CS639: Algorithmic Game Theory & Learning

Chapter 7: Mechanism Design with Money

Kirthevasan Kandasamy

UW-Madison

Mechanism design without money

In Chapter 6, we studied mechanism design *without* money:

- Legal/ethical reasons for not using money: stable matching, kidney exchange.
- Practical reasons for not using money: resource allocation.

But money is a great way to (dis-)incentivize agents!

- Using money gives rise to more powerful/flexible mechanisms.
- We can "achieve more" when we use money.

Outline

1. Simple (single-item) auctions
2. The VCG mechanism
3. Truthfulness in single-parameter mechanisms
4. Revenue-optimal truthful auctions
5. Posted price mechanisms

Slides are intended as teaching aids only and do not include all material discussed in class. Students are strongly encouraged to attend lectures and take their own notes.

Ch 7.1: Simple Auctions

A set of buyers (a.k.a bidders, agents) participate in an auction. Let us consider two common auctions:

English auction.

- Buyers start at low prices and keep increasing their bids until one “winner” is left.
- The last buyer (one who is not outbid by others) is the winner. She pays last price she bid.
- Equivalently, the seller starts at zero, and increases price. Buyers can “bow out.” Last buyer (winner) pays the amount when the second-to-last buyer bows out.

Dutch auction.

- Seller starts at a high price and decreases gradually.
- The first buyer to signal that they will buy at a price (winner) gets the item and pays the amount.

Simple Auctions (cont'd)

Question 1. Why are these normal form games?

Question 2. What is the action space of the buyers?

Question 3. Which auction is easier to play as it is more likely to be truthful?

Direct Single Item Auctions

In this course, we will primarily focus on *direct* auctions.

Direct (a.k.a. "sealed bid") single-item auction. The seller solicits bids $b = \{b_i\}_{i \in [n]}$ from the n bidders and chooses a winner ω and a payment p the winner pays. That is,

$$\omega : \mathbb{R}_+^n \to [n], \qquad p : \mathbb{R}_+^n \to \mathbb{R}.$$

More generally, we can choose a probability distribution over the winner (ω assigns a probability of winning for each agent) and a price vector $p \in \mathbb{R}^n$:

$$\omega : \mathbb{R}_+^n \to \Delta([n]), \qquad p : \mathbb{R}_+^n \to \mathbb{R}^n.$$

N.B. English/Dutch auctions are not direct auctions. Direct auctions are considered more practical as they preserve privacy of the bids and as buyers do not need to be at the same place.

Direct Single Item Auctions (cont'd)

Let us consider two common direct auctions.

First price auction. The seller gives the item to the highest bidder, and charges the winner her bid:

$$\omega(b) = \arg\max_{i \in [n]} b_i, \qquad p_i(b) = \begin{cases} \max_{j \in [n]} b_j & \text{if } i = \omega(b), \\ 0 & \text{otherwise.} \end{cases}.$$

Second price (a.k.a Vickrey) Auction. Seller gives the item to the highest bidder, but charges her the *second highest* bid:

$$\omega(b) = \arg\max_{i \in [n]} b_i, \qquad p_i(b) = \begin{cases} \max_{j \neq \omega(b)} b_j & \text{if } i = \omega(b), \\ 0 & \text{otherwise.} \end{cases}.$$

Question: Which of these direct auctions (first or second price) is like the English auction and which of these is like the Dutch auction?

Properties of the second price auction

Suppose an agent's utility is her value minus the price she pays, *i.e.*, $u_i = v_i - p_i$. (We will define this more formally soon.)

Theorem (Informal). The second price auction is:

(i) Truthful (DSIC): Bidding truthfully is a dominant strategy for all agents.

(ii) Individually Rational (IR): The utility of a truthful agent is non-negative, *i.e.*, $u_i = v_i - p_i \geq 0$, so they do not pay more than they value the item.

(iii) Efficient: When all agents are truthful, it gives the item to the buyer who values it most.

