CS 760: Machine Learning
Reinforcement Learning

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Announcements

- **Class roadmap:**
  - Last class on RL
  - Two classes on data-efficient learning
  - Exam review

Outline

- **Review: RL**
  - MDPs, policies, value function, Q-function, etc.
- **Function Approximation**
  - Value & Q-function approximations, linear, nonlinear
- **Policy-based RL**
  - Policy gradient, policy gradient theorem, REINFORCE algorithm

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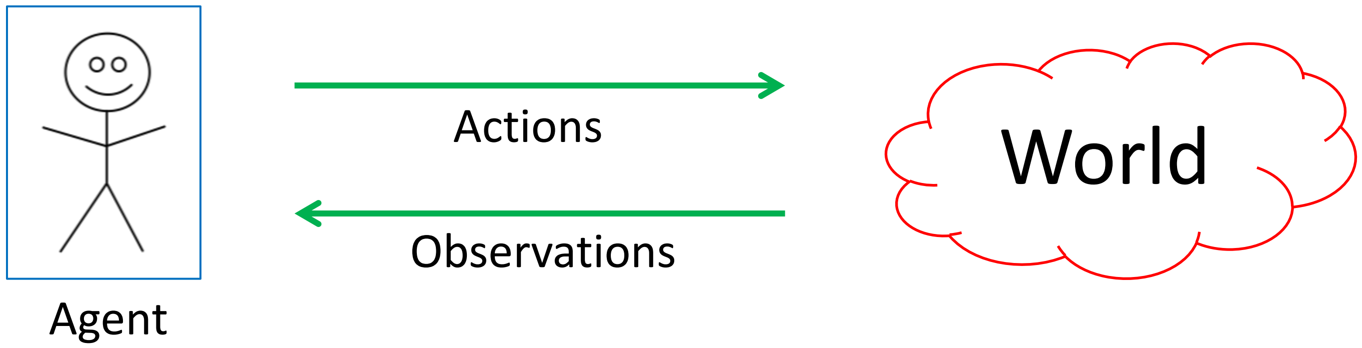
Review: General Model

We have an **agent interacting** with the **world**



- Agent receives a reward based on state of the world
  - **Goal**: maximize reward / utility **($$$)**
  - Note: **data** consists of actions & observations
    - Compare to unsupervised learning and supervised learning

Markov Decision Process (MDP)

The formal mathematical model:

- **State set** S. Initial state s_0. **Action set** A
- **State transition model**: $P(s_{t+1} | s_t, a_t)$
  - Markov assumption: transition probability only depends on s_t and a_t, and not previous actions or states.
- **Reward function:** $r(s_t)$
- **Policy**: $\pi(s) : S \rightarrow A$ action to take at a particular state.

$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \dots$$

Defining the Optimal Policy

For policy π, **expected utility** over all possible state sequences from s_0 produced by following that policy:

$$V^{\pi}(s_0) = \sum_{\substack{\text{sequences} \\ \text{starting from } s_0}} P(\text{sequence}) U(\text{sequence})$$

Called the **value function** (for π, s_0)

Discounting Rewards

One issue: these are infinite series. **Convergence**?

- Solution

$$U(s_0, s_1 \ldots) = r(s_0) + \gamma r(s_1) + \gamma^2 r(s_2) + \ldots = \sum_{t \geq 0} \gamma^t r(s_t)$$

- Discount factor γ between 0 and 1
  - Set according to how important **present** is vs. **future**
  - Note: has to be less than 1 for convergence

Bellman Equation

Let's walk over one step for the value function:

$$V^*(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V^*(s')$$

current state reward

discounted expected future **rewards**

Value Iteration

Q: how do we find $V^*(s)$?

