CS639: Algorithmic Game Theory & Learning

University of Wisconsin–Madison, Spring 2026 *Instructor: Kirthevasan Kandasamy*

Homework 2. Due 02/27/2026, 11.59 pm

Instructions:

1. Homework is due on Canvas by 11:59 pm on the due date. Please plan to submit well before the deadline. Refer to the course website for policies on late submission.
2. Homework must be typeset using appropriate software, such as LaTeX. Handwritten and scanned submissions will **not** be accepted.
3. Your solutions will be evaluated on correctness, clarity, and conciseness.
4. Unless otherwise specified, you may use any result we have already discussed in class. Clearly state which result you are using.
5. If you use any external references, please cite them in your submission.
6. **Collaboration:** You may collaborate in groups of size up to 3. If you collaborate, please indicate your collaborators at the beginning of the homework. Even if you collaborate, *you must write the solution in your own words*.

1 Computing safe strategies in simple games

Recall the following games from Problem 1 in Homework 1, for which you computed the Nash equilibria.

1. *(Coordination Game, a.k.a Battle of the Sexes)* Two friends Alice and Bob are planning to attend an event together. Alice wishes to attend a ballet (B) over a lecture (L), whereas Bob wishes to attend the lecture. However, they would both like to spend time with each other, and have no value to attending the event on their own. As they cannot communicate with each other, they have to make this decision independently. The utility function for this game is given by the following matrix.

		Bob	
	Actions	Ballet (B)	Lecture (L)
Alice	Ballet (B)	(10, 7)	(0, 0)
	Lecture (L)	(0, 0)	(7, 10)

   (a) **[4 pts]** *(Safe strategies)* Compute the safe strategies for both players.

   (b) **[1 pts]** If each player follows their safe strategy, does it constitute a Nash equilibrium?

2. *(Odd vs Even)* Alice and Bob play the following game. Both players simultaneously call out one of the numbers $\{1, 2\}$. If the sum is even, Bob pays Alice the sum, whereas if the sum is odd, Alice pays Bob the sum. For example, if Alice calls "1", and Bob calls "2", then Alice pays Bob \$3.

   (a) **[4 pts]** *(Safe strategies)* Compute the safe strategies for both players.

   (b) **[1 pts]** If each player follows their safe strategy, does it constitute a Nash equilibrium? Explain the discrepancy with your answer in part 1b.

2 Short problems on Chapters 1 and 2

1. **[8 pts]** *(Uniqueness of potential function)* Show that the potential function is unique up to an additive constant. That is, if ψ and $\widetilde{\psi}$ are potential functions of this game, then show that there exists some constant C such that, for all action profiles a, we have $\psi(a) = \widetilde{\psi}(a) + C$. Here, C may possibly depending on ψ and ψ' (but not on the action profile a).

   Hint. One option is to use an inductive argument to show that for any two action profiles a, a', we have $\psi(a) - \widetilde{\psi}(a) = \psi(a') - \widetilde{\psi}(a')$.

2. **[3 pts]** *(Every CE is a CCE)* Prove that, in any n player normal form game, every correlated equilibrium is also a coarse correlated equilibrium.

   Hint. Recall the tower property of conditional expectation: for any two random variables X, Y, we have $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$.

3. **[6 pts]** *(Uniqueness of Value/NE)* Consider the following two statements about two player games. State if the statement is true or false. Support your answer with either a proof/explanation or counter examples.

   (a) Every two player *general* sum game has a unique Nash equilibrium.

   (b) In every Nash equilibrium of a two player *general* sum game, player 1 has the same expected utility.

   (c) Every *two* player zero sum game has a unique Nash equilibrium.

   (d) In every Nash equilibrium of a *two* player zero sum game, player 1 has the same expected utility.

3 Resource usage game revisited

In this question, we will study the resource usage game from Homework 0 in further detail. You are encouraged to read Example 4.5.1 in KP before attempting this question.

