## Stochastic Bandits



# Kirthevasan Kandasamy 

Carnegie Mellon University

University of Moratuwa, Sri Lanka
August 17, 2017
Slides: www.cs.cmu.edu/~kkandasa/misc/mora-slides.pdf

## Slides are up on my webpage:

www.cs.cmu.edu/~kkandasa
home publications software misc.

## kirthevasan kandasamy

PhD Student, Carnegie Mellon University

[CV] [Google Scholar] [GitHub] [Contact]


I am a fourth year Machine Learning PhD student (now ABD) in the School of Computer Science at Carnegie Mellon University. I am co-advised by Jeff Schneider and Barnabas Poczos. I am a member of the Auton Lab and the StatML Group. Prior to CMU, I completed my B.Sc in Electronics \& Telecommunications Engineering at the University of Moratuwa, Sri Lanka.

My research interests lie in the intersection of statistical and algorithmic Machine Learning. My current research spans bandit problems, Bayesian optimisation, Gaussian processes, nonparametric statistics and graphical models. As of late, I have also hopped on the deep learning bandwagon.

I am generously supported by a Facebook PhD fellowship (2017) and a CMU Presidential fellowship (2015).

## Preprints

## Slides

Asynchronous Parallel Bayesian Optimisation via Thempson Sampling Kirthevasan Kandasamy, Akshay Krishnamurthy, Jefl Schneider, Barnabas Poczos [arxiv] [AutoML slides] [Moratuwa slides

Multi-fidelity Gaussian Process Bandit Optimisation
Kirthevasan Kandasamy, Gautam Dasarathy, Junier Oliva, Jeff Schneider, Barnabas Poczos
[arxiv] [code] [UCL slides]
Influence Functions for Machine Learning: Nonparametric Estimators for Entropies, Divergences and Mutual Informations Kirthevasan Kandasamv. Akshav Krishnamurthv. Barnabas Poczos. Larrv Wasserman. James Robins

## On-line advertising



You are given a pool of 250 ads.
Task:

- You can display one ad at a time, (say for $10^{6}$ times).
- You wish to maximise the cumulative number of clicks, i.e. identify ads with the highest click-through-rate and display them most of the time.


## The Stochastic Multi-armed Bandit

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- Goal: An algorithm (policy/strategy) which achieves "small" cumulative regret,

$$
R_{n}=\sum_{t=1}^{n} f\left(x_{\star}\right)-\sum_{t=1}^{n} f\left(x_{t}\right)=\sum_{t=1}^{n}\left(f\left(x_{\star}\right)-f\left(x_{t}\right)\right)
$$

where, $x_{\star}=\operatorname{argmax}_{x \in \mathcal{X}} f(x)$.

## Smooth Bandits

$f: \mathcal{X} \rightarrow \mathbb{R}$ is a black-box function that is accessible only via noisy evaluations. $\mathcal{X}$ is a metric space, e.g. $\mathbb{R}^{d}$.


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Cumulative Regret after $n$ evaluations

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Simple Regret after $n$ evaluations

$$
S_{n}=f\left(x_{\star}\right)-\max _{t=1, \ldots, n} f\left(x_{t}\right)
$$

## Applications



## Applications



## Applications

## $x \rightarrow$ Expensive Blackbox Function

Other Examples:

- Pre-clinical Drug Discovery
- Optimal policy in Autonomous Driving
- Synthetic gene design



## Recap

Types of arms (domain $\mathcal{X}$ )

1. $K$-armed bandit, $\mathcal{X}$ is a finite set.
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$f: \mathcal{X} \rightarrow \mathbb{R}$. On playing $x \in \mathcal{X}$ you observe $f(x)+\varepsilon, \mathbb{E} \varepsilon=0$.
Two notions of regret
4. Cumulative regret, $\quad R_{n}=\sum_{t=1}^{n} f\left(x_{\star}\right)-f\left(x_{t}\right)$.
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Other formalisms: contextual bandit, adversarial bandit, duelling bandit, linear bandit, best arm identification and several more...