- Why do we want it? Can use it to get the best policy
- Know: reward $r(s)$, transition probability $P(s'|s,a)$
- Also know $V^*(s)$ satisfies Bellman equation (recursion above)

A: Use the property. Start with $V_0(s)=0$. Then, update

$$V_{i+1}(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V_i(s')$$

Policy Iteration: Algorithm

Policy iteration. Algorithm

- Start with random policy π
- Use it to compute value function V^π : a set of linear equations

$$V^\pi(s) = r(s) + \gamma \sum_{s'} P(s'|s,a) V^\pi(s')$$

- Improve the policy: obtain π'

$$\pi'(s) = \arg\max_a r(s) + \gamma \sum_{s'} P(s'|s,a) V^\pi(s')$$

- Repeat

Q-Learning (**model-free** RL)

What if we don't know transition probability P(*s*'|*s*,*a*)?

- Need a way to learn to act without it.
- **Q-learning**: get an action-utility function Q(*s*,*a*) that tells us the value of doing *a* in state *s*
- Note: $V^*(s) = \max_a Q(s,a)$
- Now, we can just do $\pi^*(s) = \arg\max_a Q(s, a)$
  - But need to estimate *Q*!

Q-Learning Iteration

How do we get Q(*s*,*a*)?

- Similar iterative procedure

learning rate

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha[r(s_t) + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t)]$$

Idea: combine old value and new estimate of future value.

Note: We are using a policy π to take actions $a_t = \pi(s_t)$; this policy is based on Q!

Exploration Vs. Exploitation

General question!

- **Exploration:** take an action with unknown consequences
  - **Pros**:
    - Get a more accurate model of the environment
    - Discover higher-reward states than the ones found so far
  - **Cons**:
    - When exploring, not maximizing your utility
    - Something bad might happen
- **Exploitation:** go with the best strategy found so far
  - **Pros**:
    - Maximize reward as reflected in the current utility estimates
    - Avoid bad stuff
  - **Cons**:
    - Might also prevent you from discovering the true optimal strategy

Q-Learning: Epsilon-Greedy Policy

How to **explore**?

- With some 0<ε<1 probability, take a random action at each state, or else the action with highest Q(*s*,*a*) value.

$$a = \begin{cases} \text{argmax}_{a \in A} Q(s, a) & \text{uniform}(0,1) > \epsilon \\ \text{random } a \in A & \text{otherwise} \end{cases}$$

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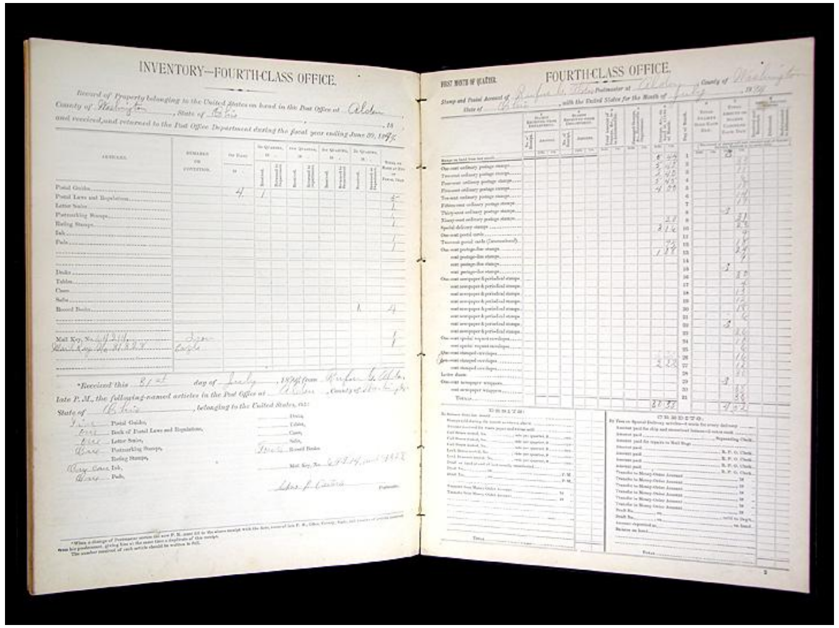
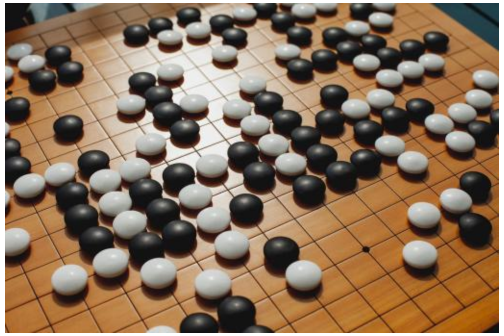
Beyond Tables



