the indifference principle to find all Nash equilibria in this game. State the expected utility for each player at each Nash equilibrium.

3 Computing Nash equilibria in more complex games

1. [10 pts] (*Pollution game*) Three firms, P1, P2, and P3, can choose to either clean (C) or pollute (P) a river. They incur a cost of 1 to clean it, but it is free to pollute. However, if two or more pollute, the water in the lake is useless and each firm incurs a cost of 4 to obtain the water from elsewhere (in addition to the cost for cleaning if they choose to do so). If at most one firm pollutes, the water is still usable and the firms incur no additional cost. The payoff matrix if P3 chooses C is as follows:

		P2	
		Clean (C)	Pollute (P)
P1	Clean (P)	(-1, -1, -1)	(-1, 0, -1)
	Pollute (P)	(0, -1, -1)	(-4, -4, -5)

The payoff matrix if P3 chooses P is as follows:

		P2	
		Clean (C)	Pollute (P)
P1	Clean (C)	(-1, -1, 0)	(-5, -4, -4)
	Pollute (P)	(-4, -5, -4)	(-4, -4, -4)

Find all Nash equilibria in this game. How many Nash equilibria are there?

Hint: You are encouraged to read Example 4.3.2 in KP (pages 82–85) or a similar example we saw in class before attempting this question.

2. [10 pts] (*Rock–paper–scissors*) Two players, P1 and P2, are playing the classical rock–paper–scissors game. Each player can choose between one of three actions, rock (R), paper (P), or scissors (S), and will receive a reward based on the action chosen by the other player. The utility function for this game is given by the matrix below. Find all Nash equilibria in this game. How many Nash equilibria are there?

		P2		
		Rock (R)	Paper (P)	Scissors (S)
P1	Rock (R)	(0, 0)	(-1, +1)	(+1, -1)
	Paper (P)	(+1, -1)	(0, 0)	(-1, +1)
	Scissors (S)	(-1, +1)	(+1, -1)	(0, 0)

3. [10 pts] (“2” vs “3”) A set of n agents play the following game. Each agent should simultaneously choose either “2” or “3”. If some of the agents choose “2”, then those who chose “2” will receive \$2, while the agents who chose “3” will receive \$3. However, if none of the agents choose “2”, then no one receives any money. An agent’s utility function in this game can be written as follows,

$$u_i(\text{“2”}, a_{-i}) = 2 \text{ for all } a_{-i}, \quad u_i(\text{“3”}, a_{-i}) = \begin{cases} 3 & \text{if “2”} \in a_{-i}, \\ 0 & \text{if “2”} \notin a_{-i}. \end{cases}$$

This game has $2^n - 1$ Nash equilibria. Find all of them.

4 Tighter bounds on the PoA for the market sharing game

Recall the market sharing game we discussed in class. There are n sport teams (players) and a set of K cities. We will assume $K \geq n$. The value (e.g population) of city $k \in [K]$ is v_k . We will assume, without loss of generality, that $v_1 \geq v_2 \geq \dots \geq v_K$. Each team should choose a city, hence the action space for each team $i \in [n]$ is $\mathcal{A}_i = [K]$. Let $N_k(a) = |\{i \in [n]; a_i = k\}|$ be the number of teams in city k , under an action profile a . All teams that choose a city, will share the value equally. Hence, the utility of a team i is $u_i(a) = \frac{v_{a_i}}{N_{a_i}(a)}$.

For any $J \subset [K]$, let $V(J) = \sum_{j \in J} v_j$ be the sum of values in J . Let $J(a) = \{k \in [K], \exists i \in [n], \text{ s.t } a_i = k\}$ be the cities selected by an action profile a . Then, we can write the welfare W as

$$W(a) = \sum_{i=1}^n u_i(a) = \sum_{j \in J(a)} v_j = V(J(a)).$$

The optimal welfare is $V(J^{\text{OPT}})$ where $J^{\text{OPT}} = \{1, \dots, n\}$ is the first n cities (recall that $v_1 \geq v_2 \geq \dots \geq v_K$). In part 1, you will show that this game is a potential game. Hence, it has at least one pure Nash equilibrium. We can therefore define the price of anarchy for this game as,

$$\text{PoA}_{\text{PNE}} = \frac{V(J^{\text{OPT}})}{\min_{a; a \text{ is a pure NE}} V(J(a))}$$

In this question, you will show that, for any market sharing game, $\text{PoA}_{\text{PNE}} \leq 2 - 1/n$.

1. **[3 pts]** (*Potential game*) Show that ψ , defined below, is a potential function for this *utility maximization* game.

$$\psi(a) = \sum_{k=1}^K \sum_{\ell=1}^{N_k(a)} \frac{v_k}{\ell}.$$

2. **[4 pts]** (*Lower bounding agent utilities*) Let $a = (a_1, \dots, a_n)$ be a pure Nash equilibrium, and let $m = |J(a)|$ be the number of cities be selected in action profile a . Show that if $J(a) \neq J^{\text{OPT}}$, then, for any agent i , we have

$$u_i(a_i, a_{-i}) \geq \frac{V(J^{\text{OPT}}) - V(J(a))}{n - m}.$$

N.B. In class, we saw that lower bounding the utility for deviations from a Nash equilibrium is one way to obtain a factor 2 bound on the price of anarchy. We will now see that lower bounding the agent's utility, and hence the welfare, at a Nash equilibrium, also allows us to bound the price of anarchy.

3. **[4 pts]** (*PoA bound*) Using part 2 or otherwise, show that for any market sharing game, $\text{PoA}_{\text{PNE}} \leq 2 - 1/n$.
4. **[6 pts]** (*Tightness*) Show that this bound cannot be improved in general.

Hint: One option is to prove the following statement: For any sufficiently small $\epsilon > 0$, there exists a market sharing game with n players such that $\text{PoA}_{\text{PNE}} = 2 - 1/n - \epsilon$.