Epitome driven 3-D Diffusion Tensor image segmentation: on extracting *specific* structures

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1 Feasible Integral Solutions

For ease of presentation, we first recall our optimization model:

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{i\sim j} c_{ij}z_{ij} + \sum_{j=1}^{n} w_{j0}(1-x_j) + \sum_{j=1}^{n} w_{j1}x_j + \lambda \sum_{b=1}^{\beta}\left(\sum_{j=1}^{n}\sum_{l=1}^{n}\mathcal{H}_b(j)\mathcal{H}_b(l)x_jx_l - 2\sum_{j=1}^{n}\mathcal{H}_b(j)x_j\hat{\mathcal{H}}_b\right) \tag{1}$$

$$\text{s.t.} \quad |x_i - x_j| \le z_{ij} \quad \forall (i\sim j) \text{ where } i,j \in \{1,\cdots,n\},$$
$$\mathbf{x},\mathbf{z} \text{ is binary,}$$

This can be slightly modified as follows:

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{i\sim j} c_{ij}z_{ij} + \sum_{j=1}^{n} w_{j0}(1-x_j) + \sum_{j=1}^{n} w_{j1}x_j + \lambda \sum_{b=1}^{\beta}\left(\sum_{j=1}^{n}\sum_{l=1}^{n}\mathcal{H}_b(j)\mathcal{H}_b(l)y_{jl} - 2\sum_{j=1}^{n}\mathcal{H}_b(j)x_j\hat{\mathcal{H}}_b\right) \tag{2}$$

$$\text{s.t.} \quad |x_i - x_j| \le z_{ij} \quad \forall (i\sim j) \text{ where } i,j \in \{1,\cdots,n\}, \tag{3}$$

$$x_j + x_l \le y_{jl} + 1 \quad \forall (j \cong l) \quad \text{i.e., } \exists b : \mathcal{H}_b(i) = \mathcal{H}_b(j) = 1 \tag{4}$$

$$\mathbf{x},\mathbf{y},\mathbf{z} \text{ is binary,} \tag{5}$$

The combinatorial solution strategy (in the main paper) provides half-integral solutions to (1). This can be verified by observing that the equivalent construction in (2) above has a constraint matrix, $\mathcal{M}$, with a 2-modular structure (i.e., determinants of all non-singular submatrices of $\mathcal{M}$ belong to $\{-2,-1,0,1,2\}$), which leads to half-integral solutions [1, 2] directly.

In order to prove an approximation, the first step is to demonstrate that the $\{0,1\}$-solution (via rounding) is feasible w.r.t. the constraints. Recall that, for the purpose of this analysis we round *all* $\frac{1}{2}$-valued variables up to 1. It is convenient to verify the feasibility of the integral solution in the context of (3)– (5) using the following results.

Lemma 1. *Rouding all the $\frac{1}{2}$-valued variables up to 1 gives a feasible solution to Prob. (2).*

Proof. A solution obtained via rounding is integral, so (5) is satisfied. Also, we can easily adjust the $\mathbf{z}$ variables as in Table 1 after rounding up the $\frac{1}{2}$-valued $\mathbf{x}$ variables.

Voxel x_i	Voxel x_j	z_{ij}(before rounding)	z_{ij}(after rounding)	$\|x_i - x_j\| \leq z_{ij}$
$\frac{1}{2}$	0	$\frac{1}{2}$	1	✓
$\frac{1}{2}$	$\frac{1}{2}$	0	0	✓
$\frac{1}{2}$	1	$\frac{1}{2}$	0	✓

Table 1: Adjustment of z_{ij} showing that the rounding results in a feasible solution

Voxel x_i	Voxel x_l	y_{jl}(before rounding)	y_{jl}(after rounding)	$x_j + x_l \leq y_{jl} + 1$
$\frac{1}{2}$	0	0	0	✓
$\frac{1}{2}$	$\frac{1}{2}$	0	1	✓
$\frac{1}{2}$	1	$\frac{1}{2}$	1	✓

Table 2: Adjustment of y_{jl} showing that the rounding results in a feasible solution

Hence, the constraint in (3) is satisfied as well. Similarly, plugging in these values for $\mathbf{y}$ variable, we get Table 2.