Properties of the second price auction (cont'd)

Proof sketch. *(ii)* and *(iii)* are straightforward to see. For *(i)*, let us fix the bids of the others. Let $b^\star_{-i} = \max_{j \neq i} b_j$. Then i's utility is:

$$u_i = v_i - p_i = \begin{cases} v_i - b^\star_{-i} & \text{if } b_i > b^\star_{-i} \text{ (she wins the item)}, \\ 0 & \text{if } b_i < b^\star_{-i}. \end{cases}$$

If $v_i > b^\star_{-i}$, then she gets positive utility by bidding $b_i > b^\star_{-i}$ and 0 utility by bidding $b_i < b^\star_{-i}$.

If $v_i < b^\star_{-i}$, then she gets negative utility by bidding $b_i > b^\star_{-i}$ and 0 utility by bidding $b_i < b^\star_{-i}$.

In either case, bidding truthfully maximizes her utilty. □

Next, in Chapter 7.2, we will formalize these ideas and generalize them to beyond a single item.

On Auctions

Practical applications of auctions

1. Government auctions (*e.g.*, wireless spectrum auctions)
2. Sponsored search and ad auctions on the internet
3. Several more: eBay, art auctions at Sotheby's *etc*.

Limitations/criticisms of auctions

1. Not practical for all use cases. But,
   - we will look at practical alternatives.
   - auction theory can still inform pricing strategies in other types of markets.
2. Not always straightforward to relate value to money: quasilinear utility (value − price) equates *value* and *willingness to pay*, but a wealthy person might be willing to pay more for something, while a poorer person may "value" it more.

Ch 7.2: The VCG Mechanism

Example 1 (A shared communication channel, E.g., 15.1.3 in KP). A set of n agents are sharing a channel of capacity C.

- Agent i's data stream has bandwidth requirement y_i (known publicly), and this agent has value v_i (known privately) for getting that data through.

- We should choose a feasible set of streams S, i.e., $\sum_{i \in S} y_i \leq C$ so as to maximize the welfare:

$$W(S) = \sum_{i=1}^{n} \mathbb{1}(i \in S) \cdot v_i = \sum_{i \in S} v_i.$$

Question: How do you truthfully elicit the values and compute the optimal S?

- Naively asking them to report their value v_i as a bid b_i, and choose S to maximize the reported welfare $\sum_{i \in S} b_i$, each agent is incentivized to overbid.

Examples (cont'd)

Example 2 (Building roads, E.g., 16.1.2 in KP). The state wants to connect cities by building roads at $10M each. Cities A and B are already connected, but C is not. We wish to solicit bids from each city for how much they value a new road, so that:

- The welfare (value created minus cost) is maximized.
- The mechanism is truthful.

Suppose the valuations are as follows (known privately to the cities, but not the state):

	A–C only	B–C only	Both	None
v_A	5	0	5	0
v_B	0	5	5	0
v_C	9	9	15	0
v_0	-10	-10	-20	0
Welfare	4	4	5	0

We should choose "Both" to maximize welfare. But we wish to elicit these valuations truthfully.

A General Model for Multi-parameter Mechanism Design

Environment. We have a set of n agents, a mechanism designer, and a set of possible outcomes Ω.

- Agent i has a valuation function $v_i : \Omega \to \mathbb{R}_+$, where $v_i(\omega)$ denotes i's value for outcome ω. All else being equal, agents prefer higher valuations.

- Let $v_0 : \Omega \to \mathbb{R}$ be the mechanism designer's valuation function.

N.B. For simplicity, we will assume that agent valuations are always non-negative while a mechanism designer's valuation may be negative. For instance, in the previous example, v_0 is the negative cost of building the bridge.

- Let $\mathcal{V}$ be the space of all agent valuation functions.

- In a mechanism, each agent i will report their valuations as a "bid" $b_i \in \mathcal{V}$, which is also a function.

A General Model for Multi-parameter Mechanism Design (cont'd)

Definition (Mechanism). A *mechanism* $M = (\omega, p)$ is a protocol to solicit bids $b = \{b_i\}_{i \in [n]}$ where $b_i \in \mathcal{V}$, and choose an outcome $\omega(b; M)$ and payment vector $p(b; M) \in \mathbb{R}^n$. That is, $\omega(\cdot; M) : \mathcal{V}^n \to \Omega$, and $p(\cdot; M) : \mathcal{V}^n \to \mathbb{R}^n$.