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Other formalisms: contextual bandit, adversarial bandit, duelling bandit, linear bandit, best arm identification and several more...
N.B: Pulling/playing an arm $=$ experiment $=$ function evaluation

## Outline

- Part I: Stochastic bandits (cont'd)

1. Gaussian processes for smooth bandits
2. Algorithms: Upper Confidence Bound (UCB) \& Thompson Sampling (TS)

- Digression: SL2College Research Collaboration Program
- Part II: My research

1. Multi-fidelity bandit: cheap approximations to an expensive experiments
2. Parallelising arm pulls

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## Gaussian (Normal) distribution <br> $\mathcal{N}\left(\mu, \sigma^{2}\right)$



- A probability distribution for real valued random variables.
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- A probability distribution for real valued random variables.
- Mean $\mu$ and variance $\sigma^{2}$ completely characterises distribution.
- For samples $X_{1}, \ldots, X_{n}$, let $\hat{\mu}=\frac{1}{n} \sum_{i} X_{i}$ be the sample mean. Then, $\hat{\mu} \pm 1.96 \frac{\sigma}{\sqrt{n}}$ is a $95 \%$ confidence interval for $\mu$.
- Can draw samples (e.g. in Matlab: mu + sigma * randn()).


## Gaussian Processes $(\mathcal{G P})$

$\mathcal{G} \mathcal{P}(\mu, \kappa)$ : A distribution over functions from $\mathcal{X}$ to $\mathbb{R}$. Completely characterised by mean function $\mu: \mathcal{X} \rightarrow \mathbb{R}$, and covariance kernel $\kappa: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$.

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Functions with no observations


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Prior $\mathcal{G P}$


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Observations


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Posterior $\mathcal{G P}$ given observations


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Posterior $\mathcal{G P}$ given observations


After $t$ observations, $\quad f(x) \sim \mathcal{N}\left(\mu_{t}(x), \sigma_{t}^{2}(x)\right)$.

## Algorithm 1: Upper Confidence Bounds in GP Bandits

Model $f \sim \mathcal{G} \mathcal{P}(\mathbf{0}, \kappa)$.
Gaussian Process Upper Confidence Bound (GP-UCB)
(Srinivas et al. 2010).
$f(x)$

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Construct upper conf. bound: $\varphi_{t}(x)=\mu_{t-1}(x)+\beta_{t}^{1 / 2} \sigma_{t-1}(x)$.

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Maximise upper confidence bound.

## GP-UCB

$$
x_{t}=\underset{x}{\operatorname{argmax}} \mu_{t-1}(x)+\beta_{t}^{1 / 2} \sigma_{t-1}(x)
$$

- $\mu_{t-1}$ : Exploitation
- $\sigma_{t-1}$ : Exploration
- $\beta_{t}$ controls the tradeoff. $\beta_{t} \asymp \log t$.


## GP-UCB

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## GP-UCB, $\kappa$ is an SE kernel (Srinivas et al. 2010)

w.h.p $\quad S_{n}=f\left(x_{\star}\right)-\max _{t=1, \ldots, n} f\left(x_{t}\right) \lesssim \sqrt{\frac{\log (n)^{d} \operatorname{vol}(\mathcal{X})}{n}}$

## GP-UCB

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## Algorithm 2: Thompson Sampling in GP Bandits

Model $f \sim \mathcal{G} \mathcal{P}(\mathbf{0}, \kappa)$.
Thompson Sampling (TS)
(Thompson, 1933).
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## Algorithm 2: Thompson Sampling in GP Bandits

Model $f \sim \mathcal{G} \mathcal{P}(\mathbf{0}, \kappa)$.
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Draw sample $g$ from posterior. Choose $x_{t}=\operatorname{argmax}_{x} g(x)$.

