Linear Algebra and Differential Equations Chapter Summaries

Chapter 1

An n-vector is a list of n numbers. An $m \times n$ matrix is a rectangular array of m rows and n columns of numbers. Vectors are often thought of as column vectors, or $n \times 1$ matrices.

A scalar is a real number. Multiplication of a vector or matrix or by a scalar is defined by multiplying each element by the scalar. Vectors or matrices of the same dimension may be added element by element.

A square matrix has the same number of rows and columns. A diagaonal matrix is a square matrix whose values are 0 if the row number and column number are different. The identity matrix is a diagaonl matrix with 1s on the diagonal. A zero matrix has 0s for all elements. The transpose of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix whose ijth entry is a_{ji}. A square matrix is symmetric if it is equal to its transpose. An upper triangular matrix has 0s below the diagonal. A strictly upper triangular matrix has 0s on and below the diagonal.

The dot product of two n-vectors x and y is defined by $x \cdot y = \sum_{i=1}^{n} x_i y_i$. The length or norm of a vector is the square-root of the sum of its squared elements.

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Note that $\|x\| = \sqrt{x \cdot x}$. The angle θ between nonzero vectors x and y satisfies

$$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$$

Two vectors are orthogonal (perpendicular) if and only if their dot product is 0.

Chapter 2

A system of m linear equations in n unknowns may be expressed in matrix notation as $Ax = b$ where A is an $m \times n$ matrix of coefficients of the equations, x is the n-vector of unknowns, and b is the m-vector of constants on the right hand side of the equations. The augmented matrix is $[A|b]$. Systems may be solved by Gaussian elimination, which uses elementary row operations to put the augmented matrix into reduced row echelon form. These steps are equivalent to solving a system of equations by the substitution method.

The three elementary row operations are: (1) swapping two rows, (2) multiplying or dividing all elements of a row by a nonzero scalar, and (3) replacing a row by itself plus a scalar times another row.

When a matrix is in row echelon form, (1) each row is either all 0s or has a 1 as its first nonzero element, (2) no row has its first 1 to the left of the first 1 of any row above it.

The leading 1 of a row in row echelon form is called a pivot. A matrix is in reduced row echelon form if it is in row echelon form and each pivot is the only nonzero element in its column.

The basic strategy of Gaussian elimination is to use the three elementary row operations to put a matrix into reduced row echelon form. Begin by swapping rows and or multiplying the top row by a scalar to put a pivot in the top row. Then replace each row with a nonzero element beneath the pivot by itself plus a multiple of the top row to set all elements below the pivot to 0. Place a pivot in the second row and continue this process until there are 0s to the left of and below all of the remaining pivots. Then, starting with the last row with a pivot, replace each row with a nonzero element above the pivot by itself plus a multiple of the last row to set those elements to 0. Continue this back up the matrix for each pivot.

It is possible to use elementary row operations to put any matrix into reduced row echelon form and this reduced row echelon form matrix is unique. The number of pivots (also the number of nonzero rows) of the reduced row echelon form of a matrix A is called the rank of the matrix A.

The reduced row echelon form of the augmented matrix $(A|b)$ from a system of linear equations $Ax = b$ expresses the general solution to the system. The system is inconsistent and has no solutions if there is any row with a pivot in the last column (corresponding to an equation $0x_1 + \cdots + 0x_n = 1$). Otherwise it is consistent. There is a unique solution if the system is consistent and there are exactly n pivots. There are an infinite number of solutions if the system is consistent and there are fewer then n pivots. The general solution of a consistent system may be parameterized with $n-r$ free parameters where x is an n-vector and r is the rank of the augmented matrix. Each x_i corresponding to a column without a pivot may be freely chosen while each x_i corresponding to a column with a pivot is determined.

Chapter 3

Multiplication of a matrix A by a vector b is defined if the number of columns of A is equal to the number of elements of b. If A is an $m \times n$ matrix and b is an n-vector, the product Ab is the m-vector whose ith element is the dot product of the ith row of A and b.

Matrix multiplication with the $m \times n$ matrix A can be thought of as a mapping from R^n to R^m where the n-vector x is mapped to Ax. A linear mapping from R^n to R^m is any mapping L that satisfies two conditions: (1) $L(x+y) = L(x)+L(y)$, and (2) $L(cx) = cL(x)$. What turns out to be true is that every linear mapping is equivalent a matrix mapping and that all matrix mappings are linear. To test if two linear mappings L_1 and L_2 are equivalent (meaning that $L_1(x) = L_2(x)$ for all x) it is sufficient to show that map the elementary vectors e_j equally.

Examples of matrix mappings include rotations. The matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

will rotate a vector counterclockwise an angle θ. Other examples include projections onto lines through the origin and reflections across lines through the origin.