N.B. Note that we are overloading notation: $\omega \in \Omega$, $p \in \mathbb{R}^n$ denote an outcome and price vector, but also (ω, p) denotes a mechanism.

Agent utility. Agent i's utility in the mechanism is (quasilinear utility):

$$u_i(b; M) = v_i\big(\omega(b; M)\big) - p_i(b; M).$$

Here $v_i(\omega(b; M))$ is i's valuation for the chosen outcome, while $p_i(b)$, the i^{th} element of $p(b)$, is the payment for agent i.

Each agent wishes to maximize her utility.

A General Model for Multi-parameter Mechanism Design (cont'd)

Mechanism designer's utility. Similarly, the mechanism designer's utility is

$$u_0(b; M) = v_0\big(\omega(b; M)\big) + \sum_{i=1}^{n} p_i(b; M).$$

Welfare. The *social welfare* of an outcome ω is:

$$W(\omega) = \sum_{i=0}^{n} v_i(\omega) = v_0(\omega) + \sum_{i=1}^{n} v_i(\omega).$$

Here $v_0(\omega)$ is the mechanism designer's value for an outcome, and $\sum_{i=1}^{n} v_i(\omega)$ is the sum of agent values.

A General Model for Multi-parameter Mechanism Design (cont'd)

Hence, the welfare under a bid profile $b = \{b_i\}_{i \in [n]}$ in a mechanism M is,

$$W(b; M) = v_0(\omega(b; M)) + \sum_{i=1}^{n} v_i(\omega(b; M)).$$

In a mechaism, this is also equal to the sum of utilities as the prices cancel out:

$$W(b; M) = \underbrace{v_0(\omega(b; M)) + \sum_{i=1}^{n} p_i(b; M)}_{=u_0(b;M)} + \sum_{i=1}^{n} \underbrace{(v_i(\omega(b; M)) - p_i(b; M))}_{=u_i(b;M)}$$
$$= \sum_{i=0}^{n} u_i(b; M).$$

Recall, agent and mechanism designer utilities:

$$u_i(b; M) = v_i\big(\omega(b; M)\big) - p_i(b; M), \qquad u_0(b; M) = v_0\big(\omega(b; M)\big) + \sum_{i=1}^{n} p_i(b; M).$$

The VCG (Vickrey–Clarke–Groves) Mechanism

VCG Mechanism. Each agent $i \in [n]$ submits her bid b_i.

- We select the outcome $\omega^\star$ which maximizes the *reported* welfare:

$$\omega(b; M) = \arg\max_{\omega \in \Omega} \left(v_0(\omega) + \sum_{i=1}^{n} b_i(\omega) \right) \quad \triangleq \omega^\star.$$

- The **payment** $p_i(b)$ for agent i is the *externality* due to agent i, i.e., the loss in total reported welfare to the others due to her presence:

$$p_i(b; M) = \underbrace{\max_{\omega \in \Omega} \left(v_0(\omega) + \sum_{j \neq i} b_j(\omega) \right)}_{\text{max welfare for others if } i \text{ were not present}} - \underbrace{\left(v_0(\omega^\star) + \sum_{j \neq i} b_j(\omega^\star) \right)}_{\text{welfare for others when } i \text{ is present}} .$$

Let us revisit the two examples from before and apply the VCG mechanism.

VCG Examples

Example 1 revisited (shared communication channel). Some agents are sharing a channel of capacity C. Agent i has bandwidth requirement y_i (public) and value v_i (private). We should choose a feasible set S, i.e., $\sum_{i \in S} y_i \leq C$ to maximize: $W(S) = \sum_{i \in S} v_i$.

Suppose $n = 3$ and the capacity be $C = 1$. Suppose (v_i, y_i) are as follows.

Agent i	bandwidth y_i (public)	valuation v_i (private)
1	0.4	1
2	0.5	2
3	0.8	2.1

The feasible outcomes are:

$$\Omega = \left\{ S \subseteq \{1, 2, 3\} : \sum_{i \in S} y_i \leq 1 \right\} = \Big\{ \varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\} \Big\}.$$

VCG Examples (cont'd)

Outcome. If bids are truthful, VCG will choose

$$\omega(v; M) = \{1, 2\} \quad \triangleq \omega^\star.$$

with $W(\omega^\star) = 3$.