Thompson Sampling (TS) in GPs


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1. Multi-fidelity bandit: cheap approximations to an expensive experiments
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## SL2College



Done with $A / L$, what's next? If you are trying to select the best option atter the Advanced Level Examination, let us guide you.

## Looking for a scholarship?

There are so many scholarships out there. We can advise you on how to grab these golden opportuntilies.

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We conneet Sti Lankan students who are seeking higher education abroed with undergracuates and graduates from respective universities, who would guide them through the process.

Willing to go that 'Extra Mile' with your research?
Submiting a paper for an International Journal / Coniference is not easy. Join our Research Collaboration programme to tind a qualitied, experienced mentor, whose expertise will take you there.


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We can guide you on Professional and Vocational qualifications required to excel in your career path.

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Planning to return to your motherland? Seeking job opportunities in Sri Lanka? We provide guidance and information for everyone who want to return or migrate to Sri Lanka

## SL2College Research Collaboration Program

-Ashwin de Silva



Samitha Samaranayake (Systems Engineering)

















## Hatre Page ?

VA. Samaranayake (Statistics)








## Hanto Page :

Keerthi Senevirathne (Nano Materials Chemistry)






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## Harce $\mathrm{Fapge}^{\prime}$

Oshani Seneviratne (Web systems)



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www.sl2college.org/research-collab research-collab@sl2college.org

## SL2College Research Collaboration Program

## How it works

We have a pool of doctoral/post-doctoral/professorial mentors (all Sri Lankan).

We connect Sri Lankan undergrads to mentors, who will guide the students on a research project.

Aim: Publish a paper (at a good venue) within a 9-15 month time frame.

## Application Process

- Fill out the application form on our webpage:
www.sl2college.org/research-collab
- mention areas of interests and preferred mentors.
- .. and email your CV to research-collab@sl2college.org.


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- We send your CV \& statement to the mentor. If he/she is interested, we initiate a collaboration.
- You report to us once every 3 months.


## SL2College Research Collaboration Team




Nuwan


Rajitha


Umashanthi


Kirthevasan
www.sl2college.org/research-collab research-collab@sI2college.org

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## Part 2.1: Multi-fidelity Bandits

Motivating question:
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## Motivating question:

What if we have cheap approximations to $f$ ?

1. Computational astrophysics and other scientific experiments: simulations and numerical computations with less granularity.

2. Hyper-parameter tuning: Train \& validate with a subset of the data.
3. Robotics \& autonomous driving: computer simulation vs real world experiment.

## Multi-fidelity Methods

For specific applications,

- Industrial design
(Forrester et al. 2007)
- Hyper-parameter tuning (Agarwal et al. 2011, Klein et al. 2015, Li et al. 2016)
- Active learning
(Zhang \& Chaudhuri 2015)
- Robotics
(Cutler et al. 2014)

Multi-fidelity bandits \& optimisation
(Huang et al. 2006,
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Multi-fidelity bandits \& optimisation
(Huang et al. 2006,
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... with theoretical guarantees
(Kandasamy et al. NIPS 2016a\&b, Kandasamy et al. ICML 2017)

Multi-fidelity Bandits
(Kandasamy et al. ICML 2017)
A fidelity space $\mathcal{Z}$ and domain $\mathcal{X}$
$\mathcal{Z} \leftarrow$ all granularity values
$\mathcal{X} \leftarrow$ space of cosmological parameters

## Multi-fidelity Bandits


(Kandasamy et al. ICML 2017)

A fidelity space $\mathcal{Z}$ and domain $\mathcal{X}$
$\mathcal{Z} \leftarrow$ all granularity values
$\mathcal{X} \leftarrow$ space of cosmological parameters
$g: \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R}$.
$g(z, x) \leftarrow$ likelihood score when performing integrations on a grid of size $z$ at cosmological parameters $x$.