A system of linear equations $Ax = b$ is homogeneous if $b = 0$. Such systems always have at least one solution, namely $x = 0$. The principle of superposition says two things: (1) if y and z are solutions to $Ax = 0$, then $y+z$ is a solution, and (2) if y is a solution and c is a scalar, then cy is a solution. Both follow from linearity of matrix mappings.

A system of linear equations $Ax = b$ is inhomogeneous if $b \neq 0$. These systems may have no solutions. If there is at least one solution y, then every solution will be of the form $y + w$ where w is a solution to the homogeneous system $Ax = 0$.

The product of two matrices is defined when their inner dimensions agree. Specifically, the $m \times n$ matrix A and the $n \times p$ matrix B have a product $C = AB$ where C is the $m \times p$ matrix whose ijth entry is the dot product of the ith row of A and the jth row of B.

Matrix multiplication is not commutative in general. The product BA may not be defined — when it is, it may not equal AB.

A summary of some properties of matrix algebra follows. All properties assume that the specified matrix multiplications are defined. Capital roman letters are matrices while greek letters are scalars.

- $(AB)C = A(BC)$
- $(A+B)C = AC + BC$
- $D(A+B) = DA + DB$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $(\alpha A)B = \alpha(AB)$
- $(AB)^t = B^t A^t$

An inverse of a square $n \times n$ matrix A is an $n \times n$ matrix B that satisfies (1) $AB = I_n$ and (2) $BA = I_n$. If A has an inverse, it is unique and denoted A^{-1}.

If A is an $n \times n$ matrix, the following are equivalent:

(a) A is invertible

(b) The system of equations $Ax = b$ has a unique solution for each b in R^n.

(c) The only solution to $Ax = 0$ is 0.

(d) A is row equivalent to I_n.

(e) $\det A \neq 0$ (we've only defined this in the 2×2 case.

If A is a square invertible matrix, then the unique solution to $Ax = b$ is $x = A^{-1}b$.

To find the inverse of a matrix A, create the augmented matrix $(A|I_n)$ and do elementary row operations to put A into reduced row echelon form. The same operations turn I_n into A^{-1}, so the resultant matrix is $(I_n|A^{-1})$

The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$. The determinant may be interpreted as the area of the parallelogram with vertices $(0,0), (a,c), (b,d), and (a+b, c+d)$. A matrix is invertible if and only if its determinant is not 0. The inverse of A is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Chapter 4

You are only responsible for sections 4.1 through 4.4.

This chapter deals with first order ordinary differential equations, which can be expressed as $\dot{x} = f(x,t)$. There are several standard f which have closed-form solutions, but more commonly there are no closed-form solutions. Examples with closed-form solutions you should recognize are listed here.

$\dot{x} = \lambda x$

All solutions to this equation are of the form $x(t) = Ke^{\lambda t}$. You can use an initial value to find a specific value for K, which is $x(0)$. The function $x(t) \equiv 0$ is a solution ($K = 0$).

Examples include continuously compounding interest and Newton's law of cooling. Note that the differential equation $\dot{x} = \lambda(x - c)$ is not in the standard form but letting $y = x - c$ gives the differential equation $\dot{y} = \lambda y$ which can be solved.

$\dot{x} = h(t)$

Integrate both sides with respect to t. This is simply integral calculus that states that the x is the antiderivative of h.

$\dot{x} = g(x)h(t)$

To solve this equation, divide both sides by $g(x)$. Integrate both sides with respect to t and use the chain rule to simplify the lefthand side. Namely,

$$\int \frac{dx}{g(x)} = \int h(t)\, dt$$

When the right-hand side of the differential equation $\dot{x} = f(x,t)$ does not depend explicitly on t, the differential equation is autonomous.

A line field shows the value of the derivative $\dot{x}$ at regularly spaced points in the (t,x) plane, and is useful for visualizing possible solutions to the differential equation.

In a nonautonomous equation, the line field may look different at different t. In an autonomous equation, the line field is the same for all t.

The autonomous differential equation $\dot{x} = g(x)$ has the solution $x(t) = x_0$ if $g(x_0) = 0$. The function $x(t) = x_0$ is called an equilibrium solution. If $g'(x_0) \neq 0$, x_0 is called a hyperbolic equilibrium. The equilibrium x_0 is asymptotically stable if $g'(x_0) < 0$ and asymptotically stable if $g'(x_0) > 0$.

Hints

Make sure that you are familiar with all of the definitions of terms used above. On the test, you will be expected to use these definitions, and not expected to only write them. Practice doing homework problems and others like them, both required and suggested. You will not be asked to prove anything.