From Table 2, we see that the constraint in (4) is satisfied. Since all the constraints from (3)-(5) are satisfied, the solution obtained after rounding is feasible, and the statement of the Lemma 1 follows. □

2 Obtaining Constant Factor Approximation

The integral rounded solution is feasible. To obtain a constant factor approximation to the rounded solution, we use the strategy of upper bounding the MRF terms and also the histogram terms in the objective. Combining these will give us the overall upper bound for the rounded solution.

The reader should note that there could be two strategies to analyze this approximation : bin-wise and voxel-wise. Since the number of bins will typically be far fewer than the number of voxels, it seems that adopting the bin-wise strategy to quantify the loss in rounding may provide sharper results. However, we found that it leads to certain difficulties in the worst-case. Therefore, we will quantify the loss due to rounding at the level of individual voxels.

Let us denote the energy of the solution given in the equation (1) by $E(\mathbf{x})$. For each voxel j, belonging to a bin b, the energy can be written as :

$$E(x_j) = \sum_{i \sim j} c_{ij}|x_i - x_j| + w_{j0}(1 - x_j) + w_{j1}x_j + \lambda x_j \mathcal{H}_b(j) \left(\sum_{l=1}^{n} \mathcal{H}_b(l) x_l - 2\hat{\mathcal{H}}_b \right) \tag{6}$$

We assume that there is at least *some* non-negative cost associated with the assignment of every voxel. In other words $E(x_j^*) \geq 0$, where $\mathbf{x}^*$ is the optimal solution. Without loss of generality (and facilitate the presentation), we offset the energy of every voxel by some value (say ρ), which makes it non-negative. Note that ρ is a constant independent of the optimization, and it is sufficient for our analysis to consider $\rho = 2\lambda \max_b(\hat{\mathcal{H}}_b)$. Denoting the data term for each voxel by α, smoothness term by β, and the histogram terms by γ, we can rewrite the equation in (7) as

$$E(x_j) = \underbrace{\sum_{i \sim j} c_{ij}|x_i - x_j|}_{\alpha_j} + \underbrace{w_{j0}(1 - x_j) + w_{j1}x_j}_{\beta_j} + \underbrace{\lambda x_j \mathcal{H}_b(j) \left(\sum_{l=1}^{n} \mathcal{H}_b(l) x_l - 2\hat{\mathcal{H}}_b \right) + \rho}_{\gamma_j} \tag{7}$$

Also, let $\alpha^*, \beta^*, \gamma^*$ denote the optimal value and $\alpha^{(0,1)}, \beta^{(0,1)}, \gamma^{(0,1)}$ denote the corresponding values for these terms after rounding. Since we analyze the terms at the voxel level, we ignore the subscript j for simplicity. Now, we can prove the following result.

Lemma 2. *Assuming α^*, β^* and $\alpha^{(0,1)}, \beta^{(0,1)}$ are as defined above, then*

$$\alpha^{(0,1)} + \beta^{(0,1)} \leq 4(\alpha^* + \beta^*) \tag{8}$$

Proof. For the *pair-wise terms* of the form $\forall(i \sim j)$ for a variable x_j, when x_j is rounded up from $\frac{1}{2}$ to 1, only the following cases are of interest :

Case 1 : if x_j was $\frac{1}{2}$ (rounded up to 1), x_i was 0 (remains 0), then $c_{ij}|x_i - x_j|$ changes from $\frac{1}{2}c_{ij}$ to c_{ij}

Case 2 : if x_j was $\frac{1}{2}$ (rounded up to 1), x_i was $\frac{1}{2}$ (rounded up to 1), then $c_{ij}|x_i - x_j|$ remains 0

Case 3 : if x_j was $\frac{1}{2}$ (rounded up to 1), x_i was 1 (remains 1), then $c_{ij}|x_i - x_j|$ changes from $\frac{1}{2}c_{ij}$ to 0

Thus, $\alpha^* = \sum_{i \sim j} \frac{1}{2}c_{ij}$ and $\alpha^{(0,1)} = \sum_{i \sim j} c_{ij}$ (in the worst case).