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Denote $f(x)=g\left(z_{\mathbf{\bullet}}, x\right)$ where $z_{\mathbf{\bullet}} \in \mathcal{Z}$. $z_{\mathbf{0}}=$ highest grid size.

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End Goal: Find $x_{\star}=\operatorname{argmax}_{x} f(x)$.

A cost function, $\lambda: \mathcal{Z} \rightarrow \mathbb{R}_{+}$. $\lambda(z)=\mathcal{O}\left(z^{p}\right) \quad$ (say).



End Goal: Find $x_{\star}=\operatorname{argmax}_{x} f(x)$.

## Multi-fidelity Simple Regret



End Goal: Find $x_{\star}=\operatorname{argmax}_{x} f(x)$.
Simple Regret after capital $\Lambda: \quad S(\Lambda)=f\left(x_{*}\right)-\max _{t: z_{t}=z_{\mathbf{e}}} f\left(x_{t}\right)$.
$\Lambda \leftarrow$ amount of a resource spent, e.g. computation time or money.

## Multi-fidelity Simple Regret



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$\Lambda \leftarrow$ amount of a resource spent, e.g. computation time or money.
No reward for pulling an arm at low fidelities, but use cheap evaluations at $z \neq z_{\bullet}$ to speed up search for $x_{\star}$.

## Algorithm: BOCA

(Kandasamy et al. ICML 2017)


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(Kandasamy et al. ICML 2017)

Model $g \sim \mathcal{G P}(0, \kappa)$ and compute posterior $\mathcal{G P}$ :
mean $\quad \mu_{t-1}: \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R}$
std-dev $\quad \sigma_{t-1}: \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R}_{+}$

## Algorithm: BOCA



Model $g \sim \mathcal{G P}(0, \kappa)$ and compute posterior $\mathcal{G P}$ : mean $\quad \mu_{t-1}: \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R}$ std-dev $\sigma_{t-1}: \mathcal{Z} \times \mathcal{X} \rightarrow \mathbb{R}_{+}$
(1) $x_{t} \leftarrow$ maximise upper confidence bound for $f(x)=g\left(z_{\mathbf{0}}, x\right)$.

$$
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## Theoretical Results for BOCA


"good"

"bad"

## Theoretical Results for BOCA


E.g.: For SE kernels, bandwidth $h_{\mathcal{Z}}$ controls smoothness.

## Theoretical Results for BOCA

## GP-UCB SE kernel,

(Srinivas et al. 2010)

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\text { w.h.p } \quad S(\Lambda) \lesssim \sqrt{\frac{\operatorname{vol}(\mathcal{X})}{\Lambda}}
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N.B: Dropping constants and polylog terms.

## Experiment: Cosmological inference on Type-1a supernovae data

Estimate Hubble constant, dark matter fraction \& dark energy fraction by maximising likelihood on $N_{\bullet}=192$ data.
Requires numerical integration on a grid of size $G_{\bullet}=10^{6}$. Approximate with $N \in[50,192]$ or $G \in\left[10^{2}, 10^{6}\right]$ (2D fidelity space).

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## Outline

- Part I: Stochastic bandits (cont'd)

1. Gaussian processes for smooth bandits
2. Algorithms: Upper Confidence Bound (UCB) \& Thompson Sampling (TS)

- Digression: SL2College Research Collaboration Program
- Part II: My research

1. Multi-fidelity bandit: cheap approximations to an expensive experiments
2. Parallelising arm pulls

## Part 2.2: Parallelising arm pulls

Sequential arm pulls with one worker


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Parallel arm pulls with $M$ workers (Asynchronous)


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Parallel arm pulls with $M$ workers (Synchronous)


## Why parallelisation?

- Computational experiments: infrastructure with 100-1000's CPUs or GPUs.
- Drug discovery: High throughput screening


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- Computational experiments: infrastructure with 100-1000's CPUs or GPUs.
- Drug discovery: High throughput screening

Prior work: (Ginsbourger et al. 2011, Janusevskis et al. 2012, Wang et al. 2016, González et al. 2015, Desautels et al. 2014, Contal et al. 2013, Shah and Ghahramani 2015, Kathuria et al. 2016, Wang et al. 2017, Wu and Frazier 2016, Hernandez-Lobato et al. 2017)

Shortcomings

- Asynchronicity
- Theoretical guarantees
- Computationally \& conceptually simple


## Review: Sequential Thompson Sampling in GP Bandits



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Draw sample $g$ from posterior. Choose $x_{t}=\operatorname{argmax}_{x} g(x)$.