There are n users (players) sharing a resource of size 1. Each player can choose some amount of resource $a_i \in [0,1]$ that she wishes to use. Hence the action space is $\mathcal{A}_i = [0,1]$. For instance, if the resources are CPUs, a_i indicates the fraction of CPUs used by each player. Player i's utility u_i is given by

$$u_i(a) = u_i(a_1, \ldots, a_n) = \begin{cases} a_i \left(1 - \sum_{j=1}^n a_j\right) & \text{if } \sum_{i=1}^n a_i < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that while a player generally stands to benefit by using more resources, her utility may also decrease when the total resource usage is very high. This is true in many real-world resource-sharing settings, such as shared compute clusters and telecommunication channels.

1. **[6 pts]** *(Pure Nash equilibria)* Find *all* pure Nash equilibria in this game.
2. **[5 pts]** *(Price of stability)* The price of stability $\mathrm{PoS}_{\mathrm{PNE}}$ of this game is defined as the ratio between the social welfare at the best pure Nash equilibrium (PNE) and the optimal social welfare. That is,

   $$\mathrm{PoS}_{\mathrm{PNE}} = \frac{\max_{a; a \text{ is a PNE}} W(a)}{\max_a W(a)}$$

   Show that, for this game, $\mathrm{PoS}_{\mathrm{PNE}} \in \mathcal{O}(1/n)$.
3. **[6 pts]** *(Potential game, exercise 4.23 in KP)* Show that this is a potential game.

   Hint: To build intuition, you can consider applying the separability lemma when $n = 2$ to derive a suitable potential function for two players. Then, see how this can be generalized to arbitrary n.
4. **[3 pts]** *(Convergence of best response dynamics)* Consider the following modified version of best response dyanamics (BRD): on each round, instead of picking an arbitrary player who could improve her utility, we randomly sample a player who could improve her utility. Then we change her action to one which maximizes her utility, while keeping the others' actions fixed. Show that this version of BRD converges to a pure Nash equilibrium in this game.

   Hint: In class, we showed that BRD converges in a *finite* potential game, so you cannot directly apply this result here. You may use the following well-known fact about coordinate ascent of concave functions.

   Coordinate ascent. Let $f : [0,1]^d \to \mathbb{R}$ be a function. Coordinate ascent refers to the following iterative procedure for maximizing f. We will start with some initial (round 0) value $x^{(0)} \in [0,1]^d$. Then, on each round we choose a random coordinate $i \in [d]$ and update $x_i^{(t-1)}$ to the maximizer x_i' of f while keeping the other coordinates fixed, i.e $x_i' = \mathrm{argmax}_{x_i \in [0,1]} f(x_1^{(t-1)}, \ldots, x_i, \ldots, x_d^{(t-1)})$. Precisely, $x_i^{(t)} \leftarrow x_i'$ and $x_j^{(t)} \leftarrow x_j^{(t-1)}$ for all $j \neq i$. If f is a concave function, then coordinate ascent will converge to the maximum value.

4 Finding an approximate Nash equilibrium via best response dynamics

Consider an n player potential game with potential function ψ. Suppose that the action space for each player has size at most m, i.e $|\mathcal{A}_i| \leq m$. Let $\mathcal{A} = \times_{i=1}^n \mathcal{A}_i$ be the space of action profiles. In the worst case, best response dynamics (BRD) may require n^m computations of best response actions, i.e computations where you keep the actions of other players fixed and find the best response for a single player. (You are encouraged to think about why this is the case.) This exponential complexity can be computationally prohibitive in some settings. In this question, we will modify BRD to find an *approximate* equilibrium.

First, we will define an ϵ–aproximate pure Nash equilibrium (ϵ–APNE). An action profile $a \in \mathcal{A}$ is an ϵ–APNE if $u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) - \epsilon$ for all actions $a_i' \in \mathcal{A}_i$ and for all players $i \in [n]$. That is, no player stands to gain more than ϵ by deviating from a. (As an aside, one can similarly define ϵ–aproximate Nash equilibria for mixed strategies.)

1. **[9 pts]** *(Algorithm design)* Propose an algorithm, which takes ϵ as a parameter, and is guaranteed to return an ϵ–APNE for any potential game, including games with infinite action spaces. It should require at most $\mathcal{O}\left(\frac{\theta}{\epsilon}\right)$ computations of best response actions, where θ may depend on any problem parameters, the potential function, or any initial conditions of your algorithm, but *not* on ϵ. You need to prove that your algorithm returns an ϵ–APNE and bound its computational complexity.