So far:

- Represent everything with a table
  - Value function **V**: table size $|S| \times 1$
  - **Q** function: table size $|S| \times |A|$

- Too big to store in memory for many tasks
  - Backgammon: 10^{20} states.
  - Go: 3^{361} states



- Need some other approach

Beyond Tables: Function Approximation

Both V and Q are functions…

- Can approximate them with models, i.e. neural networks
- So we write

$$V^\pi(s) \approx \hat{V}_\theta(s)$$

- New goal: find the weights θ
- Loss function:

$$J(\theta) = \mathbb{E}_\pi[(V^\pi(s) - \hat{V}_\theta(s))^2]$$

State **Representations** & **Models**

How do we represent a state?

- As usual, feature vectors, i.e.

$$x(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \\ x_d(s) \end{bmatrix}$$

- What kind of models could we use?
  - First, let's start with linear:

$$\hat{V}_\theta(s) = x(s)^T \theta$$

Linear VFA With an Oracle

- SGD update is

$$\alpha[(V^{\pi}(s) - \hat{V}_{\theta}(s))\nabla_{\theta}\hat{V}_{\theta}(s)]$$

- And for our linear model, we get

$$\alpha(V^{\pi}(s) - \hat{V}_{\theta}(s))x(s)$$

Step Size Prediction Error Feature Value

What if We **Don't Have** an Oracle?

Similar to what we've seen so far, use Monte-Carlo.

- We won't know $V^\pi(s_t)$
- Estimate returns $G_t = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$

- Can just run episodes and estimate, ie, get some noisy estimates as training data:

$$(s_1, G_1), (s_2, G_2), \ldots, (s_T, G_T)$$

Q-Function Approximation

Similar idea for Q-function

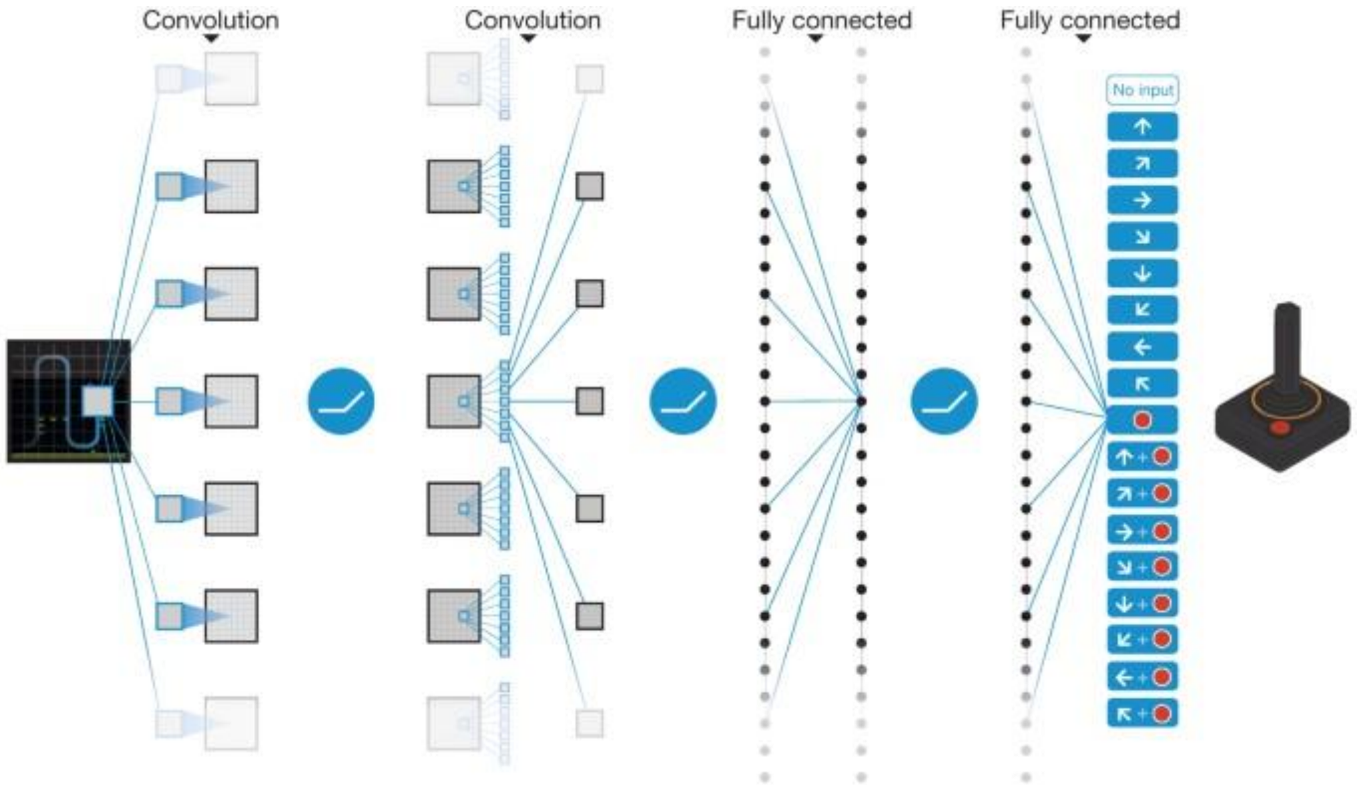
$$Q^{\pi}(s, a) \approx \hat{Q}_{\theta}(s, a)$$

Representation: use both states and values

- Can still use linear models
- Note: quite popular to use **deep models**

Q-Function Approximation: Deep Models

- Note: quite popular to use **deep models**
  - e.g. CNNs if the states are images (like in video games)



Mnih et al, "Human-level control through deep reinforcement learning"

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Policy-Based RL

So far, we either approximated V or Q

- Then use these to extract the optimal policy

Could do the same trick but with the policy

- Note: so far our policies were deterministic, now we'll allow a distribution over actions, i.e. $\pi(s) = P(a|s)$

- Want: $\pi_\theta(s, a) = P_\theta(a|s)$

Policy Gradient

Use the same idea. We'll define an objective $J(\theta)$

- And then can get gradients:

$$\nabla_\theta \pi_\theta(s,a) = \pi_\theta(s,a) \underbrace{\nabla_\theta \log \pi_\theta(s,a)}_{\text{Score Function}}$$

- Example: continuous action space.
  - Gaussian policy $a \sim \mathcal{N}(x(s)^T\theta), \sigma^2)$
  - has score: $(a - x(s)^T\theta)x(s)/\sigma^2$

Policy Gradient

Set our objective to be

$$J(\theta) = \underbrace{\sum_s P(s|\pi_\theta)}_{\text{stationary distribution}} \sum_a \pi_\theta(s,a) Q^\pi(s,a)$$

- Compute the gradient via the **policy gradient theorem**

$$\nabla_\theta J(\theta) = \sum_s P(s|\pi_\theta) \sum_a \nabla_\theta \pi_\theta(s,a) Q^\pi(s,a)$$

REINFORCE Algorithm

So, to learn a policy, we can run SGD (actually ascent)

- Compute gradients via policy gradient theorem

$$\nabla_\theta J(\theta) = \sum_s P(s|\pi_\theta) \sum_a \nabla_\theta \pi_\theta(s,a) Q^\pi(s,a)$$

- Just need $Q^\pi(s,a)$ estimates.

- How? Monte-Carlo again: Use G_t for our estimates.

Thanks Everyone!

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