## Parallelised Thompson Sampling

Asynchronous: asyTS
At any given time,

1. $\left(x^{\prime}, y^{\prime}\right) \leftarrow$ Wait for a worker to finish.
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4. Re-deploy worker at $\operatorname{argmax} g$.


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At any given time,

1. $\left\{\left(x_{m}^{\prime}, y_{m}^{\prime}\right)\right\}_{m=1}^{M} \leftarrow$ Wait for all workers to finish.
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3. Draw $M$ samples
$g_{m} \sim \mathcal{G} \mathcal{P}, \forall m$.
4. Re-deploy worker $m$ at $\operatorname{argmax} g_{m}, \forall m$.


## Theoretical Results: number of evaluations

Sequential TS, SE Kernel (Russo \& van Roy 2014)

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\mathbb{E}\left[S_{n}\right] \lesssim \sqrt{\frac{\operatorname{vol}(\mathcal{X}) \log (n)^{d}}{n}}
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Theorem: synTS \& asyTS, SE Kernel (Kandasamy et al. Arxiv 2017)

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\mathbb{E}\left[S_{n}\right] \lesssim \frac{M \log (M)^{2 d}}{n}+\sqrt{\frac{\operatorname{vol}(\mathcal{X}) \log (n)^{d}}{n}}
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$n \leftarrow$ \# completed arm pulls by all workers.

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- A sequential algorithm can make use of information from all previous rounds to determine where to evaluate next.
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## Theoretical Results: Simple regret with time



Synchronous


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Asynchronous


Theorem (Informal)
If evaluation times are the same, asyTS $\approx$ synTS.
Otherwise, bounds for asyTS is strictly better than synTS. More the variability in evaluation times, the bigger the difference.

## Theoretical Results: Simple regret with time

Asynchronous


Synchronous


## Theorem (Informal)

If evaluation times are the same, asyTS $\approx$ synTS.
Otherwise, bounds for asyTS is strictly better than synTS. More the variability in evaluation times, the bigger the difference.

- Bounded tail decay: constant factor
- Sub-gaussian tail decay: $\sqrt{\log (M)}$ factor
- Sub-exponential tail decay: $\log (M)$ factor


## Experiment: Currin-Exponential-14D $\quad M=35$

Evaluation time sampled from a Pareto-3 distribution


Experiment: Hyper-parameter tuning in Cifar10 $M=4$
Tune \# filters in in range $(32,256)$ for each layer in a 6 layer CNN. Time taken for an evaluation: 4-16 minutes.


## Summary

- Bandits are a framework for studying exploration vs exploitation trade-offs when optimising black-box functions.
- Smooth bandit formulations are more common in practical applications.
- Several algorithms: UCB, TS, Index based policies, $\epsilon$-greedy etc.


## Summary

- Bandits are a framework for studying exploration vs exploitation trade-offs when optimising black-box functions.
- Smooth bandit formulations are more common in practical applications.
- Several algorithms: UCB, TS, Index based policies, $\epsilon$-greedy etc.
- Multi-fidelity Bandits: Allows us to use cheap approximations to a an expensive experiment to quickly find the optimum.
- Parallelised TS: Simple and intuitive way to deal with multiple workers.


Akshay


Barnabás


Gautam


Jeff


Junier

## Thank You

Slides: www.cs.cmu.edu/~kkandasa/misc/mora-slides.pdf