2. *(Bounding θ.)* Using the algorithm above, bound θ for the following games.

   (a) **[3 pts]** The congestion game we discussed in class (you may also refer to example 4.4.1 in KP). Assume that there are n players, R edges in the road network, and let $\ell_{\max} = \max_{r,N} \ell_r(N)$ be the maximum latency on any road. You should bound θ in terms of n, R, and $\ell_{\max}$.

   (b) **[3 pts]** The consensus game we discussed in class (you may also refer to example 4.4.7 in KP). You should bound θ in terms of the parameters of the graph $G = (V, E)$.

   (c) **[3 pts]** The game in problem 3.

5 PNE vs MNE vs CE vs CCE

Recall the load balancing game we discussed in class. There are $n = 4$ players sharing $m = 6$ computers. Each player should decide which of the m machines to run their job on. Hence, $\mathcal{A}_i = [m]$. The load $\ell_j(a)$ on machine j under an action profile a is is the number of players on that machine, i.e $\ell_j(a) = |\{i \in [n]; a_i = j\}|$. The cost of player i is the load of the machine she is running her job on, i.e $c_i(a) = \ell_{a_i}(a)$. Let $\mathrm{cost}(a) = \sum_{i=1}^{n} c_i(a)$ be the total cost. (This is a simplified version of the load balancing game we saw in class where all players have equal job size.)

1. **[3 pts]** *(Pure Nash equilibrium (PNE).)* Identify at least one pure Nash equilibrium in this game. What is the total cost (sum of player costs) in this PNE?

2. **[3 pts]** *(Mixed Nash equilibrium (MNE).)* Identify at least one MNE that is not a PNE. What is the expected total cost in this MNE?

3. *(Correlated equilibrium (CE).)* A cluster manager decides to allocate the machines as follows. He chooses one machine uniformly at random, and then assigns two players (also chosen uniformly at random) to this machine. The remaining two players receive their own machine, both chosen uniformly at random from the remaining five. We will show that this is a CE but not a Nash equilibrium.

   To show that it is a CE, let s be the joint distribution induced by the cluster manager's allocation scheme. Show that $\mathbb{E}_{a\sim s}[c_i(a)|a_i] \le \mathbb{E}_{a\sim s}[c_i(a'_i, a_{-i})|a_i]$ for all $i \in [n]$, and all $a_i, a'_i \in [m]$, where $a'_i \neq a_i$. In other words, knowing the distribution s *a priori*, and then only the machine a_i assigned to her after sampling $a \sim s$, the player does not benefit by choosing a different machine a'_i.

   (a) **[1 pts]** First show that $\mathbb{E}_{a\sim s}[c_i(a)|a_i] = 3/2$.

   (b) **[3 pts]** Next, show that for any other action a'_i player i may consider *after a_i is revealed to her*, we have $\mathbb{E}[c_i(a'_i, a)|a_i] = 3/2$. Thereby conclude that this is a CE.

   (c) **[1 pts]** What is the expected total cost in this CE?

   (d) **[1 pts]** Explain why this is not a Nash equilibrium (pure or mixed).

4. **[6 pts]** *(Coarse Correlated equilibrium (CE).)* A cluster manager decides to allocate the machines as follows. He first chooses either the machines $\{1, 2, 3\}$ or $\{4, 5, 6\}$ with equal probability. From this chosen set of three machines, he chooses one machine uniformly at random, and then assigns two players (also chosen uniformly at random) to this machine. The other two players receive the remaining two machines from the initially chosen set of three machines. Show that this is a CCE but not a CE. What is the expected total cost in this CCE?

   To show that it is a CCE, let s be the joint distribution induced by the cluster manager's allocation scheme. Show that $\mathbb{E}_{a\sim s}[c_i(a)] \le \mathbb{E}_{a\sim s}[c_i(a'_i, a_{-i})]$ for all $i \in [n]$, and all $a'_i \in [m]$. In other words, knowing the distribution s *a priori*, but prior to knowing the machine assigned to her, the player does not benefit by unilaterally choosing a machine a'_i on her own.