Payments. The payment for each agent is,

$$p_1 = \max_{\omega \in \Omega} \big(v_2(\omega) + v_3(\omega)\big) - \big(v_2(\omega^\star) + v_3(\omega^\star)\big) = 2.1 - 2 = 0.1,$$
$$p_2 = \max_{\omega \in \Omega} \big(v_1(\omega) + v_3(\omega)\big) - \big(v_1(\omega^\star) + v_3(\omega^\star)\big) = 2.1 - 1 = 1.1,$$
$$p_3 = \max_{\omega \in \Omega} \big(v_1(\omega) + v_2(\omega)\big) - \big(v_1(\omega^\star) + v_2(\omega^\star)\big) = 3 - 3 = 0.$$

Note that for agents 1 and 2, the payment is less than their value. This is not a coincidence; we will show that VCG is IR.

VCG Examples (cont'd)

Example 2 revisited (building roads). The state wants to connect cities by building roads at $10M each. Cities A and B are already connected, but C is not. We wish to truthfully solicit bids from each city for how much they value a new road, so that the welfare is maximized.

Suppose the valuations are as follows (known privately to the cities, but not the state):

	A–C only	B–C only	Both	None
v_A	5	0	5	0
v_B	0	5	5	0
v_C	9	9	15	0
v_0	-10	-10	-20	0
Welfare	4	4	5	0

VCG Examples (cont'd)

Outcome. The VCG mechanism will choose outcome $\omega^\star =$ "Both" with $W(\omega^\star) = 5$.

Payments. The payment for city A is,

$$p_A = \max_\omega \left(v_0(\omega) + \sum_{j \neq A} v_j(\omega) \right) - \left(v_0(\omega^\star) + \sum_{j \neq A} v_j(\omega^\star) \right) = 4 - 0 = 4.$$

Here, the optimal welfare for others without A is 4 (choosing B–C only: $v_0 + v_B + v_C = -10 + 5 + 9 = 4$), and the welfare of others in $\omega^\star$ is $-20 + 5 + 15 = 0$. Similarly, $p_B = 4$.

The price for city C is \$10. *(try at home)*

N.B. Observe that $p_A + p_B + p_C = 18$. The state does not recover the cost (20) to build the roads! In general, this is not guaranteed by VCG.

VCG Examples (cont'd)

Observation. In a single-item auction, the VCG mechanism is the second price (Vickrey) auction.

Proof. You will show this in the homework.

Properties of the VCG Mechanism

Theorem (Properties of VCG). The VCG mechanism M is,

(i) **Efficient:** It maximizes the social welfare when all agents report truthfully.

(ii) **Individually Rational (IR):** An agent does not stand to lose by participating in the mechanism. That is, regardless of the others' bids, a truthful agent has non-negative utility: for all $b_{-i} \in \mathcal{V}^{n-1}$,

$$u_i(v_i, b_{-i}; M) \geq 0 \quad \iff \quad v_i\big(\omega(b; M)\big) \geq p_i(b; M).$$

(iii) **Truthful (DSIC):** Reporting her bid truthfully is a dominant strategy for each agent, *i.e.*,

$$u_i(v_i, b_{-i}; M) \geq u_i(b_i, b_{-i}; M) \qquad \text{for all } b_i \in \mathcal{V}, b_{-i} \in \mathcal{V}^{n-1}.$$

Properties of the VCG Mechanism (cont'd)

Recall, the VCG mechanism

$$\text{Outcome:} \quad \omega(b; M) = \arg\max_{\omega\in\Omega} \left(v_0(\omega) + \sum_{i=1}^{n} b_i(\omega) \right) \quad \triangleq \omega^\star.$$

$$\text{Prices:} \quad p_i(b; M) = \max_{\omega\in\Omega} \left(v_0(\omega) + \sum_{j\neq i} b_j(\omega) \right) - \left(v_0(\omega^\star) + \sum_{j\neq i} b_j(\omega^\star) \right).$$