Hence, we get :

$$\alpha^{(0,1)} \leq 4\alpha^* \tag{9}$$

Similarly, for the *Unary Term*, we see that when x_j is rounded up, the data term changes from $\frac{1}{2}(w_{j0} + w_{j1})$ to w_{j1}. Hence, we get :

$$\beta^{(0,1)} \leq 4\beta^* \tag{10}$$

Combining the equations (9) and (10),we obtain the statement of the lemma, which is :

$$\alpha^{(0,1)} + \beta^{(0,1)} \leq 4(\alpha^* + \beta^*) \tag{11}$$

□

Note that a 2 approximation for the MRF terms can be obtained as shown in [3, 4]

Lemma 3. *Assuming γ^* and $\gamma^{(0,1)}$ are as defined above, then*

$$\gamma^{(0,1)} \leq 4\gamma^* \tag{12}$$

Proof. Now, let us consider the remaining histogram terms. When $x_j = \frac{1}{2}$ and belongs to a bin b, notice that $\mathcal{H}_b(j)x_j = \frac{1}{2}$, since $\mathcal{H}_b$ is a given binary coefficient. Hence,

$$\gamma^* = \lambda\frac{1}{2}\left(\sum_{l=1}^{n}\mathcal{H}_b(l)x_l - 2\hat{\mathcal{H}}_b\right) + \rho \tag{13}$$

Also, let us suppose that w_b and f_b are the number of voxels being assigned to 1 and $\frac{1}{2}$ in the bin b. Thus, we can write the above equation as :

$$\gamma^* = \lambda\frac{1}{2}w_b + \lambda\frac{1}{4}f_b - \lambda\hat{\mathcal{H}}_b + \rho \tag{14}$$

When x_j is rounded up, we see that

$$\gamma^{(0,1)} = \lambda w_b + \lambda f_b - 2\lambda\hat{\mathcal{H}}_b + \rho \tag{15}$$

Now, since $\rho = 2\lambda \max_b(\hat{\mathcal{H}}_b)$:

$$\rho - 2\lambda\hat{\mathcal{H}}_b \leq \rho - \lambda\hat{\mathcal{H}}_b \tag{16}$$

Both the sides of the inequality are positive, so we get :

$$\rho - 2\lambda\hat{\mathcal{H}}_b \leq 4(\rho - \lambda\hat{\mathcal{H}}_b) \tag{17}$$

And,

$$\lambda r_b + \lambda s_b \leq 4(\lambda\frac{1}{2}r_b + \lambda\frac{1}{4}s_b) \tag{18}$$

Combining (17) and (18), we get,

$$\gamma^{(0,1)} \leq 4\gamma^* \tag{19}$$

From (19), the lemma stated follows directly. □

Putting Lemmas 1-3 together leads to our main result below.

Theorem 1. *The rounding strategy described gives a factor* 4 *approximation to the Problem (1).*

3 Tightness of Approximation

It is interesting to check if the approximation obtained above can be improved for the rounding strategy adopted. We can construct examples when the approximation ratio becomes tight.

Example 1 : Consider a problem consisting of only 2 voxels, both assigned a value of $\frac{1}{2}$ in the optimal solution. Also, let $\hat{\mathcal{H}}_b = 0$. ρ can take any arbitary value. Clearly, for this optimal solution, the energy of any of the voxel, denoted by $E(v^*)$ can be given by :

$$E(v^*) = \frac{1}{2}(w_{j0} + w_{j1}) + \lambda(\frac{1}{4} + \frac{1}{4} - 2 \cdot 0) + \rho \tag{20}$$

$$= \frac{1}{2}(w_{j0} + w_{j1}) + \frac{1}{2}\lambda + \rho \tag{21}$$

Rounding the above solution, we get

$$E(v^{(0,1)}) = w_{j1} + \lambda(1 + 1 - 2 * 0) + \rho \tag{22}$$

$$= w_{j1} + 2\lambda + \rho \tag{23}$$

Let $\rho = 0$, then comparing the second terms, we see that there is at least a factor-4 approximation in this case.

Example 2 : Consider a problem consisting of 4 voxels, with an optimal solution vector of $0, 1, 1, \frac{1}{2}$ and $\hat{\mathcal{H}}_b = 1$, where all these voxels belong to only one bin. In this case, for the $\frac{1}{2}$ valued term, the energy before rounding (considering the rest 3 as neighbours) will be

$$E(v^*) = \frac{1}{2}(w_{j0} + w_{j1}) + \frac{3}{2}c + \lambda(\frac{1}{4} + 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} \cdot 1) \tag{24}$$

Where as, after rounding the energy will be :

$$E(v^{(0,1)}) = w_{j1} + c + \lambda(1 + 2 \cdot 1 - 2 \cdot 1 \cdot 1) \tag{25}$$

Clearly, the value in (25) is at least 4 times of (24).

Therefore, we have the following result :

Theorem 2. *For the presented rounding strategy, the approximation factor of* 4 *is tight.*

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