Proof. Here, statement *"(i) efficiency"* is by design. For *(ii) IR*, let us fix the bids b_{-i} of other agents and assume i is truthful. Then, the chosen allocation is

$$\omega(v_i, b_{-i}) = \arg\max_{\omega\in\Omega} \left(v_0(\omega) + v_i(\omega) + \sum_{j\neq i} b_j(\omega) \right) \triangleq \omega_1. \tag{1}$$

Her payment is:

$$p_i(v_i, b_{-i}) = \max_{\omega} \left(\sum_{j\neq i} b_j(\omega) + v_0(\omega) \right) - \left(v_0(\omega_1) + \sum_{j\neq i} b_j(\omega_1) \right).$$

Properties of the VCG Mechanism (cont'd)

Therefore her utility is:

$$
\begin{aligned}
u_i(v_i, b_{-i}; M) &= v_i(\omega_1) - p_i(v_i, b_{-i}). \\
&= \left(v_0(\omega_1) + v_i(\omega_1) + \sum_{j \neq i} b_j(\omega_1) \right) - \max_{\omega} \left(v_0(\omega) + \sum_{j \neq i} b_j(\omega) \right). \\
&= \max_{\omega \in \Omega} \left(v_0(\omega) + v_i(\omega) + \sum_{j \neq i} b_j(\omega) \right) - \max_{\omega} \left(v_0(\omega) + \sum_{j \neq i} b_j(\omega) \right).
\end{aligned}
$$

Here, the last step uses the definition of ω_1 in (1).

IR follows from the observation that $v_i(\omega) \geq 0$, and the fact that when $f \geq g$, we have $\max_\omega f(\omega) \geq \max_\omega g(\omega)$.

Properties of the VCG Mechanism (cont'd)

For *(iii) DSIC*, we need to show, $u_i(v_i, b_{-i}; M) \geq u_i(b_i, b_{-i}; M)$ for all b_i, b_{-i}. Fix the reports b_{-i} of others. First suppose i reports truthfully.

Recall, from the calculations we did for IR, we have

$$\omega(v_i, b_{-i}) = \arg\max_{\omega \in \Omega} \widetilde{W}(\omega) \triangleq \omega_1 \text{ (say)}, \qquad \text{where, } \widetilde{W}(\omega) = \left(v_0(\omega) + v_i(\omega) + \sum_{j \neq i} b_j(\omega) \right).$$

$$u_i(v_i, b_{-i}; M) = \underbrace{\left(v_0(\omega_1) + v_i(\omega_1) + \sum_{j \neq i} b_j(\omega_1) \right)}_{= \widetilde{W}(\omega_1)} - \underbrace{\max_{\omega} \left(v_0(\omega) + \sum_{j \neq i} b_j(\omega) \right)}_{\triangleq \widetilde{W}^{\star}_{-i}}.$$

Now suppose i reports $b_i \neq v_i$. Then VCG chooses,

$$\omega(b_i, b_{-i}; M) = \arg\max_{\omega} \left(v_0(\omega) + b_i(\omega) + \sum_{j \neq i} b_j(\omega) \right) \triangleq \omega_2.$$

Properties of the VCG Mechanism (cont'd)

Then, i's payment and utility are

$$p_i(b_i, b_{-i}; M) = \underbrace{\max_{\omega}\left(v_0(\omega) + \sum_{j\neq i} b_j(\omega)\right)}_{=\widetilde{W}^{\star}_{-i}} - \underbrace{\left(v_0(\omega_2) + \sum_{j\neq i} b_j(\omega_2)\right)}_{=\widetilde{W}(\omega_2) - v_i(\omega_2)},$$

$$u_i(b_i, b_{-i}; M) = v_i(\omega_2) - p_i(b_i, b_{-i}; M) = \widetilde{W}(\omega_2) - \widetilde{W}^{\star}_{-i}.$$

We therefore have,

$$u_i(v_i, b_{-i}; M) - u_i(b_i, b_{-i}; M) = \widetilde{W}(\omega_1) - \widetilde{W}(\omega_2) \geq 0,$$

where the inequality uses the fact that ω_1 maximizes $\widetilde{W}$. □

To be updated.